FLUID DYNAMICS

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Contents

Part I

Description of Fluids

1 Conservation of Mass

Conservation of mass is a fundamental tenet of classical mechanics.

In this chapter, we review the classical theory of currents and densities. We then use this theory to establish the continuity equation for conserved and for non-conserved quantities, and interpret the results physically. Lastly, we apply the theory to the description of conservation of mass in the fluid continuum. In later chapters, we also apply the theory developed here to the description of the conservation of energy, momentum, and angular momentum in fluids.

1.1 Continuity Equation for Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.1}$$

1.2 Alternative Forms of the Continuity Equation

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \; \nabla \cdot \vec{v} = 0 \tag{1.2}$$

or

$$\frac{D\rho}{Dt} + \rho \; \nabla \cdot \vec{v} = 0 \tag{1.3}$$

or

$$\frac{1}{\rho} \; \frac{D\rho}{Dt} + \nabla \cdot \vec{v} = 0 \tag{1.4}$$

or

$$\frac{D \ln \rho}{Dt} + \nabla \cdot \vec{v} = 0 \tag{1.5}$$

1.3 Incompressibility

Fluids can be broadly divided into two cate3gories: liquids, and gases. Gases are highly compressible and their density depends on how much matter has been compressed into the volume that they currently occupy. Density in gas flows can vary greatly from point to point in the gas. In contrast, fluids are extremely difficult to compress. It can be done, but huge pressures are required to achieve minute changes of volume. The volume of a given mass of gas is also sensitive to temperature changes, with the volume increasing with increasing temperature. However, these changes of volume are generally very small. Thus, to good approximation, the density of liquids may be regarded as constant. Strangely, the density of gases which are moving at speeds much less than the speed of sound in the gas may also be regarded as constant. For the density in a gas to change, one would require its density gradient to be large, and thus can only happen if the gas is travelling at near to the speed of sound.

****(Make this clearer!)****

It is therefore common to approximate many flows by modelling them as incompressible.

We have seen in our study of fluid kinematics that the rate at which a fluid element is being compressed is given by

$$\theta = \lim_{V\to 0} \frac{1}{V}\frac{dV}{dt} \quad = \nabla\cdot\vec{v} \tag{1.6}$$

This result is consistent with the continuity equation derived above, and could have been derived from it. The mass δm of a fluid element is constant, so dividing V by this mass and taking the limit as the fluid element is made vanishingly small yields the specific volume v of the fluid element. Thus,

$$\theta \; = \; \lim_{V\to 0} \frac{1}{V}\frac{dV}{dt} = \lim_{V\to 0} \frac{\delta m}{V}\frac{d(V/\delta m)}{dt} = \frac{1}{v}\frac{dv}{dt} = \nabla\cdot\vec{v}$$

Then using the fact that $v = 1/\rho$ we get,

$$\theta = \frac{1}{v}\frac{dv}{dt} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

Combining, this gives

$$-\frac{1}{\rho}\frac{d\rho}{dt} = \nabla\cdot\vec{v}$$

which is the continuity equation.

We can now state clearly what we mean by saying that a fluid is incompressible. Consider some given fluid element. The element is said to be incompressible if, as it flows, it does not change its volume. Thus, incompressibility means that

$$\nabla\cdot\vec{v} = 0$$

In turn, through the continuity equation, this also means that

$$\frac{1}{\rho}\frac{d\rho}{dt} = 0 \tag{1.7}$$

that is, it does not change its density as it moves.

It is important to understand that this, by itself, *does not mean* that the density ρ of the fluid is constant. It only means that the density *of the given fluid element* remains constant as it moves. Some fluids may be like your mother's gravy: lumpy. That is, at any given time, the density of the fluid changes from position to position. As it moves, the lumps move with it. If the fluid is incompressible, the density of any portion of this lumpy fluid does not change as it moves. This constancy of the density of the fluid lumps is also described by the condition that $\nabla \cdot \vec{v} = 0$.

The origin of lumps in gravy is an inhomogeneity in the chemical composition of the fluid at the initial instant. In this course, we deal only with fluids of homogeneous chemical composition. These are described by the condition that $\rho =$ constant. From the equation of continuity, if the fluid has uniform density initially, *and* the fluid is incompressible, then the density of the fluid is constant. For us, therefore, an incompressible fluid will mean, unless otherwise stated, a fluid with

$$\rho = \text{constant} \tag{1.8}$$

Note that this is a *stronger* condition than $\nabla \cdot \vec{v} = 0$. Clearly,

$$\rho = \text{constant} \Rightarrow \nabla \cdot \vec{v} = 0$$

but not conversely. Fluids of constant density are thus a special case of incompressible fluids. The general case is your mother's gravy.

It should be noted that a fluid need not be incompressible in order for its flow to behave as if it were incompressible. For a compressible fluid to have different densities at different points, it is necessary to have different pressures at different points. Pressure inhomogeneities which are not balanced by other forces, like gravity, drive mechanical motions in the fluid whose natural tendency is to eliminate the gradients, except insofar as they are needed to balance other forces. Pressure inhomogeneities propagate through the fluid at the speed of sound. So, if the fluid is flowing at speeds considerably less than the speed of sound, these disturbances, if present, will disappear rapidly, and the density variations with them. So, unless the fluid motions are at speed comparable to the speed of sound, the fluid can be considered to be incompressible. In this case, we say that the *flow is incompressible*, even though technically the fluid is not.

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2 Equation of Motion

The equations of motion for a fluid can be expressed in a variety of different forms. The form one chooses depends on what effects one wishes to describe, and on what specialised assumptions one makes. However, all of the different forms are special cases of a single, general form of the equations of motion and can be obtained from it.

In this chapter, I first construct the equation of motion in its most general form. I then deduce from the the most common of the specialised forms that one encounters, stating explicitly what additional assumptions have gone into their construction.

2.1 General Form

The equations of motion for a fluid are obtained by applying Newton's second law of motion to an infinitesimally small element of fluid. This requires us to determine what forces act on the element, and then use Newton second law to determine its acceleration.

This is not the only way to obtain the equations of motion, but it is the most direct. An alternative route uses the principle of the conservation of momentum. This yields a continuity equation for the fluid momentum. Taken together with the equation for the conservation of mass, the momentum continuity equation is equivalent to the equation of motion that we will derive here.

An ordinary fluid is subject to only two forces which act on it. These are the contact force exerted on it by contact with the surrounding fluid or solid boundaries, and the gravitational force. In some more specialised studies, other forces may also act on the fluid. For example, if the fluid consists of charged particles and moves in an electromagnetic field, then the fluid will experience also an electromagnetic force which must be added into the equations of motion. This leads to a specialised branch of fluid mechanics, called *magnetohydrodynamics*. It is a difficult subject, because of the peculiar nature of electromagnetic forces, and is best studied as a special topic. Another example is that of a fluid in which energy is transported by radiation. The interaction of the fluid material with a radiation field produces a radiation force, which must be taken into account in the equations of motion. This too is a specialised study, and is best left to a chapter devoted to the investigation of these effects.

In this chapter, therefore, we will assume that the fluid is subject to only two forces: contact forces arising from its contact with the surrounding fluid and boundaries, and the force of gravity. At this stage, we assume nothing about the nature of the contact forces. We therefore describe them by a general stress tensor σ^{ij}. The various specialised forms of the equations of motion all arise as we make different assumptions about the precise form of σ^{ij}.

Consider a small element of fluid. The fluid element consists of a fixed number of fluid points and is easily defined by specifying a small domain in the Lagrangian coordinate system for the fluid. Since this domain is well defined, we can use it to track the motion of the fluid element through space as the fluid moves. The element is thus a perfectly well defined entity in the theory.

Because we have imposed on the fluid the requirement of mass conservation, the fluid element has a fixed mass δm as it moves. Suppose that at time t it occupies a volume δV in space. Then $\delta m = \rho\, \delta V$. The total contact force acting on this fluid element is given by

$$\delta F_{\text{contact}} = \partial_j \sigma^{ij}\ \delta V$$

while the total gravitational force on the element is given by

$$\delta F^i_{\text{gravitational}} = \delta m\ g^i$$

By Newton's second law, the sum of these forces must equal the mass of the element multiplied by its acceleration. Thus,

$$\begin{aligned} \delta m\ \frac{Dv^i}{Dt} &= \delta F_{\text{contact}} + \delta F^i_{\text{gravitational}} \\ &= \partial_j \sigma^{ij}\ \delta V + \delta m\ g^i \end{aligned}$$

Inserting the expression for δm and cancelling the common factor δV, this gives

$$\rho\ \frac{Dv^i}{Dt} = \partial_j \sigma^{ij} + \rho g^i \tag{2.1}$$

or, equivalently, with the acceleration written out in full,

$$\rho \left(\frac{\partial v^i}{\partial t} + v^k \partial_k v^i \right) = \partial_j \sigma^{ij} + \rho g^i \tag{2.2}$$

These are the equations of motion for the fluid written out in their most general form. Sone authors add a force per unit mass, f^i, to the right hand side to indicate explicitly that any other forces that have not been mentioned explicitly can be added into the equations. They thus write these equations as,

$$\rho \left(\frac{\partial v^i}{\partial t} + v^k \partial_k v^i \right) = \partial_j \sigma^{ij} + \rho g^i + \rho f^i \tag{2.3}$$

where the factor ρ in front of f^i is needed to convert a force per unit mass into a force per unit volume, to match the remaining terms on the right hand side.

Other authors prefer to work directly in terms of force per unit mass, which is more natural. This is accomplished by removing the factor ρ from all terms in the equation, to get

$$\frac{\partial v^i}{\partial t} + v^k \partial_k v^i = \frac{1}{\rho}\, \partial_j \sigma^{ij} + g^i + f^i \tag{2.4}$$

However, all of these forms are trivial reorganisations of the fundamental equation, which is

$$\boxed{\rho\, \frac{Dv^i}{Dt} = \partial_j \sigma^{ij} + \rho g^i + \rho f^i}$$

Equations of Motion

The general equation for fluids derived in the previous section contains 13 unknown fields. They are the three components of the fluid velocity v^i, the fluid density ρ, the nine components of the stress tensor σ^{ij}. In many problems, the components g^i of the gravitational field and the three components f^i of any additional external force that acts on the fluid are regarded as known. If they are dynamical participants in the motion of the fluid, affected by how the fluid moves, then these are 6 more unknowns for which we shall need to solve, bringing the total of unknown fields up to 19. So far, we have only for equations for them, three components of the equation of motion and the continuity equation for mass. It is clear that we will need to search for more equations. We will obtain some of these equations from dynamical principles, like the conservation of energy and of angular momentum. For the remaining equations, we will need to consider the properties of the fluid material. Some of these equations are fundamental, such as the equation of state and the heat equation for the material, and the laws of radiation. Others are semi-empirical and take the form of constitutive relations, like the relationship between stress and strain-rates, heat conduction, diffusion, and so on.

2.2 Conservation of Momentum

2.3 Conservation of Energy

2.4 Conservation of Angular Momentum

2.5 Newtonian Fluids

2.6 Euler's Equations

To construct Euler's equations, we assume that the fluid is free from friction. The most general form for the stress tensor for a fluid is,

$$\sigma^{ij} = -P\delta^{ij} + \pi^{ij}$$

where the tensor π^{ij} represents the action of viscosity on the fluid. To say that the fluid is free from friction is to say that $\pi^{ij} = 0$. Thus, the stress tensor becomes

$$\sigma^{ij} = -P\delta^{ij}$$

and hence

$$\partial_j \sigma^{ij} = \partial_j(-P\delta^{ij}) = -(\partial_j P)\delta^{ij} = -\partial^i P$$

The equation of motion thus becomes,

$$\rho \left(\frac{\partial v^i}{\partial t} + v^k \partial_k v^i \right) = -\partial^i P + \rho g^i + \rho f^i \tag{2.5}$$

Commonly, these equations are written in vector notation, giving

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla P + \rho \vec{g} + \rho \vec{f} \tag{2.6}$$

or, equivalently,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\, \nabla P + \vec{g} + \vec{f} \tag{2.7}$$

Of course, no real fluid is free from the action of viscosity. A fluid with no viscosity exists only in our minds, as an idea, and has no reality. For this reason, such fluids are said to be *ideal*, and Euler's equations are thus said to be the equations of motion for an *ideal fluid*.

Note that we have not here defined an ideal fluid. We have simply noted that, if a fluid is ideal, then it must obey Euler's equations. In a later chapter, we will consider more carefully what properties are needed to define an ideal fluid, and we will give a precise definition of this concept.

Though all real fluids are viscous, their motions are influenced by the action of viscosity in different degrees under different circumstances. In some situations, the effects of viscosity are negligible when compared to the action of the other forces like pressure or gravity. In these situations, Euler's equations are a good approximation to the correct equations of motion and we can treat the fluid as if it were ideal. In most flows, the action of viscosity becomes important only near solid boundaries, where velocity gradients are significant,, but behave like ideal fluids away from the boundaries. This fact has given rise to a specialised study called *boundary layer theory*, in which the effects of viscosity are taken into account only in the vicinity of the boundaries, and ignored in the regions away from the boundaries. The solutions for the different regions are then matched at the interface. This technique gives a good description of the effects of viscosity in very many situations of practical interest.

The fact that in many flows the effects of viscosity are negligible, and the success of boundary layer theory, means that Euler's equations are not merely an academic abstraction with no practical use. Their study thus forms an important part of fluid mechanics.

2.7 Navier-Stokes Equations

The most general form of the viscous stress tensor for isotropic Newtonian fluids is given by

$$\pi_{ij} = \mu \left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\, \delta_{ij} (\nabla \cdot \vec{v}) \right) + \lambda \delta_{ij} (\nabla \cdot \vec{v}) \tag{2.8}$$

where μ is the coefficient of viscosity, and λ the coefficient of bulk viscosity. In their original work, both Navier and Stokes assumed,

- The density of the fluid is constant. This is a very good approximation for liquids. an immediate consequence of this assumption is that $\nabla \cdot \vec{v} = 0$.
- The viscosity μ is constant.

These two assumptions give,

$$\pi_{ij} = \mu \left(\partial_i v_j + \partial_j v_i \right) \tag{2.9}$$

and hence

$$\partial_j \pi^{ij} = \mu \nabla^2 v^i \tag{2.10}$$

The equations of motion then become,

$$\rho \left(\frac{\partial v^i}{\partial t} + v^k \partial_k v^i \right) = -\partial^i P + \mu \nabla^2 v^i + \rho g^i + \rho f^i \tag{2.11}$$

or, in vector form,

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla P + \mu \nabla^2 \vec{v} + \rho \vec{g} + \rho \vec{f} \tag{2.12}$$

This equation is most commonly expressed in the form,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{v} + \vec{g} + \vec{f} \tag{2.13}$$

These are the Navier-Stokes equations in their original form. Most authors, when referring to the Navier-Stokes equations, have these equations in mind. Note that, with the above assumptions, both ρ and μ are constants. Their ratio,

$$\nu = \frac{\mu}{\rho} \tag{2.14}$$

is thus also a constant. Since ρ and μ are properties of the fluid material, so also is ν, called the *kinematic viscosity* of the fluid.

The Navier-Stokes equations look very similar to the Euler equations. The only difference between them is the additional term $\nabla^2 \vec{v}$. As mathematical entities, however, these equations are very different. The presence of second derivatives with respect to the space variables in the Navier-Stokes equations alters radically the boundary conditions that need to be specified in order to define unique solutions, and dramatically changes also the mathematical techniques that need to be used to solve them. The new term is therefore not a minor modification of the equations. The Euler and the Navier-Stokes equations therefore require separate study and must be regarded as distinct chapters in the theory of fluid dynamics.

2.8 More General Equations of Motion

We obtain the most general form of the equations of motion for isotropic Newtonian fluids by removing the special assumptions made by Navier and Stokes. This means that we must use the viscous stress tensor for isotropic Newtonian fluids in its most general form, making no assumptions about either the fluid density or the coefficients of viscosity. This gives, for the stress tensor,

$$\begin{aligned}
\partial_j \pi^{ij} &= \partial_j \left[\mu \left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\, \delta^{ij} (\nabla \cdot \vec{v}) \right) \right] + \delta^{ij} \partial_j \left[\lambda (\nabla \cdot \vec{v}) \right] \\
&= \partial_j \left[\mu \left(\partial^i v^j + \partial^j v^i \right) \right] + \partial^i \left[\left(\lambda - \frac{2}{3}\, \mu \right) (\nabla \cdot \vec{v}) \right]
\end{aligned}$$

The equations of motion then become,

$$\begin{aligned}
\rho \left(\frac{\partial v^i}{\partial t} + v^k \partial_k v^i \right) &= -\partial^i P + \partial_j \left[\mu \left(\partial^i v^j + \partial^j v^i \right) \right] \\
&\quad + \partial^i \left[\left(\lambda - \frac{2}{3}\, \mu \right) (\nabla \cdot \vec{v}) \right] + \rho g^i + \rho f^i
\end{aligned} \tag{2.15}$$

This is the most general equation of motion for an isotropic Newtonian fluid.

A common specialisation of this equation assumes that the viscosities μ and

λ are constants. Then,

$$\begin{aligned}\partial_j \pi^{ij} &= \mu \left[\partial_j \left(\partial^i v^j + \partial^j v^i\right)\right] + \left(\lambda - \frac{2}{3}\,\mu\right) \partial^i (\nabla \cdot \vec{v}) \\ &= \mu \left[\partial^i (\nabla \cdot \vec{v}) + \nabla^2 v^i\right] + \left(\lambda - \frac{2}{3}\,\mu\right) \partial^i (\nabla \cdot \vec{v}) \\ &= \mu \nabla^2 v^i + \left(\lambda + \frac{1}{3}\,\mu\right) \partial^i (\nabla \cdot \vec{v})\end{aligned}$$

so that the equation of motion becomes,

$$\rho \left(\frac{\partial v^i}{\partial t} + v^k \partial_k v^i\right) = -\partial^i P + \mu \nabla^2 v^i + \left(\lambda + \frac{1}{3}\,\mu\right) \partial^i (\nabla \cdot \vec{v}) + \rho g^i + \rho f^i$$

or,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho}\, \nabla P + \mu \nabla^2 \vec{v} + \frac{1}{\rho} \left(\lambda + \frac{1}{3}\, \frac{\mu}{\rho}\right) \nabla (\nabla \cdot \vec{v}) + \vec{g} + \vec{f} \tag{2.16}$$

2.9 Non-Newtonian Fluids

The above catalogue of equations exhausts the equations that are possible for Newtonian fluids. The Newtonian assumption is that the viscous stress tensor is a linear function of the rate of strain tensor. Many real fluids do not have the Newtonian property. Those that do not are called *non-Newtonian.*

In a non-Newtonian fluid, the relation between the shear stress and the strain rate is not linear homogeneous, and can even be time-dependent. The general form of the stress-rate of strain relationships of some important types of non-Newtonian fluids are shown in Figure 2.1. Examples of non-Newtonian fluids

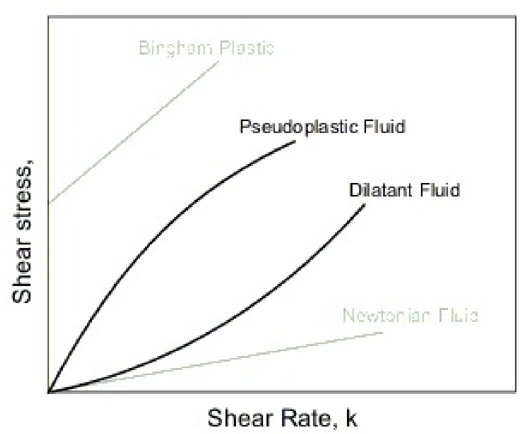

Figure 2.1 Typical stress-rate of strain curves for some important types of non-Newtonian fluids.

include polymer solutions, molten polymers and many common fluids such as gravy, paint, detergents and human blood.

Non-Newtonian fluids do not lend themselves to the elegant and precise methods analysis developed for Newtonian fluids. They can nevertheless be modelled within the framework of the general equation of motion introduced in the first section of this Chapter. Successful modelling of their properties reduces in essence to finding a relationship between stress and rate of strain that adequately reproduces their properties.

2.10 Summary

General Equation of motion	$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\ \partial_j \sigma^{ij} + g^i + f^i$
Euler's Equation	$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\ \nabla P + \vec{g} + \vec{f}$
Navier-Stokes Equation	$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{v} + \vec{g} + \vec{f}$
Newtonian Fluid (Const. λ, μ)	$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\ \nabla P + \frac{\mu}{\rho}\nabla^2\vec{v} + \frac{1}{\rho}\left(\lambda + \frac{1}{3}\ \mu\right)\nabla(\nabla\cdot\vec{v}) + \vec{g} + \vec{f}$
General Newtonian Fluid	$\rho\left(\frac{\partial v^i}{\partial t} + v^k\partial_k v^i\right) = -\partial^i P + \partial_j\left[\mu\left(\partial^i v^j + \partial^j v^i\right)\right] + \partial^i\left[\left(\lambda - \frac{2}{3}\ \mu\right)(\nabla\cdot\vec{v})\right] + \rho g^i + \rho f^i$

3 Energy Equation

3.1 Work-Energy Theorem

Newton's second law applied to an infinitesimal element of fluid leads to the equations of motion for the fluid, given by

$$\rho\,\frac{Du^i}{Dt} = \rho X^i + \partial_k \tau^{ki} \tag{3.1}$$

This equation describes the action on the fluid element of the forces that are acting on it. $\vec{X}$ is the total force per unit mass exerted by an external force field on the fluid element, so $\rho\vec{X}$ is the total force per unit volume exerted on the element by this field. Also $\partial_k \tau^{ki}$ is the total force per unit volume exerted on the element by the adjacent fluid through contact. The right hand side therefore represents the total force per unit volume acting on the element. The left hand side is the mass per unit volume of the element, multiplied by its acceleration. Equation (3.1) is therefore Newton's second law for fluids.

When applied to point particles, Newton's second law leads by necessary consequence to the work-energy theorem. This theorem states that the power delivered to the particle by the impressed forces is equal to the rate of increase of its kinetic energy. We now use equation (3.1) to deduce an analogous result for fluids.

The power per unit mass delivered by the external force field $\vec{X}$ to a given fluid element is given by $\vec{X}\cdot\vec{u}$. The power per unit volume delivered by the applied forces to the fluid is therefore $\rho\vec{X}\cdot\vec{u}$. This fact suggests that we can obtain a work-energy theorem by taking the dot product of equation (3.1) with the fluid velocity field $\vec{u}$. This gives,

$$\rho u_i\,\frac{Du^i}{Dt} = \rho u_i X^i + u_i \partial_k \tau^{ki} \tag{3.2}$$

Using the product rule of differentiation, this equation can be written as

$$\rho\,\frac{D}{Dt}\left(\frac{1}{2}\,u_i u^i\right) = \rho u_i X^i + \partial_k(u_i \tau^{ki}) - \tau^{ki}\partial_k u_i \tag{3.3}$$

Consider now a finite fluid element of volume V and with boundary surface S as it moves with the fluid. Integrating (3.3) over the volume V, we get

$$\int_V \rho\,\frac{D}{Dt}\left(\frac{1}{2}\,u_i u^i\right) dV = \int_V \rho u_i X^i\;dV + \int_V \partial_k(u_i\tau^{ki})\;dV - \int_V \tau^{ki}\partial_k u_i\;dV \tag{3.4}$$

The integral on the left hand side can be rewritten by noting that the mass $dm = \rho\ dV$ of an infinitesimal fluid element does not change with time as it moves, so that

$$\frac{D}{Dt}\,(dm) = \frac{D}{Dt}\,(\rho\ dV) = 0 \tag{3.5}$$

and hence

$$\int_V \rho\ \frac{D}{Dt}\left(\frac{1}{2}\ u_i u^i\right) dV = \int_V\ \frac{D}{Dt}\left(\frac{1}{2}\ \rho u_i u^i\ dV\right) = \frac{D}{Dt}\ \int_V\ \frac{1}{2}\ \rho u_i u^i\ dV \tag{3.6}$$

The second integral on the right-hand side of (3.4) can be converted into a surface integral by the divergence theorem,

$$\int_V\ \partial_k(u_i \tau^{ki})\ dV = \oint_S\ u_i \tau^{ki}\ dS_k \tag{3.7}$$

so that (3.4) becomes

$$\frac{D}{Dt}\ \int_V\ \frac{1}{2}\ \rho u_i u^i\ dV = \int_V\ \rho u_i X^i\ dV + \oint_S\ u_i \tau^{ki}\ dS_k - \int_V\ \tau^{ki}\partial_k u_i\ dV \tag{3.8}$$

The integral on the left-hand side of this equation is the total kinetic energy of the finite fluid element at any instant t. The derivative D/Dt of this integral is therefore the rate of increase of kinetic energy of the fluid element as it moves under the action of the impressed forces. Equation (3.8) is therefore the work-energy theorem for the finite fluid element. To understand its content, we need to interpret the terms occurring on the right hand side.

The first two terms are easy to interpret. Since $\rho u_i X^i$ is the power delivered per unit volume by the force fields to the fluid, we have

$$\int_V\ \rho u_i X^i\ dV = \begin{cases} \text{Total power delivered by the force fields} \\ \text{to the finite fluid element} \end{cases}$$

Similarly, in the second integral, $\tau^{ki}\ dS_k$ is the force exerted on the element of area dS_k of the boundary surface of the fluid element through contact with the surrounding fluid. Thus, $u_i \tau^{ki}\ dS_k$ is the power delivered to the element by the surrounding fluid by virtue of its contact with the element of area dS_k. Integrating over the entire boundary surface of the finite element, we obtain the *total* power delivered to the element by the surrounding fluid through contact,

$$\oint_S\ u_i \tau^{ki}\ dS_k = \begin{cases} \text{Total power delivered by surrounding fluid} \\ \text{to the finite fluid element through contact} \end{cases}$$

Were the fluid element a point particle, all of the delivered power would contribute to the increase of its kinetic energy and there would be no other term present in the equation. However, the element is not a point particle. A fluid is made up of an huge number of microscopic particles, each with its own degrees of freedom. The fluid element thus has internal structure and possesses a multitude of internal microscopic degrees of as well as its macroscopic degrees of freedom. Each internal degree of freedom is able to accept or reject energy.

The fluid element is therefore a thermodynamic system. The power delivered to the element therefore need not all be used to increase its kinetic energy. Part of it may be channelled into raising the internal energy of the element and in compressing it. The last integral therefore represents the action of the applied forces in compression of the element and in raising its internal energy,

$$\int_V \tau^{ki}\partial_k u_i \; dV = \left\{ \begin{array}{l} \text{Total power delivered by the action of applied} \\ \text{and contact forces, used to compress the finite} \\ \text{fluid element and to increase its internal energy} \end{array} \right.$$

It should be remembered that this term represents only the contribution to the internal energy of the element by the action of the applied forces. There are other mechanisms for increasing its internal energy, including the influx of heat by conduction, the deposition of heat by radiation, and the generation of heat by chemical and nuclear reactions.

To see in detail the significance of this last term, we need to make use of the constitutive relation that relates the fluid stresses to its rate of strain. In a Newtonian fluid, this is given by

$$\tau_{ij} = (-p + \lambda\theta)\delta_{ij} + \mu e_{ij} \tag{3.9}$$

Here, θ and e_{ij} are abbreviations for

$$\theta = \nabla \cdot \vec{u} \tag{3.10}$$

$$e_{ij} = \partial_i u_j + \partial_j u_i \tag{3.11}$$

and represent the volumetric strain rate (rate of change of volume of the convecting element per unit volume), and the total strain rate. Also, p is the pressure corresponding to the local density and temperature. This is the thermodynamic pressure, or hydrostatic pressure. It is the only pressure present when the fluid is at rest and is due to the random motions of the molecules that make up the fluid. When the fluid is in motion, there is an additional contribution to the normal stress on the surface of the fluid element due to the motion. Though sometimes called a ‘pressure’ because it is a contribution to the normal stress on the surface of the fluid element, it is not related to the internal state of the fluid material, and so cannot be counted as a part of the thermodynamic pressure due to the internal constitution of the fluid material. *(Is this assertion correct???)* The quantities λ and μ are physical properties of the fluid. This means that their values are independent of the state of motion of the fluid and depend only on the thermodynamic state of the fluid element. Some books state that they are “physical constants”, but this may be a misrepresentation since they might be functions of the density and temperature of the element. *(Is this assertion correct???)*

Using the constitutive relation (3.9), the integrand of the last integral in (3.8) becomes

$$\begin{aligned}\tau^{ki}\partial_k u_i &= \left[(-p+\lambda\theta)\delta^{ki}+\mu e^{ki}\right]\partial_k u_i\\ &= (-p+\lambda\theta)(\delta^{ki}\partial_k u_k)+\mu e^{ki}\partial_k u_i\\ &= (-p+\lambda\theta)\theta+\mu e^{ki}\partial_k u_i\end{aligned}$$

This last term can be rewritten using the symmetry of e^{ki} in the form

$$e^{ki}\partial_k u_i = \frac{1}{2}\, e^{ki}(\partial_k u_i+\partial_i u_k) = \frac{1}{2}\, e^{ki}e_{ki}$$

Explicitly,

$$e^{ki}\partial_k u_i = (e_{11})^2+(e_{22})^2+(e_{33})^2+(e_{12})^2+(e_{23})^2+(e_{31})^2$$

which is seen to be positive definite. So,

$$\tau^{ki}\partial_k u_i = -p\ \theta+\lambda\theta^2+\frac{1}{2}\mu\ e^{ki}e_{ki} \tag{3.12}$$

and hence

$$\int_V \tau^{ki}\partial_k u_i\ dV = -\int_V p\ \theta\ dV+\int_V\left(\lambda\theta^2+\frac{1}{2}\mu\ e^{ki}e_{ki}\right)dV$$

The physical significance of the first term on the right hand side is easily recognised. Its integrand, $-p\,\theta$ is the rate per unit volume at which work is done by the forces on an infinitesimal fluid element compress it,

$$-p\ \theta = -p\ \nabla\cdot\vec{u} = -\ \frac{1}{\delta V}\ p\ \frac{d\ \delta V}{dt} = \begin{cases}\text{Rate of doing work on an infinitesimal}\\ \text{fluid element of volume } \delta V \text{ to compress it}\end{cases}$$

so that

$$-\int_V p\ \theta\ dV = \begin{cases}\text{Rate of doing work on the finite fluid}\\ \text{element of volume } V \text{ to expand it}\end{cases}$$

The remaining integral is positive definite, and vanishes when the relative motion between adjacent fluid elements vanishes, that is, when $e_{ij} = \partial_i u_j + \partial_j u_i = 0$ (from which it also follows that $\theta = 0$). is meaning is seen by rewriting the work energy theorem in the form,

$$\int_V \rho u_i X^i\ dV+\oint_S u_i\tau^{ki}\ dS_k = \frac{DK}{Dt}-\int_V p\ \theta\ dV+\int_V\left(\lambda\theta^2+\frac{1}{2}\mu\ e^{ki}e_{ki}\right)dV \tag{3.13}$$

where

$$K = \int_V \frac{1}{2}\ \rho u_i u^i\ dV$$

is the total kinetic energy of the finite fluid element. The left-hand side of equation (3.13) now represents the total power delivered by the external force fields and the contact forces to the fluid element. This power is used for three things, as indicated by the presence of three terms on the right-hand side. The first two terms are, respectively, the power needed to increase the total kinetic energy of the fluid element, and the power needed to compress it. The final term, which is always positive (or zero when the fluid is not strained), must therefore represent the rate at which mechanical energy is dissipated in, or lost from, the fluid element. Together, the energy of compression and the energy dissipated, constitute the energy that goes into raising the internal energy of the fluid element. Therefore, from the point of view of the thermodynamics of the fluid element, the dissipation of mechanical energy from the element has the same effect as an irreversible flow of heat into the element.

The work energy theorem (3.13) appears in the literature in a variety of equivalent forms, all of them based on the following ways of rewriting its various terms. First, we note that fundamental force fields are all conservative, and so can be written in terms of a potential per unit mass. We thus write

$$X^i = -\partial^i \phi \tag{3.14}$$

This means that

$$\int_V \rho u_i X^i \, dV = -\int_V \rho u_i \partial^i \phi \, dV \tag{3.15}$$

In many cases of interest, the force fields are time-independent and hence

$$\frac{\partial \phi}{\partial t} = 0$$

Hence,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u^i \partial_i \phi = u^i \partial_i \phi$$

so that (3.15) becomes

$$\int_V \rho u_i X^i \, dV = -\int_V \rho \, \frac{D\phi}{Dt} \, dV = -\int_V \frac{D}{Dt}(\rho\phi \, dV) = -\frac{D}{Dt}\int_V \rho\phi \, dV \tag{3.16}$$

The integral on the right-hand side is easily interpreted: it is the total potential energy of the finite fluid element in the force fields, that is, the total amount of energy needed to assemble the fluid element in the given force fields,

$$P = \int_V \rho\phi \, dV = \left\{ \begin{array}{l} \text{total potential energy of the finite} \\ \text{fluid element in the force fields} \end{array} \right.$$

With this notation, (3.13) becomes

$$\oint_S u_i \tau^{ki} \, dS_k = \frac{D}{Dt}(K + P) - \int_V p \, \theta \, dV + \int_V \left(\lambda\theta^2 + \frac{1}{2}\mu \, e^{ki} e_{ki} \right) dV \tag{3.17}$$

We now define the configurational energy I of the fluid element. This is the total energy 'locked up' in the fluid element by virtue of its configuration, measured relative to some standard reference configuration. Then, the rate of change of configurational energy as the fluid element moves with the fluid is given by

$$\frac{DI}{Dt} = -\int_V p \, \theta \, dV = \left\{ \begin{array}{l} \text{total power delivered to the finite} \\ \text{fluid element to change its configuration} \end{array} \right.$$

Denote the integrand of the final integral by Φ. Thus,

$$\begin{aligned}\Phi &= \lambda\theta^2 + \frac{1}{2}\mu\ e^{ki}e_{ki} \\ &= \lambda\theta^2 + \frac{1}{2}\mu[(e_{11})^2 + (e_{22})^2 + (e_{33})^2 + (e_{12})^2 + (e_{23})^2 + (e_{31})^2]\end{aligned}$$

With these notations, (3.17) becomes, for fluids moving time-independent force fields,

$$\oint_S u_i\tau^{ki}\ dS_k = \frac{D}{Dt}(K + P + I) + \int_V \Phi dV \qquad (3.18)$$

Note that, if the force fields are explicitly time-dependent, the above expression for P is incorrect and must be suitably modified.

In the special case where the fluid expands isotropically, we have

$$e_{11} = e_{22} = e_{33} = e \text{ say, and } e_{ij} \neq 0 \text{ for } i \neq j \qquad (3.19)$$

Then,

$$\theta = \nabla \cdot \vec{u} = \frac{1}{2}(e_{11} + e_{22} + e_{33}) = \frac{3}{2}\ e \qquad (3.20)$$

so that

$$\begin{aligned}\Phi &= \lambda\theta^2 + \frac{1}{2}\mu[3e^2] \\ &= \lambda\theta^2 + \frac{3}{2}\mu\left(\frac{2}{3}\theta\right)^2 \\ &= \left(\lambda + \frac{2}{3}\mu\right)\theta^2\end{aligned}$$

In this case, the fluid element suffers no shear deformation and only undergoes volumetric expansion with time. For this reason, the quantity

$$\lambda + \frac{2}{3}\mu$$

is often called the *volume viscosity*, or the *bulk viscosity.*

If the fluid is incompressible, then $\theta = \nabla \cdot \vec{u} = 0$, and so

$$\Phi = \frac{1}{2}\ \mu e^{ij}e_{ij}$$

3.2 Energy Continuity Equation

The results of the previous section can be understood from a different point of view. Instead of considering the power delivered by the forces to a given element of fluid as it convects, we consider instead how the energy of the fluid filling a fixed volume in space changes with time. For this, we write the equations of motion in the form

$$\rho \left(\frac{\partial u^i}{\partial t} + u^k \, \frac{\partial u^i}{\partial x^k} \right) = \rho X^i + \partial_k \tau^{ki} \tag{3.1}$$

Now take the dot product of this equation with the fluid velocity,

$$\rho u_i \, \frac{\partial u^i}{\partial t} + \rho u^k u_i \, \frac{\partial u^i}{\partial x^k} = \rho u_i X^i + u_i \partial_k \tau^{ki} \tag{3.2}$$

We now rewrite this equation in the form of a continuity equation. The left hand side becomes, using the continuity equation,

$$\begin{aligned} \rho \, \frac{\partial}{\partial t} \left(\frac{u^2}{2} \right) + \rho u^k \, \frac{\partial}{\partial x^k} \left(\frac{u^2}{2} \right) &= \frac{\partial}{\partial t} \left(\rho \frac{u^2}{2} \right) - \frac{u^2}{2} \, \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k} \left(\rho \, \frac{u^2}{2} \, u^k \right) - \frac{u^2}{2} \, \frac{\partial}{\partial x^k} \left(\rho u^k \right) \\ &= \frac{\partial}{\partial t} \left(\rho \frac{u^2}{2} \right) + \nabla \cdot \left(\rho \, \frac{u^2}{2} \, u^k \right) \end{aligned}$$

so that,

$$\frac{\partial}{\partial t} \left(\rho \frac{u^2}{2} \right) + \nabla \cdot \left(\rho \, \frac{u^2}{2} \, u^k \right) = \rho u_i X^i + u_i \partial_k \tau^{ki} \tag{3.3}$$

This result is in the form of a continuity equation. The quantity

$$\rho_{\mathrm{KE}} = \rho \, \frac{u^2}{2}$$

is the kinetic energy per unit volume of the fluid, that is, the density of kinetic energy of the fluid. The equation thus identifies the quantity

$$\vec{J}_{\mathrm{KE}} = \rho_{\mathrm{KE}} \vec{u} = \rho \, \frac{u^2}{2} \, \vec{u}$$

as corresponding kinetic energy current density. The kinetic energy of the fluid thus convects with the fluid with velocity $\vec{u}$, as expected. The terms on the right represent the source of the fluid's kinetic energy. Kinetic energy is created in the fluid by the action of the external force field $\vec{X}$ at a rate of $\rho u_i X^i$ per unit volume, and by the contact forces on the fluid element at a rate of $u_i \partial_k \tau^{ki}$ per unit volume. The continuity equation (3.3) is therefore an alternative statement of the work energy theorem, seen from the point of view of a fluid-filled volume element that is fixed in space, rather than one that convects with the fluid.

If the external force field has a potential ϕ, that is, if

$$\vec{X} = -\nabla \phi \tag{3.4}$$

then, using the continuity equation,

$$\rho u_i X^i = -\rho u^i \partial_i \phi = -\partial_i (\rho \phi u^i) + \phi \partial_i (\rho u^i) = -\partial_i (\rho \phi u^i) - \phi \frac{\partial \rho}{\partial t} = -\partial_i (\rho \phi u^i) - \frac{\partial}{\partial t} (\rho \phi) - \rho \, \frac{\partial \phi}{\partial t}$$

Putting this result into the continuity equation gives,

$$\frac{\partial}{\partial t} \left(\rho \frac{u^2}{2} + \rho \phi \right) + \nabla \cdot \left[\left(\rho \, \frac{u^2}{2} + \rho \phi \right) u^k \right] = -\rho \, \frac{\partial \phi}{\partial t} + u_i \partial_k \tau^{ki} \tag{3.5}$$

This too has the form of a continuity equation, with density

$$\rho_{\mathrm{ME}} = \rho \frac{u^2}{2} + \rho\phi$$

which we can interpret as the density of mechanical energy of the fluid, that is, the sum of kinetic and potential energy density of the fluid. The current-density of mechanical energy in the fluid is therefore

$$\vec{J}_{\mathrm{ME}} = \rho_{\mathrm{ME}}\vec{u} = \left(\rho \frac{u^2}{2} + \rho\phi\right)\vec{u}$$

The sources of mechanical energy in the fluid are given by the terms on the right hand side. In the common special case where the external field does not change with time, $\partial\phi/\partial t = 0$, and then

$$\frac{\partial}{\partial t}\left(\rho \frac{u^2}{2} + \rho\phi\right) + \nabla \cdot \left[\left(\rho\ \frac{u^2}{2} + \rho\phi\right) u^k\right] = u_i \partial_k \tau^{ki} \tag{3.6}$$

As in the previous section, for Newtonian fluids, the right hand side can be expressed in the form

$$u_i \partial_k \tau^{ki} = \partial_k(u_i \tau^{ki}) - \tau^{ki}\partial_k u_i = \partial_k(u_i \tau^{ki}) + p\ \theta - \lambda\theta^2 - \frac{1}{2}\mu\ e^{ki} e_{ki}$$

so that

$$\frac{\partial}{\partial t}\left(\rho \frac{u^2}{2} + \rho\phi\right) + \nabla \cdot \left[\left(\rho\ \frac{u^2}{2} + \rho\phi\right) u^k\right] = \partial_k(u_i \tau^{ki}) + p\ \theta - \lambda\theta^2 - \frac{1}{2}\mu\ e^{ki} e_{ki} \tag{3.7}$$

The source of fluid mechanical energy is thus seen to consist of four contributions, represented by the four terms on the right hand side. The first term $\partial_k(u_i \tau^{ki})$ is the rate per unit volume at which the surface stresses deliver energy to the fixed volume element. The second, $p\ \theta$, is the rate per unit volume at which the compressive stresses deliver energy to the volume. The remaining two terms are always negative, and represent a drain of mechanical energy from the fluid. The term $\lambda\theta^2$ is the rate per unit volume at which mechanical energy is dissipated from the fluid by bulk friction as the fluid expands or contracts, while $\mu e^{ki} e_{ki}$ is the rate per unit volume at which mechanical energy is dissipated from the fluid by fluid shearing. As we shall see in a later section, in both cases the fluid mechanical energy is being converted into internal energy of the fluid.

4 The Energy Equation

4.1 The Need for an Energy Equation

The work-energy theorem describes how the kinetic energy of a fluid changes with time under the action of both external forces and of internal stresses. However, fluids do not only possess kinetic energy. The fluid material has internal structure and is therefore able to absorb energy into, and to expel it from, both its macroscopic configurational degrees of freedom and its microscopic internal degrees of freedom. This means that more energy processes can occur inside a fluid than those which are described by the work-energy theorem. These processes include, among others, heat conduction, deposition of heat by radiation, and the generation or depletion of heat by chemical and nuclear reactions and by changes of state such as ionisation.

These additional energy processes are not described by the fluid equations developed so far. The equations of motion, whether Euler's or the Navier-Stokes equations, treat the fluid elements as if they were point particles and ignore the fact that each element is a thermodynamic system in its own right. They processes that we now want to describe involve new physics that is not described by the equations considered thus far. We therefore require new equations to describe them.

In this chapter, we will set up an *energy equation* for the fluid. Essentially, this requires us to identify the energy processes that occur in the fluid of interest, write down the equations that govern these processes, and then construct an equation from them that reflects the principle of the conservation of energy for the system.

The energy equation for a system is not general. It is specific to the system being modelled. It will depend on the particular physical characteristics of the fluid material, and the processes that the material is likely to undergo. Different systems require different energy equations.

The energy equation that we will construct assumes that the fluid material, whether liquid or gas, is homogeneous in chemical composition and that this composition does not change in time through chemical reactions or other reactions. We also assume that the only energy processes that are taking place are the deposition and extraction of energy through work done by external forces and internal stresses, and the conduction of heat from one part of the fluid to another.

4.2 Construction of the Energy Equation

In this section, we will consider the energy contained in the fluid occupying a given volume V that is fixed in space. Denote its boundary surface by S. Divide this fixed volume into a large number of fixed volume elements of size δV. The total energy contained in one such element is then given by the sum of its kinetic and internal energies. Denote its internal energy per unit mass by ε. Then the total internal energy of an element with volume δV is given by

$$dE_{\text{internal}} = \varepsilon\ \delta m = \rho\varepsilon\ \delta V \tag{4.1}$$

Similarly, its total kinetic energy is given by

$$dE_{\text{kinetic}} = \frac{1}{2}\ \delta m\ v^2 = \frac{1}{2}\ \rho v^2\ \delta V \tag{4.2}$$

The total energy of the fixed element is then

$$dE = dE_{\text{kinetic}} + dE_{\text{internal}} = \left(\frac{1}{2}\ \rho v^2 + \rho\varepsilon\right) \delta V \tag{4.3}$$

The total energy contained in the fixed volume V is therefore

$$E = \int_V\ dE = \int_V \left(\frac{1}{2}\ \rho v^2 + \rho\varepsilon\right) dV \tag{4.4}$$

The total energy contained in the volume V will change with time due to the energy and other processes taking place in the fluid. First, the fluid is not static. Fluid is both entering and leaving the volume V through the surface S, thereby transporting internal energy through V. Second, the action of both external force fields and of contact forces on the fluid will alter its kinetic and internal energies. Last, energy will be transported through the fluid by conduction. All of these processes will act to change the total energy of the fluid contained in the fixed volume V with time. We may therefore write, by the principle of the conservation of energy, the rate at which the energy of the fluid in volume V increases is given by

$$\frac{dE}{dt} = P_{\text{external forces}} + P_{\text{contact forces}} + P_{\text{conduction}} + P_{\text{convection}} \tag{4.5}$$

where

$$\begin{aligned}
P_{\text{external forces}} &= \text{Power delivered to fluid in } V \text{ by external force fields}\\
P_{\text{contact forces}} &= \text{Power delivered to fluid in } V \text{ by contact forces}\\
P_{\text{conduction}} &= \text{Power delivered to fluid in } V \text{ by conduction of heat}\\
P_{\text{convection}} &= \text{Power delivered to fluid in } V \text{ by convection}
\end{aligned}$$

We now find explicit expressions for the various powers listed above.

Consider first the action of the external force fields on the fluid. These act individually on each element of the fluid in the fixed volume V, so we must consider the power delivered to each element of fluid and sum these to calculate the total power delivered by these forces to the fluid in V. Denote the sum of all forces per unit mass due to external force fields on the fluid by $\vec{X}$. Then the total force acting on a fixed element of volume δV is given by

$$\delta\vec{F} = \vec{X}\ \delta m = \rho\vec{X}\ \delta V \tag{4.6}$$

The power delivered to this element by the external forces is therefore

$$\delta\vec{F}\cdot\vec{v} = \rho\vec{X}\cdot\vec{v}\ \delta V \tag{4.7}$$

and the total power delivered by the external forces to the fluid in the fixed volume V is

$$P_{\text{externalforces}} = \int_V d\vec{F}\cdot\vec{v} = \int_V \rho\vec{X}\cdot\vec{v}\ dV \tag{4.8}$$

Consider next the action of the contact forces on the volume V. These forces act only at the surface of the volume V, so we subdivide the surface S into elements $\delta\vec{S}$ of surface. Then the total contact force exerted on this element of surface is

$$\delta F^i = \tau^{ij}\delta S_j \tag{4.9}$$

and the power delivered to the fluid in V by contact with this element of surface is given by

$$\delta F^i v_i = v_i\tau^{ij}\delta S_j \tag{4.10}$$

The total power delivered by contact to the fluid in V is therefore

$$P_{\text{contact forces}} = \oint_S v_i\tau^{ij}\ dS_j \tag{4.11}$$

The right hand side can be converted to a volume integral using the divergence theorem. This gives

$$P_{\text{contact forces}} = \int_V \partial_j(v_i\tau^{ij})\ dV \tag{4.12}$$

Now consider the rate at which energy is delivered to the fluid in V by conduction. Heat enters the volume V through the surface S. According to Fourier's Law, the heat current density at any point is given by

$$\vec{J}_Q = -k\ \nabla T \tag{4.13}$$

where k is the coefficient of heat conduction of the material. Note that k is a property of the material and, like all material properties, can be expected to depend on the thermodynamic state of the material. In principle, therefore, we expect k to be a function of ρ and T. In many materials, the dependence of k on

ρ and T is very weak, and so k can be considered to be a constant. The rate of heat energy transfer across an element of surface $\delta\vec{S}$ *in the direction of the unit normal* $\vec{n}$ to the surface is given by

$$\delta\dot{Q} = \vec{J}_Q \cdot \delta\vec{S} = -k\ \nabla T \cdot \delta\vec{S} \tag{4.14}$$

Hence the rate at which heat energy flows *out* of the volume V is given by

$$\dot{Q} = \oint_S -k\ \nabla T \cdot d\vec{S} \tag{4.15}$$

Therefore the power delivered *to* the fluid in V is given by

$$P_{\text{conduction}} = \oint_S k\ \nabla T \cdot d\vec{S} \tag{4.16}$$

or, using the divergence theorem,

$$P_{\text{conduction}} = \int_V \nabla \cdot (k\ \nabla T)\ dV \tag{4.17}$$

Finally, consider the rate at which energy is delivered to the fluid in volume V by the motion of the fluid itself. This energy transport is the result of the motion of the fluid through the surface S, and is due to the fact that the fluid carries its kinetic and internal energy with it. Thus, the fluid leaving the volume V at any instant is depleting the total energy in V. Similarly, fluid flowing into V is increasing it. The total energy per unit volume of the fluid is given by

$$\rho_E = \frac{1}{2}\ \rho v^2 + \rho\varepsilon \tag{4.18}$$

this energy is transported at each point of the fluid at the fluid velocity at that point. The energy current density of the fluid, due to the motion of the fluid, is therefore

$$\vec{J}_E = \rho_E \vec{v} = \frac{1}{2}\ \rho v^2 \vec{v} + \rho\varepsilon\vec{v} \tag{4.19}$$

The rate at which this energy is transported across an element of surface $\delta\vec{S}$ *in the direction of the unit normal* $\vec{n}$ to the surface is given by

$$\delta\dot{E} = \vec{J}_E \cdot \delta\vec{S} = \left(\frac{1}{2}\ \rho v^2 \vec{v} + \rho\varepsilon\vec{v}\right) \cdot \delta\vec{S} \tag{4.20}$$

Therefore the rate at which energy is *leaving* the volume V as the fluid moves is given by

$$\dot{E} = \oint_S \vec{J}_E \cdot d\vec{S} = \oint_S \left(\frac{1}{2}\ \rho v^2 \vec{v} + \rho\varepsilon\vec{v}\right) \cdot d\vec{S} \tag{4.21}$$

The rate at which energy is *entering* the volume V is therefore

$$P_{\text{convection}} = -\oint_S \left(\frac{1}{2}\ \rho v^2 \vec{v} + \rho\varepsilon\vec{v}\right) \cdot d\vec{S} \tag{4.22}$$

Using the divergence theorem, we get

$$P_{\text{convection}} = -\int_V \nabla \cdot \left(\frac{1}{2}\, \rho v^2 \vec{v} + \rho\varepsilon\vec{v} \right)\, dV \tag{4.23}$$

We now assemble all of these terms into the energy equation. Each of these processes contributes to the rate of increase of the total energy in the fixed volume V. The total energy in V is

$$E = \int_V \left(\frac{1}{2}\, \rho v^2 + \rho\varepsilon \right) dV \tag{4.24}$$

Since the volume V is *fixed* in space, the rate of increase of this energy is given by

$$\frac{dE}{dt} = \frac{d}{dt} \int_V \left(\frac{1}{2}\, \rho v^2 + \rho\varepsilon \right) dV = \int_V \frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho v^2 + \rho\varepsilon \right) dV$$

and hence,

$$\int_V \frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho v^2 + \rho\varepsilon \right) dV = \int_V \rho\vec{X} \cdot \vec{v}\; dV + \oint_S v_i \tau^{ij}\; dS_j + \oint_S k\; \nabla T \cdot d\vec{S} - \oint_S \left(\frac{1}{2}\, \rho v^2 \vec{v} + \rho\varepsilon\vec{v} \right) \cdot d\vec{S} \tag{4.25}$$

This is the heat equation in integral form. We can deduce from it an equation in differential form if we convert all surface integrals into volume integrals using the divergence theorem, and then equate the integrands. This gives

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho v^2 + \rho\varepsilon \right) = \rho v_i X^i + \partial_j (v_i \tau^{ij}) + \nabla \cdot (k\; \nabla T) - \partial_j \left(\frac{1}{2}\, \rho v^2 v^j + \rho\varepsilon v^j \right) \tag{4.26}$$

Inspection of this equation shows that several terms on the right-hand side are terms that appear also in the work-energy theorem derived previously. The left hand side contains the time derivative of the kinetic energy density, which is the rate of increase of kinetic energy in the fixed volume V and hence can be expressed in terms of the work-energy theorem. We therefore expect to be able to simplify the energy equation using the equations of motion and the continuity equation for the fluid. To achieve, this simplification, proceed as follows. First, rewrite the derivative of the kinetic energy density as follows,

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho v^2 \right) = \frac{1}{2}\, \frac{\partial \rho}{\partial t}\, v^2 + \rho v_i\, \frac{\partial v^i}{\partial t}$$

Then use the equations of motion for the fluid,

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho v^2 \right) &= \frac{1}{2}\, \frac{\partial \rho}{\partial t}\, v^2 + \rho v_i \left(-v^k \partial_k v^i + \rho X^i + \partial_k \tau^{ki} \right) \\
&= \frac{1}{2}\, \frac{\partial \rho}{\partial t}\, v^2 - \rho v^k \partial_k \left(\frac{1}{2}\, v^2 \right) + \rho v_i X^i + v_i \partial_k \tau^{ki} \\
&= \frac{1}{2}\, \frac{\partial \rho}{\partial t}\, v^2 - \partial_k \left(\rho v^k \frac{1}{2}\, v^2 \right) + \frac{1}{2}\, v^2 \partial_k \left(\rho v^k \right) + \rho v_i X^i + \partial_k (v_i \tau^{ki}) - \tau^{ki} \partial_k v_i \\
&= \frac{1}{2}\, v^2 \left[\frac{\partial \rho}{\partial t} + \partial_k (\rho v^k) \right] - \partial_k \left(\rho v^k \frac{1}{2}\, v^2 \right) + \rho v_i X^i + \partial_k (v_i \tau^{ki}) - \tau^{ki} \partial_k v_i
\end{aligned}$$

The bracketed term on the right-hand side is zero by the continuity equation, while the last term has already been evaluated in equation (3.12),

$$\tau^{ki}\partial_k u_k = -p\ \theta + \left(\lambda\theta^2 + \frac{1}{2}\mu\ e^{ki}e_{ki}\right) = -p\ \theta + \Phi \tag{4.27}$$

where $\theta = \nabla \cdot \vec{v}$ and $e_{ij} = (\partial_i v_j + \partial_j v_i)$. Thus

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\ \rho v^2\right) = -\partial_k\left(\rho v^k \frac{1}{2}\ v^2\right) + \rho v_i X^i + \partial_k(v_i\tau^{ki}) + p\ \theta - \Phi \tag{4.28}$$

If we now subtract (4.28) from (4.26), we get

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \partial_j\left(\rho\varepsilon v^j\right) = \nabla \cdot (k\ \nabla T) - p\ \theta + \Phi \tag{4.29}$$

This is a simplified form of the energy equation. It is a continuity equation for the internal energy per unit volume of the fluid. Internal energy is distributed continuously in the fluid with density $\rho_\varepsilon = \rho\varepsilon$ per unit volume, and convects with energy current-density $\vec{J_\varepsilon} = \rho\varepsilon\vec{v}$.

Inspection of (4.29) shows that the rate of increase of internal energy is due to three sources, represented by the three terms on the right-hand side of the equation representing the rate of creation of internal energy per unit volume by three different mechanisms.

The first term represents the rate of supply of energy to the fluid by heat conduction. The rate per unit volume at which heat flows into an infinitesimal element of fluid at given fixed position is $\nabla \cdot (k\ \nabla T)$.

The second term, $-p\,\theta = -p\nabla \cdot \vec{v}$, represents the rate at which work is done per unit volume by the surrounding fluid on a given infinitesimal fixed volume of fluid. This is clear from the fact that

$$\theta = \nabla \cdot \vec{v} = \frac{1}{V}\frac{dV}{dt}$$

so that

$$-p\theta = -\frac{1}{V}\frac{pdV}{dt} = -\frac{1}{V}\frac{dW}{dt}$$

where $dW = p\ dV$ is the work done *by* the fluid *on* the surroundings when its volume increases by amount dV. Thus $-p\ dV$ is the work done *by* the surroundings *on* the fluid when its volume increases by amount dV.

The third term Φ, given by

$$\Phi = \lambda\theta^2 + \frac{1}{2}\mu\ e^{ki}e_{ki} = \lambda\theta^2 + \frac{1}{2}\mu[(e_{11})^2 + (e_{22})^2 + (e_{33})^2 + (e_{12})^2 + (e_{23})^2 + (e_{31})^2]$$

is positive definite and represents a rate of increase in internal energy that is not due to the expansion of the fluid, i.e., it is not due to work being done on the fluid. It must therefore be classified, by the definition of heat, as an influx of heat into the fluid. However, this heat has not been conducted into the fluid. It is an influx of heat due to processes other than conduction. When considering the work-energy theorem, we arrived at this very same term, and saw that it represents the rate at which kinetic energy is removed from the fluid because of the action of viscosity. The effect of viscosity is therefore to degrade the kinetic energy of the fluid, which is due to organised collective motion of the fluid particles, converting it into disorganised random thermal motions of the constituent particles, where it shows up in the thermodynamic description as heat. We say that the kinetic energy has been dissipated into heat.

Equation (4.29) may be further simplified as follows. The left-hand side can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}(\rho\varepsilon) + \partial_j\left(\rho\varepsilon v^j\right) &= \frac{\partial\rho}{\partial t}\,\varepsilon + \rho\,\frac{\partial\varepsilon}{\partial t} + \varepsilon\partial_j(\rho v^j) + \rho v^j\partial_j\varepsilon \\ &= \varepsilon\left[\frac{\partial\rho}{\partial t} + \partial_j(\rho v^j)\right] + \rho\left[\frac{\partial\varepsilon}{\partial t} + v^j\partial_j\varepsilon\right]\end{aligned}$$

The first bracketed term on the right-hand side is zero by the continuity equation. So,

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \partial_j\left(\rho\varepsilon v^j\right) = \rho\left[\frac{\partial\varepsilon}{\partial t} + v^j\partial_j\varepsilon\right] = \rho\,\frac{D\varepsilon}{Dt}$$

and hence equation (4.29) becomes

$$\rho\,\frac{D\varepsilon}{Dt} = \nabla\cdot(k\,\nabla T) - p\,\theta + \Phi \tag{4.30}$$

This is the energy equation in its simplest form. It deals directly with the internal energy per unit mass rather than the internal energy per unit volume. The internal energy per unit mass is a property of the fluid material and is therefore a more natural quantity to use in an energy equation than the internal energy per unit volume. This is reflected in the greater simplicity of equation (4.30) when compared with (4.29). In contrast, the internal energy per unit volume is a property not only of the fluid material, but also of the state of the material since it involves also the volume occupied by a given mass of material.

Interestingly, equation (4.30) presents the rate of increase of internal energy per unit mass from the point of view of a convecting element of fluid rather than that of a fixed element of volume in space. This is to be expected. The fluid material is convecting and the conduction properties are most simply expressed when the heat flux is referred to the instantaneous rest frame of the material rather than to an inertial frame relative to which the material is in motion.

4.3 Thermodynamic Properties of the Fluid

The equation of motion for the fluid and the continuity equation together form a set of four equations for five unknown functions: $\vec{v} = (v^1, v^2, v^3)$, p and ρ. The energy equation introduces two further variables: the internal energy per unit mass, ε, and the temperature T. We therefore now have five equations for seven variables. Inspection of the variables, however, shows that four of the variables are thermodynamic: p, ρ, ε, and T. Since the fluid we are considering has fixed chemical composition, the fluid considered as a thermodynamic system has only two degrees of freedom, so only two of the variables p, ρ, ε, and T are independent. The other two may be expressed in terms of them. This means that we have in total five partial differential equations for five independent functions which, together with suitably chosen boundary conditions, completely determines the system. All we now therefore are two equations that express two of the thermodynamic variables in terms of two independent ones. These equations are holonomic in form, not differential, and are provided by the theory of thermodynamics.

Viewed as a thermodynamic system, the fundamental equation of the fluid can be expressed in differential form as

$$T\ ds = d\varepsilon + p\ dv \tag{4.1}$$

where s, ε and v are respectively the entropy, internal energy and volume per unit mass. These quantities are also sometimes called the *specific entropy, internal energy and volume*, or the *mass-specific entropy, internal energy and volume*. It is more convenient to work in terms of the density ρ of the fluid rather than its specific volume. These are related according to,

$$\rho = \frac{1}{v} \tag{4.2}$$

In terms of ρ, the fundamental equation becomes

$$d\varepsilon = T\ ds + \frac{p}{\rho^2}\ d\rho \tag{4.3}$$

According to this relation

5 Properties of Inviscid Fluids

5.1 Definition of an Inviscid Fluid

In Part B, we study some general properties of inviscid fluids. An *inviscid fluid* is one with no viscosity. This means that both μ, the coefficient of viscosity, and λ, the coefficient of bulk viscosity are zero. The equation of motion appropriate to an inviscid fluid is Euler's equation.

5.2 Euler's Equation

Derive separately for further insight? (Maybe put the derivation into an exercise.)

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g} \tag{5.1}$$

rewrite the advective term to get a second useful form of E's equation

5.3 Inviscid Fluid vs. Ideal Fluid

Some authors call any fluid that obeys Euler's equation an *ideal fluid*. This is not correct. More is needed to define an ideal fluid than the simple requirement that it obey Euler's equation. Landau and Lifschitz (2003) define the ideal fluid as follows: p 3, "In deriving the equations of motion we have taken no account of the process of energy dissipation, which may occur in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it. ... motions of fluids in which thermal conductivity and viscosity are unimportant ... are said to be ideal." This statement is further amplified on p 3 as follows: "The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. The motion of an ideal fluid must necessarily be supposed adiabatic. In adiabatic motion the entropy of any particle of fluid remains constant as the particle moves about in space." Thus, according to Landau and Lifschitz, an ideal fluid is defined as one that obeys Euler's equations, the equation of continuity, and in which the entropy of every fluid element remains constant; that is, the entropy per unit mass of the fluid obeys the conservation law,

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Using the continuity equation, this can also be rewritten in the form of a continuity equation for entropy,

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s \vec{v}) = 0$$

where ρs is the entropy per unit volume of the fluid, and $\rho s \vec{v}$ is the entropy flux density.

Landau and Lifschitz (2003, p 4) add also the following comment: "The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout the volume of the fluid at some initial instant, it retains everywhere that same constant value at all times and for any subsequent motion of the fluid. In this case we write the adiabatic equation simply as

$$s = \text{constant}$$

... Such a motion is said to be *isentropic*." The case where s is a constant is thus a special case of an ideal fluid.

In Part B, we explicitly assume only that the fluid obeys Euler's equation and the equation of continuity for mass. We make no explicit assumptions about the thermal conductivity of the fluid, though in some of the specialised discussions we will make further assumptions, some of which are equivalent to assuming that the fluid is ideal.

6 Bernoulli's Equation

6.1 Exact Integrals of Euler's Equation

Euler's equation is non-linear. The non-linearity occurs in the advection term $\vec{v} \cdot \nabla\vec{v}$. The presence if this non-linear makes the equations of fluid dynamics difficult to solve.

In spite of being non-linear, there are many situations in which it is possible to obtain an *exact first integral* of the equations. The first such integral was discovered by Daniel Bernoulli in 1738 and is called *Bernoulli's equation*.

Since Bernoulli published his equation, many other similar exact integrals have been found. When deducing his equation, Bernoulli assumed that the density of the fluid is constant, and that its flow is irrotational. Bernoulli's equation is therefore valid only for fluids of constant density in irrotational flow. The new exact integrals were found by either relaxing or changing Bernoulli's original assumptions.

These new integrals all look very similar to Bernoulli's. This has led to some authors also calling all of them "Bernoulli's equation". This is not useful. Though they all look similar, there are subtle differences between them. Some are exact integrals of Euler's equation that are valid everywhere; others are integrals along a streamline and are valid only along that streamline. Also, the conditions under which they are valid are different. It is better therefore not to use the same name for them. They are best regarded as different results, with different domains of applicability, in spite of their similar appearance.

We could consider them to be 'generalisations' of Bernoulli's original equation, but this does not help us to name them. There is more than one generalisation.

In desperation, some authors resort to referring to these integrals as "Bernoulli's equation for ..." and they then append the specific assumptions about the nature of the flow that were made when deriving them. This might be a solution. However, it does lead to very clumsy names.

Unfortunately, there is no "most general" form of this result from which all others can be deduced. So, there is no unified framework for their presentation. I this chapter, I will show you how to derive several of these integrals, how to interpret them, and how to use them in practical calculations. When you have seen a sufficient number of them, you should be able to work out the others for yourself.

One item of good news is that, in spite of there being no unifying framework, there is nonetheless something that is common to all: they are deduced from Euler's equations by applying one of only *two methods of integration.* I will explain these two methods in the following sections. The differences in the various results arise because of the different assumptions that can be made about the nature of the flow.

6.2 Bernoulli's Equation

Derivation from Euler's equation

Bernoulli's original equation describes a fluid of constant density in steady irrotational flow in a gravitational field. Translating these words in mathematical conditions, we assume,

$$\rho = \text{constant} \qquad \text{constant density}$$

$$\frac{\partial \vec{v}}{\partial t} = 0 \qquad \text{steady flow}$$

$$\nabla \times \vec{v} = 0 \qquad \text{irrotational flow}$$

$$\vec{g} = -\nabla \phi \qquad \text{gravitational field}$$

We write Euler's equation in the form,

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\,\nabla^2 \vec{v} - \vec{v} \times (\nabla \times \vec{v}) = -\frac{1}{\rho}\,\nabla p + \vec{g} \tag{6.1}$$

which, with the above assumptions, becomes

$$\frac{1}{2}\,\nabla v^2 = -\nabla \left(\frac{p}{\rho}\right) - \nabla \phi$$

or

$$\nabla \left(\frac{1}{2}\,v^2 + \frac{p}{\rho} + \phi\right) = 0$$

If the gradient of a function is zero, the function is independent of position. This means that it can at most depend on time. The requirement that the flow be steady, however, requires that all the fields in the problem, including p and ϕ be time independent. So we get

$$\frac{1}{2}\,v^2 + \frac{p}{\rho} + \phi = C \tag{6.2}$$

Here C is an arbitrary constant of integration. We have thus obtained an equation relating p, ρ and $\vec{v}$ which is valid in every region of space that is occupied by the fluid. This result is exact. It involves no approximations. It also involves the fields $\vec{v}$ and p directly, and not their derivatives. It is therefore an exact integral of Euler's equation. This integral is called *Bernoulli's equation*.

Bernoulli's equation takes on a more familiar appearance if we multiply through by ρ. Since ρ is a constant and C is an arbitrary constant, the product ρC is also an arbitrary constant. To avoid proliferating mathematical symbols, denote the new constant by C also. We thus obtain,

$$\frac{1}{2}\,\rho v^2 + p + \rho\phi = C \tag{6.3}$$

Physical interpretation

Equation (6.3) is clearly an energy equation. The term $\rho v^2/2$ is the kinetic energy per unit volume of a fluid element at the position $\vec{x}$ where this function is evaluated, and $\rho\phi$ is the potential energy per unit volume, measured relative to some implicitly chosen reference for the potential, of the fluid element.

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6.3 Bernoulli's Theorem for Steady Flows

Euler's equation is strongly non-linear. The most obvious non-linearity is found in the term $\vec{v}\cdot\nabla\vec{v}$, but there are also others which are not immediately evident and which arise from the equations needed to complete Euler's equations such as the continuity equation for mass and the equation of state for the fluid.

In spite of this non-linearity, there nevertheless exists an exact first integral for important special cases of the motion. The existence of this integral is called *Bernoulli's Theorem*, and the equation that is provided by this theorem is called *Bernoulli's equation*.

Unfortunately, in the literature, the name *Bernoulli's Theorem* does not refer to a single result. Different authors derive a "Bernoulli Theorem" using different special assumptions and arrive at similar looking results that are valid for different types of flow. Before applying any particular version of Bernoulli's Theorem, it is important that you check the assumptions under which the result was derived before you attempt to apply it. The reason that these different results are all called by the same name appears to be that they are derived in the same way from Euler's equations using essentially the same technique.

The result that we derive in this section makes the following assumptions:

- The flow is steady, that is, $\partial\vec{v}/\partial t = 0$.
- The external force acting on the fluid is conservative, that is, $\vec{f} = -\nabla\phi$.

Then we can write Euler's equation in the form

$$\nabla\left(\frac{1}{2}\,v^2\right) - \vec{v}\times(\nabla\times\vec{v}) = -\,\frac{1}{\rho}\,\nabla p - \nabla\phi \tag{6.4}$$

Now take the dot product of this equation with the fluid velocity $\vec{v}$. Noting that

$$\vec{v} \cdot [\vec{v} \times (\nabla \times \vec{v})] = 0 \tag{6.5}$$

this gives

$$\vec{v} \cdot \nabla \left(\frac{1}{2} \, v^2 + \phi \right) + \frac{1}{\rho} \, \vec{v} \cdot \nabla p = 0 \tag{6.6}$$

This almost gives us our result. The operator $\vec{v} \cdot \nabla$ is the directional derivative in the direction of $\vec{v}$. It therefore measures the rate of change of the function on which it operates in the direction of a streamline. If this derivative is zero, then the function does not change at all along the streamline. The only term in our result that is not a directional derivative is the last one on the left hand side. The culprit here is the factor $1/\rho$, which is outside of the directional derivative of p. If, somehow, we can write this term in the same for as the others, then we have the desired result. Different authors use different ways to accomplish this. Some of the diversity in the results that are called "Bernoulli's theorem" arises because of the way in which different authors resolve this problem. (The remainder of the diversity arises because of the different assumptions that are made when initially specialising the form of equations of motion of the fluid.) I will now indicate some of the ways in which different authors proceed from here.

1. A number of authors, for example Lamb (1993), assume that the equation of state for the fluid has the form

$$\rho = \rho(p) \tag{6.7}$$

or, equivalently,

$$p = p(\rho) \tag{6.8}$$

An equation of state of this form is said to be *barotropic*. Being barotropic may be an intrinsic property of the fluid, in which case the fluid is said to be barotropic, or it may be a property of the process undergone by the fluid material during a given type of flow. In this case, we say that the flow is barotropic. Examples of barotropic fluids are the degenerate gases, like the degenerate electron gas in a white dwarf star or the degenerate neutron fluid in a neutron star. Examples of barotropic flows are isothermal or isentropic flows. In these cases, it is the process undergone by the fluid during the flow, rather than the properties of the fluid material itself, that makes the equation of state barotropic.

In this case, we can perform the indefinite integral

$$\int \frac{1}{\rho(p)} \, dp = F(p) \tag{6.9}$$

to get a function $F(p)$ of p which has the property that

$$\frac{dF}{dp}(p) = \frac{1}{\rho(p)} \tag{6.10}$$

and hence

$$\nabla F(p) = \frac{dF}{dp}(p)\ \nabla p = \frac{1}{\rho(p)}\ \nabla p \tag{6.11}$$

This means that we can write

$$\frac{1}{\rho(p)}\ \vec{v}\cdot\nabla p = \vec{v}\cdot\nabla F(p) = \vec{v}\cdot\nabla\left(\int \frac{dp}{\rho(p)}\right) \tag{6.12}$$

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Now form the line integral of each side of this equation along a streamline. This gives

$$\int_\Gamma \left[\nabla\left(\frac{1}{2}\ v^2\right) - \vec{v}\times(\nabla\times\vec{v})\right]\cdot d\vec{\ell} = -\int_\Gamma \left[\frac{1}{\rho}\ \nabla p - \nabla\phi\right]\cdot d\vec{\ell} \tag{6.13}$$

where Γ denotes a segment of a streamline, and $d\vec{\ell}$ is an element of path along it, so that $d\vec{\ell} = \vec{v}\ dt$. This means that

$$\vec{v}\times(\nabla\times\vec{v})\cdot d\vec{\ell} = 0 \tag{6.14}$$

and hence the line integral becomes

$$\int_\Gamma \nabla\left[\frac{1}{2}\ v^2 + \frac{1}{\rho}\ \nabla p + \nabla\phi\right]\cdot d\vec{\ell} = 0 \tag{6.15}$$

Now, for any function f, we have

$$\int_\Gamma (\nabla f)\cdot d\vec{\ell} = \int_\Gamma \frac{\partial f}{\partial x^i}\ dx^i \ = \ \int_\Gamma df \ = \ f(\vec{x}_1) - f(\vec{x}_0) \tag{6.16}$$

so the integrated equation becomes

$$\int_\Gamma d\left[\frac{1}{2}\ v^2 + \phi\right] + \int_\Gamma \frac{1}{\rho}\ dp = 0 \tag{6.17}$$

or

$$\left[\frac{1}{2}\ v^2 + \phi\right]_0^1 + \int_\Gamma \frac{1}{\rho}\ dp = 0 \tag{6.18}$$

The second term can be expressed as

$$\int_\Gamma \frac{1}{\rho}\ dp = \left[\int \frac{1}{\rho}\ dp\right]_0^1 \tag{6.19}$$

where the integral inside the square brackets on the right hand side is no longer a path integral, but the indefinite integral of the function $1/\rho$ with respect to p. Since the endpoints of the chosen streamline are arbitrary, this result tells us that the value of the function

$$\frac{1}{2}\ v^2 + \phi + \int \frac{1}{\rho}\ dp \tag{6.20}$$

takes the same value all along the streamline. We thus obtain Bernoulli's theorem, that along any given streamline, we have

$$\frac{1}{2}\, v^2 + \phi + \int \frac{1}{\rho}\, dp = \text{constant} \tag{6.21}$$

From Wikipedia:
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In fluid dynamics, Bernoulli's principle states that for an inviscid flow, an increase in the speed of the fluid occurs simultaneously with a decrease in pressure or a decrease in the fluid's potential energy.[1][2] Bernoulli's principle is named after the Dutch-Swiss mathematician Daniel Bernoulli who published his principle in his book Hydrodynamica in 1738.[3]

Bernoulli's principle can be applied to various types of fluid flow, resulting in what is loosely denoted as Bernoulli's equation. In fact, there are different forms of the Bernoulli equation for different types of flow. The simple form of Bernoulli's principle is valid for incompressible flows (e.g. most liquid flows) and also for compressible flows (e.g. gases) moving at low Mach numbers. More advanced forms may in some cases be applied to compressible flows at higher Mach numbers (see the derivations of the Bernoulli equation).

Bernoulli's principle can be derived from the principle of conservation of energy. This states that in a steady flow the sum of all forms of mechanical energy in a fluid along a streamline is the same at all points on that streamline. This requires that the sum of kinetic energy and potential energy remain constant. If the fluid is flowing out of a reservoir the sum of all forms of energy is the same on all streamlines because in a reservoir the energy per unit mass (the sum of pressure and gravitational potential ? g h) is the same everywhere.[4]

Fluid particles are subject only to pressure and their own weight. If a fluid is flowing horizontally and along a section of a streamline, where the speed increases it can only be because the fluid on that section has moved from a region of higher pressure to a region of lower pressure; and if its speed decreases, it can only be because it has moved from a region of lower pressure to a region of higher pressure. Consequently, within a fluid flowing horizontally, the highest speed occurs where the pressure is lowest, and the lowest speed occurs where the pressure is highest.

7 Fluid Kinematics

As a fluid flows, each of its elements translate, rotate, and change shape. The shape changes are due to distortions of the fluid material that include elongation, shearing, and dilatation. Since the motion of the fluid is fully described by its velocity field $\vec{v}$, all information about its motions and shape changes must be encoded in some way in the field $\vec{v}$. In this section, we examine how this information is contained in the velocity field, and learn how to to extract it systematically.

7.1 Relative Displacement

A fluid element is a collection of fluid points. Rotation and distortion of shape of a given fluid element are not caused by the absolute motion of its fluid points relative to the ambient space, but by their motion relative to each other. So to quantify rotation and distortion, we need first to quantify relative motion.

Let P be a given fluid point. Denote its position at time t by $\vec{x}$. We regard P as a given reference point in the fluid that moves with the fluid. Let Q be any arbitrary fluid point adjacent to P. Denote its position at time t by $\vec{x} + \delta\vec{x}$. Note the abuse of notation in this statement. Unlike rectilinear systems of coordinates, a general curvilinear system has no single well-defined universal set of basis vectors that is used for the description of vectors everywhere. Rather, the coordinate system defines a *local basis* at each point of the space. This local basis is used to decompose vectors at that point only. A position vector, by definition, is a vector at the origin of coordinates which extends to the point whose position it defines. Its components are therefore those which it has in the basis defined by the coordinate system at the origin of coordinates. These are not simply related to the curvilinear coordinates of the point it describes. Position vectors are thus not useful in coordinate systems that are not rectilinear. Consequently, we should not denote positions in them by the symbol $\vec{x}$, but by their coordinates x^i or by some other symbol that designates points in space, like P or $P(x^i)$. However, this is inconvenient. It increases unreasonably the number of symbols that we need to use. We get around this by agreeing to use the symbol $\vec{x}$ to abbreviate the phrase *the point P whose coordinates are x^i*. This is an abuse of vector notation, but no ambiguity arises from it and so it is tolerable.

The relative displacement of Q from P at time t is the vector $\delta\vec{x}$. Its components in the local basis $\vec{e}_i(\vec{x})$ at P are, to correct to first order, δx^i. Thus the displacement vector $\delta\vec{x}$ is given by

$$\delta\vec{x} = \delta x^i\,\vec{e}_i(\vec{x})$$

7.2 Relative Velocity

We are interested in how the relative displacement of Q from P changes with time as these points move with the fluid. The velocity of P at time t is $\vec{v}(\vec{x},t)$. So, in time δt, P will displace by $\vec{v}(\vec{x},t)\,\delta t$ to new position

$$\vec{x}' = \vec{x} + \vec{v}(\vec{x},t)\,\delta t$$

The velocity of Q at time t, however, is not $\vec{v}(\vec{x},t)$, but $\vec{v}(\vec{x}+\delta\vec{x},t)$. It will therefore displace by $\vec{v}(\vec{x}+\delta\vec{x},t)\,\delta t$ to position

$$\vec{x}'' = \vec{x} + \delta\vec{x} + \vec{v}(\vec{x}+\delta\vec{x},t)\,\delta t$$

The relative displacement of Q from P at time $t+\delta t$ is therefore

$$\delta\vec{x}' = \vec{x}'' - \vec{x}' = \delta\vec{x} + [\vec{v}(\vec{x}+\delta\vec{x},t) - \vec{v}(\vec{x},t)]\,\delta t$$

To calculate this vector in component form in a general system of coordinates x^i, we need to take into account not only the way that the components v^i of the velocity field change from point to point, but also how the vectors of the coordinate basis change. We therefore have,

$$\begin{aligned}\vec{v}(\vec{x}+\delta\vec{x},t) &= v^i(\vec{x}+\delta\vec{x},t)\,\vec{e}_i(\vec{x}+\delta\vec{x}) \\ &= \left[v^i(\vec{x},t) + \frac{\partial v^i}{\partial x^j}(\vec{x},t)\,\delta x^j + \cdots\right]\left[\vec{e}_i(\vec{x}) + \frac{\partial\vec{e}_i}{\partial x^k}(\vec{x})\,\delta x^k + \cdots\right]\end{aligned}$$

Using the fact that

$$\frac{\partial\vec{e}_i}{\partial x^k}(\vec{x}) = \Gamma^j{}_{ik}(\vec{x})\,e_j(\vec{x})$$

we get

$$\begin{aligned}\vec{v}(\vec{x}+\delta\vec{x},t) &= v^i(\vec{x},t)\,\vec{e}_i(\vec{x}) + \left[\frac{\partial v^j}{\partial x^k}(\vec{x},t) + v^i(\vec{x},t)\Gamma^j{}_{ik}(\vec{x})\right]\vec{e}_j(\vec{x})\,\delta x^k + \cdots \\ &= \vec{v}(\vec{x},t) + \left(v^j{}_{;k}(\vec{x},t)\,\delta x^k\right)\vec{e}_j(\vec{x}) + \cdots\end{aligned}$$

where we have used the standard notation for the components of the covariant derivative,

$$\frac{\partial v^j}{\partial x^k} + v^i\Gamma^j{}_{ik} = v^j{}_{;k}$$

This can be expressed in coordinate-free form by noting that

$$\left(v^j{}_{;k}\,\delta x^k\right)\vec{e}_j = \delta\vec{x}\cdot\nabla\vec{v}$$

is the directional derivative of $\vec{v}$. Thus

$$\vec{v}(\vec{x}+\delta\vec{x},t)-\vec{v}(\vec{x},t)=\delta\vec{x}\cdot\nabla\vec{v}(\vec{x},t)+\cdots$$

and the relative displacement of Q from P at time $t+\delta t$ is given by

$$\delta\vec{x}'=\delta\vec{x}+\delta\vec{x}\cdot\nabla\vec{v}(\vec{x},t)\,\delta t+\cdots$$

The velocity of Q relative to P is rate of change of its relative displacement from P. So,

$$\frac{d}{dx}\,\delta\vec{x}=\lim_{\delta t\to 0}\frac{\delta\vec{x}'-\delta\vec{x}}{\delta t}$$

and it is therefore given, correct to first order in $\delta\vec{x}$, by

$$\frac{d}{dx}\,\delta\vec{x}=\delta\vec{x}\cdot\nabla\vec{v}(\vec{x},t) \tag{7.1}$$

This is the principle result of this section. Expressed in component form, it is

$$\frac{d}{dx}\,\delta x^i={v^i}_{;k}(\vec{x},t)\;\delta x^k \tag{7.2}$$

The information about relative velocities of adjacent points in the fluid at position $\vec{x}$ at time t is thus encoded in the rank-two tensor ${v^i}_{;k}$.

Note that the right hand side of (7.2) is a linear transformation of the infinitesimal displacements δx^i. The relative velocity of a fluid point at position $\vec{x}+\delta\vec{x}$ at time t is thus (correct to first order) linearly related to its displacement $\delta\vec{x}$ from the reference point $\vec{x}$.

The velocity of fluid points relative to their neighbours determines the rotation rate of the fluid at that point, and also its strain and distortion rates. The tensor ${v^i}_{;k}$ thus encodes all of these properties. We need now to determine precisely how they are encoded, and to devise ways of extracting them from it. We begin by considering the rate of linear strain of the fluid.

7.3 Rate of Linear of Strain

In this section, we consider the rate at which fluid points separate as the fluid flows. In elementary physics, strain is defined as the elongation per unit length suffered by a material on deformation. On deformation, an elastic solid reaches static equilibrium when deformed. Strain defined in this way is therefore an useful concept. A fluid, however, is constantly in motion. The separation between fluid points is continually changing, so it is not useful to speak of an elongation per unit length. A given elongation occurs over some time interval δt, and the elongation of the separation vector depends on the time interval considered. The longer the interval δt, the greater the elongation. To take this into account, we introduce the concept of *strain rate*. The strain rate of a given separation vector δx^i in the fluid is its elongation per unit length per unit time.

Consider two fluid points separated at time t by vector δx^i. Denote the distance between them at this time by δl. Then

$$\delta l^2 = g_{ij}\delta x^i \delta x^j$$

Their rate of change of this distance can then be found by differentiating this relation,

$$2\,\delta l\,\frac{d}{dt}\,\delta l = \frac{d}{dt}\left(g_{ij}\delta x^i\delta x^j\right)$$

Note that $g_{ij;k} = 0$, so

$$\frac{d}{dt}\,g_{ij} = g_{ij;k}v^k = 0$$

giving, correct to first order in the δx^i,

$$\begin{aligned}
2\,\delta l\,\frac{d}{dt}\,\delta l &= g_{ij}\left(\frac{d}{dt}\,\delta x^i\right)\delta x^j + g_{ij}\delta x^i\left(\frac{d}{dt}\,\delta x^j\right)\\
&= g_{ij}v^i{}_{;r}\delta x^r\delta x^j + g_{ij}\delta x^i v^j{}_{;s}\delta x^s + \cdots\\
&= v_{j;r}\delta x^r\delta x^j + v_{i;s}\delta x^i\delta x^s + \cdots\\
&= \left(v_{s;r} + v_{r;s}\right)\delta x^r\delta x^s + \cdots
\end{aligned}$$

This result shows that it is not directly the tensor $v^i{}_j$ that determines the rate of linear strain of fluid elements, but rather its symmetric part,

$$\epsilon_{rs} = v_{s;r} + v_{r;s}$$

We therefore call ϵ_{ij} the *strain tensor* for the fluid. We shall see that all other strain rates of interest are also determined by ϵ_{ij}. This tensor therefore contains all the information necessary for determining all strain rates and rates of deformation of the fluid. The strain rate is therefore given, correct to first order in the δx^i, by

$$\frac{1}{\delta l}\,\frac{d}{dt}\,\delta l = \frac{1}{2}\,\epsilon_{rs}\,\frac{\delta x^r}{\delta l}\,\frac{\delta x^s}{\delta l} + \cdots$$

The vector $n^i = \delta x^i/\delta l$ is the unit vector in the direction of the separation δx^i. The strain rate can thus be written as

$$\frac{1}{\delta l}\,\frac{d}{dt}\,\delta l = \frac{1}{2}\,\epsilon_{rs}\,n^r n^s + \cdots \tag{7.3}$$

We get the linear strain rate by taking the limit of this expression as the distance δl between the fluid becomes infinitesimally small,

$$\text{Strain Rate } = \lim_{\delta l\to 0}\,\frac{1}{\delta l}\,\frac{d}{dt}\,\delta l$$

Thus the linear strain rate of the fluid is given by

$$\lim_{\delta l\to 0}\,\frac{1}{\delta l}\,\frac{d}{dt}\,\delta l = \frac{1}{2}\,\epsilon_{rs}\,n^r n^s \tag{7.4}$$

This result is no longer approximate, but exact. Note that the linear strain rate depends through ϵ_{ij} on where we are in the fluid, and at what time, as well as on the *direction* n^i in which we consider the rate of strain. The strain rate of the fluid is therefore, in general, not isotropic but depends strongly on the direction considered.

7.4 Shearing Rate

As the fluid flows, no only do the separations of the fluid points change with time, but also their relative orientations. Changes of relative orientation give rise to shearing effects. These too are encoded in the tensor ϵ_{ij}. This is seen as follows.

Consider three adjacent fluid points, P, Q and R. Take P as the reference point, and denote the separation vectors of Q and R respectively by δX^i and δY^i. Denote the angle formed by these separation vectors at P by θ. As the fluid flows, θ will change. The rate of change of θ determines he shearing rate. To calculate it, note first that

$$\delta\vec{X} \cdot \delta\vec{Y} = l_X l_Y \cos\theta$$

where $l_X = |\vec{X}|$ and $l_Y = |\vec{Y}|$ are the separation distances of Q and R from P respectively. Differentiating with respect to time, we get

$$\frac{d}{dt}\left(g_{ij}\delta X^i \delta Y^j\right) = \frac{d}{dt}(l_X)\, l_Y \cos\theta + l_X \frac{d}{dt}(l_Y)\cos\theta - l_X l_Y \sin\theta\, \frac{d\theta}{dt}$$

The left hand side is easily evaluated correct to first order, as before, using (7.2) together with the fact that $g_{ij;k} = 0$ to give

$$\begin{aligned}\frac{d}{dt}\left(g_{ij}\delta X^i \delta Y^j\right) &= g_{ij}v^i{}_r \delta X^r \delta Y^j + g_{ij}\delta X^i v^j{}_s \delta Y^s \\ &= (v_{sr} + v_{rs})\, \delta X^r \delta Y^s \\ &= \epsilon_{rs}\delta X^r \delta Y^s\end{aligned}$$

The right hand side can then be evaluated, correct to first order, using (7.4). We thus get

$$\epsilon_{rs}\delta X^r \delta Y^s = \left(l_X \frac{1}{2}\,\epsilon_{rs}\, n^r n^s\right) l_Y \cos\theta + l_X \left(l_Y \frac{1}{2}\,\epsilon_{rs}\, m^r m^s\right)\cos\theta - l_X l_Y \sin\theta\, \frac{d\theta}{dt}$$

where we have denoted the unit vectors in the directions of $\delta\vec{X}$ and $\delta\vec{Y}$ by $\vec{n}$ and $\vec{m}$ respectively. Thus,

$$\epsilon_{rs} n^r m^s = \frac{1}{2}\,\epsilon_{rs}\,(n^r n^s + m^r m^s)\cos\theta - \sin\theta\, \frac{d\theta}{dt} \tag{7.5}$$

This is the general result determining $d\theta/dt$ for arbitrarily selected displacements $\delta\vec{X}$ and $\delta\vec{Y}$ at P. It is complicated. We can simplify it considerably if, instead of considering arbitrary $\delta\vec{X}$ and $\delta\vec{Y}$, we agree to consider only displacements that are mutually orthogonal. Then $\cos\theta = 0$, $\sin\theta = 1$, and we have

$$-\frac{d\theta}{dt} = \epsilon_{rs} n^r m^s \tag{7.6}$$

Here, $d\theta/dt$ is the rate at which the angle between orthogonal displacements *increases* as the fluid flows. So, $-d\theta/dt$ is the rate at which θ *decreases*.

In elementary physics, the shearing strain is defined as the *decrease in angle* between initially orthogonal segments of the material produced when the material is deformed. In a fluid, the shearing of the fluid occurs throughout the flow, so it makes sense to define the shearing strain per unit time. We may therefore define the *rate of shearing strain* as the rate of decrease of angle between orthogonal fluid segments. This shearing strain rate is therefore given by

$$\text{Shearing Strain Rate } = -\frac{d\theta}{dt} = \epsilon_{rs} n^r m^s \tag{7.7}$$

This shows that the shearing strain rate information is also contained in the strain tensor ϵ_{ij}.

7.5 Volumetric Strain Rate

As the fluid flows, the movement of adjacent fluid points causes the volume of a fluid element to change. The rate of change of volume per unit volume of a given element is its the *volumetric strain rate.* This strain rate is calculated as follows.

Let P be a reference point in the fluid, and let Q, R and S be adjacent fluid points separated from P by displacement vectors $\delta\vec{X}$, $\delta\vec{Y}$ and $\delta\vec{Z}$. The volume of the parallelepiped defined by these vectors is

$$\delta V = \Omega_{ijk} \delta X^i \delta Y^j \delta Z^k$$

where Ω_{ijk} is the volume element in the given system of coordinates. The rate of change of volume as the fluid flows is therefore

$$\frac{d}{dt}\delta V = \frac{d}{dt}\left(\Omega_{ijk}\delta X^i \delta Y^j \delta Z^k\right)$$

We calculate this rate of change by noting that $\Omega_{ijk;r} = 0$, so that

$$\frac{d}{dt}\Omega_{ijk} = \Omega_{ijk;r} v^r = 0$$

Thus

$$\frac{d}{dt}\delta V = \Omega_{ijk}\frac{d}{dt}\left(\delta X^i \delta Y^j \delta Z^k\right)$$

ppppp

8 Convective Derivatives

8.0.1 Convection of Fluid Points

Consider the convection of a fluid point. As the fluid moves, the fluid point moves (or, convects) with it. Of course, the rate at which the fluid point changes position must be the fluid velocity at that position. So we expect that

$$\frac{D}{Dt}\, x^i = v^i \tag{8.1}$$

It is instructive however to deduce this from an intuitive physical argument that will be useful for deducing how other quantities of interest change with time as they convect.

Suppose a given fluid point is at position x^i at time t. Its velocity at that instant is then $\vec{v}(\vec{x}, t)$. In time Δt, fluid point will convect to a new position x'^i. Since it is moving with the fluid, we have approximately

$$x'^i = x^i + v^i(\vec{x}, t)\Delta t + \cdots \tag{8.2}$$

The change in its position is therefore

$$x'^i - x^i = v^i(\vec{x}, t)\Delta t + \cdots \tag{8.3}$$

and so the rate at which it changes position is

$$\frac{x'^i - x^i}{\Delta t} = v^i(\vec{x}, t) + \cdots \tag{8.4}$$

So, in the limit as $\Delta t \to 0$, we have

$$\frac{D}{Dt}\, x^i = \lim_{\Delta t \to 0} \frac{x'^i - x^i}{\Delta t} = v^i(\vec{x}, t) \tag{8.5}$$

as anticipated.

Interestingly, we can obtain the same result by yet another method. Applying the directional derivative operator to each of the coordinates x^i of the fluid point, we get

$$\left(\vec{v} \cdot \nabla + \frac{\partial}{\partial t}\right) x^i = v^k \partial_k x^i + \frac{\partial x^i}{\partial t} = v^k \delta^i_k + 0 = v^i$$

as above. We may thus regard each of the position coordinates x^i of the fluid point as a scalar valued function of the type $f(\vec{x}, t)$ considered previously. The rate at which the fluid point position x^i changes is thus obtainable from

$$\frac{D}{Dt}\, x^i = \left(\vec{v}\cdot\nabla + \frac{\partial}{\partial t}\right) x^i = v^i \tag{8.6}$$

8.0.2 Convection of Fluid Curves

Consider a curve drawn in the fluid at some given time t. Each point of this curve is occupied at this instant by some fluid point. Now, each of these fluid points convects with the fluid. So, at some later time $t + \Delta t$, each will have convected to a new position. The entire set of fluid points thus defines a new curve at this later time. We may regard this new curve as the original curve moved into a new position by the motion of the fluid. In other words, we are considering the convection of an entire curve with the fluid. We call this the *convective transport* of the curve by the fluid.

For ease of visualisation, imagine the original fluid points marked with dye, so that they are easily identifiable within the body of the fluid. The original curve is thus a line of dye in the fluid. As the fluid convects, the line of dye moves with it. At each time t therefore, the line of dye occupies a new position, which we can regard as the position of the curve as it convects with the fluid.

As the curve convects, each portion of the curve will translate, turn, and stretch. Description of the motion of the entire curve is thus complicated. One useful way to keep track of what happens to it during convection is by tracking infinitesimal segments of it. In this section therefore, we will focus on an infinitesimal segment of a curve, and follow its progress as it convects.

An infinitesimal section of a fluid curve is best described by its endpoints. We may then regard the vector joining these endpoints as a good approximation to the curve segment. Suppose the endpoints of a given segment occupy positions x^i and $x^i + \delta x^i$ at time t. The vector δx^i may thus be taken to be the infinitesimal segment of curve at this time. At a later time $t + \Delta t$, the endpoints will occupy new positions. Denote the coordinates of these by x'^i and $x'^i + \delta x'^i$. The infinitesimal segment of curve may thus be represented at this time by the vector $\delta x'^i$. Since each endpoint convects with the fluid, each will move in time Δt with the velocity of the fluid at its position. Thus

$$x'^i = x^i + v^i(\vec{x}, t)\Delta t + \cdots \tag{8.7}$$

where the dots indicate terms of order higher than 1 in Δt. Similarly,

$$\begin{aligned} x'^i + \delta x'^i &= (x^i + \delta x^i) + v^i(\vec{x} + \delta\vec{x}, t)\Delta t \\ &= x^i + \delta x^i + \left[v^i(\vec{x}, t) + \frac{\partial v^i}{\partial x^k}(\vec{x}, t)\ \delta x^k + \cdots\right]\Delta t \\ &= x^i + v^i(\vec{x}, t)\Delta t + \delta x^i + \frac{\partial v^i}{\partial x^k}(\vec{x}, t)\ \delta x^k \Delta t + \cdots \end{aligned}$$

so that

$$\delta x'^i = \delta x^i + \frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k \Delta t + \cdots \tag{8.8}$$

The rate of change of the infinitesimal separation vector δx^i as the fluid convects is therefore given by

$$\begin{aligned}\frac{D}{Dt}\ \delta x^i &= \lim_{\Delta t \to 0} \frac{\delta x'^i - \delta x^i}{\Delta t} \\ &= \lim_{\Delta t \to 0} \left[\frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k + \Delta t(\cdots)\right] \\ &= \frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k\end{aligned}$$

This result can be expressed in a variety of other useful ways. First, note that

$$\frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k = \delta v^i$$

or equivalently,

$$\frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k = \delta \vec{x} \cdot \nabla v^i$$

Note also that

$$\frac{D}{Dt}\ x^i = v^i$$

We may therefore write the above result as

$$\frac{D}{Dt}\ \delta \vec{x} = \delta \vec{v}$$

or as,

$$\frac{D}{Dt}\ \delta \vec{x} = \delta \vec{x} \cdot \nabla \vec{v}$$

More useful still, we note that

$$\frac{D}{Dt}\ x^i = v^k \partial_k x^i + \frac{\partial x^i}{\partial t} = v^i$$

so that we can write the result in the form

$$\frac{D}{Dt}\ \delta \vec{x} = \delta\ \frac{D}{Dt}\ \vec{x}$$

This last form has the advantage of showing that we can regard the infinitesimal difference operator δ and the convective derivative operator D/Dt as commuting operators.

9 Flow and Circulation

9.1 Flow

definition

$$\Gamma = \int_C \vec{v}(\vec{x}, t) \cdot d\vec{l} \tag{9.1}$$

This yields a function of t. This is seen as follows. Let the curve C have equations $x^i = \xi^i(\lambda)$. Then the infinitesimal displacement $d\vec{l}$ may be written as

$$d\vec{l} = dx^i E_i = \frac{d\xi^i}{d\lambda}(\lambda)\ d\lambda\ E_i = \frac{d\vec{\xi}}{d\lambda}(\lambda)\ d\lambda \tag{9.2}$$

so that

$$\vec{v}(\vec{x}(\lambda), t) \cdot d\vec{l} = \vec{v}(\vec{x}(\lambda), t) \cdot \frac{d\vec{\xi}}{d\lambda}(\lambda)\ d\lambda \tag{9.3}$$

and hence

$$\Gamma = \int_C \vec{v}(\vec{x}(\lambda), t) \cdot \frac{d\vec{\xi}}{d\lambda}(\lambda)\ d\lambda \tag{9.4}$$

The value obtained depends on t because the fluid velocity field depends on t, and so $\Gamma = \Gamma(t)$.

9.2 Rate of Change of Flow on Convecting Curves

Γ is a measure of the nett flow of the fluid along the curve C. In general, we expect Γ to depend on t. To see the form of this dependence, we need to calculate its rate of change.

As stated, this is not a well-posed problem. We have not specified what the curve C is doing in time. It could remain fixed in a given position in space within the fluid, it could move through the fluid, or it could convect with the fluid in such a way that it remains stationary relative to the fluid points. The value of Γ at time t clearly depends on where the curve is at that time, so its rate of change must depend strongly on how C moves.

A curve C in arbitrary motion through the fluid may be of interest when considering the action of the fluid on some object moving relative to it. However, it is of no interest when discussing the properties of the flow. So, the only cases of a curve that is stationary in space and of one that convects with the fluid are of interest here. The rate of change of flow on a stationary curve yields information on the flow through a fixed region of space. That on a convecting curve gives information about the flow of a fixed element of fluid as it moves through space.

In this section, we are interested in the intrinsic properties of the flow. We shall therefore consider the rate of change of flow on a curve that consists of fluid points and which convects with the fluid. Let C be a curve of this type. Then

$$\frac{D\Gamma}{Dt}(t) = \frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} \tag{9.5}$$

To evaluate the right hand side, we first calculate the rate of change of the flow along an infinitesimal segment of the convecting curve.

Denote the positions of the end-points of the infinitesimal segment at time t by x^i and $x^i + \delta x^i$. The flow along this segment is then approximately

$$\delta\Gamma(t) \approx \vec{v}(\vec{x},t)\cdot\delta\vec{x} \tag{9.6}$$

The δ in front of the Γ indicates that this is the contribution of a small section of the curve C to the flow along it. As time advances, each fluid point on the curve segment convects to a new position. The point at position x^i at time t will thus be at a new position x'^i at time $t+\Delta t$, given by

$$x'^i = x^i + v^i(\vec{x},t)\ \Delta t + \cdots \tag{9.7}$$

while the point at position $x^i + \delta x^i$ at time t will be at the new position x''^i given by

$$x''^i = x^i + \delta x^i + v^i(\vec{x}+\delta\vec{x},t)\ \Delta t + \cdots \tag{9.8}$$

The segment of curve we are considering will thus be described at time $t+\Delta t$ by the vector

$$\begin{aligned}\delta x'^i = x''^i - x'^i &= x^i + \delta x^i + v^i(\vec{x}+\delta\vec{x},t)\ \Delta t - x^i - v^i(\vec{x},t)\ \Delta t\\ &= x^i + \delta x^i + \left[v^i(\vec{x},t) + \frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k + \cdots\right]\Delta t - x^i - v^i(\vec{x},t)\ \Delta t\\ &= \delta x^i + [(\delta\vec{x}\cdot\nabla)v^i(\vec{x},t)]\Delta t + \cdots\end{aligned}$$

All fields are now evaluated at $\vec{x}$ and t, so we can omit the arguments without ambiguity so, in vector notation, we get

$$\delta\vec{x}' = \delta\vec{x} + [(\delta\vec{x}\cdot\nabla)\vec{v}]\Delta t + \cdots \tag{9.9}$$

The term $(\delta\vec{x}\cdot\nabla)\vec{v}$ represents the amount by which the fluid velocity field $\vec{v}$ changes across the curve segment $\delta\vec{x}$. It is sometimes denoted by $\delta\vec{v}$. However, this notation does not display explicitly what change in $\vec{v}$ is under consideration, so I do not use it here. The flow along this segment at time $t + \Delta t$ is therefore given by

$$\begin{aligned}
\delta\Gamma(t+\Delta t) &\approx \vec{v}(\vec{x}', t+\Delta t)\cdot\delta\vec{x}' \\
&= v_i\left(\vec{x}+\vec{v}(\vec{x},t)\ \Delta t+\cdots, t+\Delta t\right)\left(\delta x^i + [(\delta\vec{x}\cdot\nabla)v^i(\vec{x},t)]\ \Delta t+\cdots\right) \\
&= \left[v_i(\vec{x},t)+\frac{\partial v_i}{\partial x^k}\ \delta v^k(\vec{x},t)\ \Delta t+\frac{\partial v_i}{\partial t}(\vec{x},t)\ \Delta t+\cdots\right]\left(\delta x^i+[(\delta\vec{x}\cdot\nabla)v^i(\vec{x},t)]\ \Delta t+\cdots\right) \\
&= \left[v_i(\vec{x},t)+\left((\vec{v}(\vec{x},t)\cdot\nabla)v_i(\vec{x},t)+\frac{\partial v_i}{\partial t}(\vec{x},t)\right)\Delta t+\cdots\right]\left(\delta x^i+[(\delta\vec{x}\cdot\nabla)v^i(\vec{x},t)]\ \Delta t+\cdots\right) \\
&= \left[v_i(\vec{x},t)+\frac{Dv_i}{Dt}(\vec{x},t)\ \Delta t+\cdots\right]\left(\delta x^i+[(\delta\vec{x}\cdot\nabla)v^i(\vec{x},t)]\ \Delta t+\cdots\right)
\end{aligned}$$

All fields are now evaluated at $\vec{x}$ and t so we can omit all arguments without ambiguity. This result thus becomes, in vector notation,

$$\delta\Gamma(t+\Delta t) \approx \vec{v}\cdot\delta\vec{x}+\left[\frac{D\vec{v}}{Dt}\cdot\delta\vec{x}+\vec{v}\cdot(\delta\vec{x}\cdot\nabla)\vec{v}\right]\Delta t+\cdots \tag{9.10}$$

$$= \delta\Gamma(t)+\left[\frac{D\vec{v}}{Dt}\cdot\delta\vec{x}+\vec{v}\cdot(\delta\vec{x}\cdot\nabla)\vec{v}\right]\Delta t+\cdots \tag{9.11}$$

so that the rate of change of flow on the given segment as it convects is

$$\frac{D\Gamma}{Dt}(t) = \lim_{\Delta t\to 0}\frac{\delta\Gamma(t+\Delta t)-\delta\Gamma(t)}{\Delta t} \tag{9.12}$$

$$= \frac{D\vec{v}}{Dt}\cdot\delta\vec{x}+\vec{v}\cdot(\delta\vec{x}\cdot\nabla)\vec{v} \tag{9.13}$$

Interestingly, the second term is the convective rate of change of the infinitesimal vector $\delta\vec{x}$,

$$\frac{D}{Dt}\ \delta\vec{x} = (\delta\vec{x}\cdot\nabla)\vec{x} \tag{9.14}$$

We have therefore shown that

$$\frac{D}{Dt}\ \vec{v}\cdot d\vec{x} = \left(\frac{D}{Dt}\ \vec{v}\right)\cdot\delta\vec{x}+\vec{v}\cdot\left(\frac{D}{Dt}\ \delta\vec{x}\right) \tag{9.15}$$

This fact may be used in future calculations to arrive at results more easily. Returning to (9.13), we note that

$$\vec{v}\cdot(\delta\vec{x}\cdot\nabla)\vec{v} = \delta\vec{x}\cdot\nabla\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) = \delta\vec{x}\cdot\nabla\left(\frac{v^2}{2}\right)$$

This term represents the change in $v^2/2$ across the curve segment $\delta\vec{x}$. The rate of change of flow along the given curve segment is thus

$$\frac{D\Gamma}{Dt}(t) = \frac{D\vec{v}}{Dt}\cdot\delta\vec{x}+\delta\vec{x}\cdot\nabla\left(\frac{v^2}{2}\right) \tag{9.16}$$

Thus the change of flow with time along an infinitesimal segment of convecting path is driven by two effects. The first is the component of the fluid acceleration in the direction of the segment. The second is the gradient of $v^2/2$ along the segment.

This result is completely general. It makes no assumptions about the nature of the fluid, or about the nature of the flow. It therefore applies to all fluids and to all flows alike. In particular, it applies to both viscid and inviscid fluids, and to barotropic and baroclinic flows.

We can now calculate the rate of change of flow along the entire convecting curve C. Since the total flow on C is the sum of flows on each of its infintesimal segments, the rate of change of flow will be the sum of the rates of change on each of its segments,

$$\frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} = \frac{D}{Dt}\left[\lim_{N\to\infty}\sum_{A=1}^{N} \vec{v}(\vec{x}_A,t)\cdot\delta\vec{x}_A\right] = \lim_{N\to\infty}\sum_{A=1}^{N} \frac{D}{Dt}[\vec{v}(\vec{x}_A,t)\cdot\delta\vec{x}_A] = \int_C \frac{D}{Dt}[\vec{v}\cdot d\vec{l}]$$

so that

$$\begin{aligned}\frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} &= \int_C \frac{D}{Dt}[\vec{v}\cdot d\vec{l}]\\ &= \int_C \left[\left(\frac{D}{Dt}\,\vec{v}\right)\cdot d\vec{l} + \vec{v}\cdot\left(\frac{D}{Dt}\,d\vec{l}\right)\right]\\ &= \int_C \left[\frac{D\vec{v}}{Dt}\cdot d\vec{l} + d\vec{l}\cdot\nabla\left(\frac{v^2}{2}\right)\right]\end{aligned}$$

Thus, finally,

$$\frac{D\Gamma}{Dt}(t) = \int_C \frac{D\vec{v}}{Dt}\cdot d\vec{l} + \left[\frac{v^2}{2}\right]_i^f \tag{9.17}$$

The interpretation of this result is similar to that of equation (9.16). The change of flow in time along a convecting curve is driven by two effects. The first is the integrated (or, nett) component of acceleration of the fluid in the direction of the curve. This is given by the line-integral of the dot-product $(D\vec{v}/Dt)\cdot d\vec{l}$, which is the component of $D\vec{v}/Dt$ in the direction $d\vec{l}$ scaled by the length of $d\vec{l}$. The second is the integrated (or, nett) component of the gradient of $v^2/2$ in the direction of the curve. This is given by the line-integral of the dot-product $\nabla(v^2/2)\cdot d\vec{l}$, which is the component of $\nabla(v^2/2)$ in the direction $d\vec{l}$ scaled by the length of $d\vec{l}$. Because $\nabla(v^2/2)\cdot d\vec{l} = d(v^2/2)$ is an exact differential, the integral is independent of the path C of integration and evaluates to the difference in values of $v^2/2$ at its end-points.

Like (9.9), result (9.17) is completely general. It makes no assumptions at all about the nature of the fluid, or about the nature of the flow, and so applies to all fluids, and to all flows, irrespective of their particular properties. It applies therefore both to viscid and inviscid fluids, and to barotropic and baroclinic flows.

In the above derivation, we have proved some general results that reduce this (and other) calculations to the application of routine operations. The results of interest, that we collect here for covenience are, first,

$$\frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} = \int_C \frac{D}{Dt}\left(\vec{v}\cdot d\vec{l}\right) \tag{9.18}$$

that is, the derivative D/Dt commutes with the line integral $\int_C$. Second, the action of D/Dt on the integrand obeys a product rule of the form

$$\frac{D}{Dt}\left(\vec{v}\cdot d\vec{l}\right) = \left(\frac{D\vec{v}}{Dt}\right)\cdot d\vec{l} + \vec{v}\cdot\left(\frac{D}{Dt}\,d\vec{l}\right) \tag{9.19}$$

The strange appearance of (9.19) is due to the fact that the infinitesimal element $d\vec{l}$ must also be differentiated. Ordinarily, one does not consider line integrals along moving curves, so this term does not appear. It occurs here because, as the curve convects, each segment $d\vec{l}$ of the curve will translate, rotate and stretch as it convects. The rate of change of the integrand $\vec{v}\cdot d\vec{l}$ is thus determined not only by the rate of change of $\vec{v}$, but also by the rate of change of the segment $d\vec{l}$ as it convects. Third, we need the rate of change of $d\vec{l}$ as it cnvects

$$\frac{D}{Dt}\,d\vec{l} = (d\vec{l}\cdot\nabla)\vec{v} \tag{9.20}$$

Combining these results, we get that

$$\begin{aligned}
\frac{D\Gamma}{Dt}(t) &= \frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} \\
&= \int_C \frac{D}{Dt}\left(\vec{v}\cdot d\vec{l}\right) \\
&= \int_C \left[\left(\frac{D\vec{v}}{Dt}\right)\cdot d\vec{l} + \vec{v}\cdot\left(\frac{D}{Dt}\,d\vec{l}\right)\right] \\
&= \int_C \left(\frac{D\vec{v}}{Dt}\right)\cdot d\vec{l} + \int_C \vec{v}\cdot\left((d\vec{l}\cdot\nabla)\vec{v}\right) \\
&= \int_C \left(\frac{D\vec{v}}{Dt}\right)\cdot d\vec{l} + \int_C d\vec{l}\cdot\nabla\left(\frac{v^2}{2}\right)
\end{aligned}$$

We thus have

$$\frac{D\Gamma}{Dt}(t) = \int_C \left(\frac{D\vec{v}}{Dt}\right)\cdot d\vec{l} + \left[\frac{v^2}{2}\right]_i^f \tag{9.21}$$

The change of flow in time along a convecting curve is thus driven by two effects. The first is the integrated (or, nett) acceleration of the fluid in the direction of the curve, given by the line-integral of the component of $D\vec{v}/Dt$ in the direction $d\vec{l}$. The second is the integrated (or, nett) gradient of $v^2/2$ in the direction of the curve which, because it is an exact differential, evaluates to the difference in values of $v^2/2$ at the end-points of the curve.

Note that result (9.21) is completely general. It makes no assumptions at all about the nature of the fluid, or about the nature of the flow. It therefore applies to all fluids, and to all flows, irrespective of their particular properties.

In particular, it applies both to viscid and inviscid fluids, and to barotropic and baroclinic flows.

APPENDIX TO SECTION 9.2
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9.3 Rate of Change of Circulation

We now evaluate the derivative

$$\frac{D}{Dt}\oint_C \vec{v}\cdot d\vec{l} \tag{9.22}$$

of the circulation around a closed circuit C of fluid points. The simplest and most transparrent method for doing this is by breaking up the problem as follows. First, replace the closed circuit C by an open curve. We thus evaluate the rate of change

$$\frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} \tag{9.23}$$

on an open curve C. The case of a closed circuit is then a special case of this more general result. Then replace the finite open curve C by an infinitesimal one. The rate of change of the integral over a finite open curve is then a sum of the rates of change of integrals over these infinitesimal segments.

So, consider first an infinitesimal curve consisting of fluid points, stretching from a fluid point that at time t occupies position $\vec{x}$ to one that at the same time t occupies position $\vec{x}+\delta\vec{x}$. The required integral then becomes approximately

$$\int_C \vec{v}\cdot d\vec{l} \approx \vec{v}(\vec{x},t)\cdot d\vec{x} \tag{9.24}$$

The fluid point at position $\vec{x}$ at time t has velocity $\vec{v}(\vec{x},t)$, so at a later time $t+\delta t$ it occupies position

$$\vec{x}' = \vec{x} + \vec{v}(\vec{x},t)\ \delta t$$

Similarly, the fluid point at position $\vec{x}+\delta\vec{x}$ at time t has velocity $\vec{v}(\vec{x}+\delta\vec{x},t)$, so at a later time $t+\delta t$ it occupies position

$$\vec{x}'' = (\vec{x}+\delta\vec{x}) + \vec{v}(\vec{x}+\delta\vec{x},t)\ \delta t$$

or, in coordinate form,

$$x''^i = x^i + \delta x^i + \left(v^i(\vec{x},t) + \frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j\right)\delta t$$

The infinitesimal curve C of fluid points, approximated by the vector $\delta\vec{x}$ thus convects in time δt into an infinitesimal curve C', consisting of the *same* fluid points, approximated by the infinitesimal vector

$$\begin{aligned}\delta x'^i = x''^i - x'^i &= x^i + \delta x^i + v^i(\vec{x},t)\ \delta t + \frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j \delta t - x^i - v^i(\vec{x},t)\ \delta t \\ &= \delta x^i + \frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j \delta t\end{aligned}$$

The value of the required integral at time $t + \delta t$ is thus

$$\begin{aligned}\int_{C'} \vec{v}\cdot d\vec{l} &\approx \vec{v}(\vec{x}', t+\delta t)\cdot \delta\vec{x}' \\ &= v_i\left(\vec{x} + \vec{v}(\vec{x},t)\ \delta t, t+\delta t\right)\left(\delta x^i + \frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j \delta t\right) \\ &= \left(v_i(\vec{x},t) + \frac{\partial v_i}{\partial x^j}(\vec{x},t)\ v^j(\vec{x},t)\ \delta t + \frac{\partial v_i}{\partial t}(\vec{x},t)\ \delta t\right)\left(\delta x^i + \frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j \delta t\right) \\ &= v_i(\vec{x},t)\ \delta x^i + v_i(\vec{x},t)\frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j \delta t + \frac{\partial v_i}{\partial x^j}(\vec{x},t)\ v^j(\vec{x},t)\ \delta t\ \delta x^i \\ &\quad + \frac{\partial v_i}{\partial x^j}(\vec{x},t)\ v^j(\vec{x},t)\ \frac{\partial v^i}{\partial x^k}(\vec{x},t)\ \delta x^k(\delta t)^2 + \frac{\partial v_i}{\partial t}(\vec{x},t)\ \delta t\ \delta x^i + \frac{\partial v_i}{\partial t}(\vec{x},t)\ \frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j(\delta t)^2\end{aligned}$$

Since we will be dividing the difference of these two integrals by δt and taking the limit $\delta t \to 0$, we can write this as

$$\begin{aligned}\int_{C'} \vec{v}\cdot d\vec{l} \approx v_i(\vec{x},t)\ \delta x^i + v_i(\vec{x},t)\frac{\partial v^i}{\partial x^j}(\vec{x},t)\ \delta x^j \delta t + \frac{\partial v_i}{\partial x^j}(\vec{x},t)\ v^j(\vec{x},t)\ \delta t\ \delta x^i \\ + \frac{\partial v_i}{\partial t}(\vec{x},t)\ \delta t\ \delta x^i\end{aligned}$$

Thus

$$\begin{aligned}\frac{D}{Dt}\int_C \vec{v}\cdot d\vec{l} &= \lim_{\delta t\to 0}\left[\frac{\int_{C'} \vec{v}\cdot d\vec{l} - \int_C \vec{v}\cdot d\vec{l}}{\delta t}\right] \\ &= \lim_{\delta t\to 0}\left[v_i\frac{\partial v^i}{\partial x^j}\ \delta x^j + \frac{\partial v_i}{\partial x^j}\ v^j\ \delta x^i + \frac{\partial v_i}{\partial t}\ \delta x^i\right] \\ &= \left[v_j\frac{\partial v^j}{\partial x^i} + v^j\frac{\partial v_i}{\partial x^j} + \frac{\partial v_i}{\partial t}\right]\ \delta x^i\end{aligned}$$

where are fields are to be evaluated at $(\vec{x}, t)$.

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9.4 Kelvin's Circulation Theorem

10 Irrotational Flow

10.1 Definition

A flow is said to be *irrotational* at time t if the fluid velocity field at time t satisfies the condition

$$\nabla \times \vec{v} = 0 \tag{10.1}$$

This condition is equivalent to demanding that the circulation around every *infinitesimally small* closed curve C in the fluid is zero.

Note however that this condition does *not* guarantee that the circulation around *every* closed curve C, finite or infinitesimal, is zero. This latter condition requires the circulation to be zero for a larger class of closed curves than is guaranteed by (10.1) alone. It is therefore a stronger condition for which (10.1) is necessary but not sufficient. Sufficient conditions for this stronger requirement include ones of a topological nature which will be discussed with the various cases that arise. The existence of a potential for the velocity field is closely related to these additional conditions.

10.2 Persistence of Irrotational Flow

The irrotationality condition (10.1) is a constraint on the spatial partial derivatives of the fluid velocity field and says nothing about its partial derivative with respect to time. Since the development in time of the motion of the fluid is determined by a different and independent set of equations, it makes sense to ask whether a flow that is irrotational at one instant in time will remain irrotational at all times. In other words, do the equations that govern fluid flow preserve irrotationality, or do they destroy it?

This is an important question. If irrotationality is destroyed by the equations of motion, the irrotational flow is at best a transient phenomenon that can occur at a single instant only. It would therefore be, at best, a chance occurrence and thus not worthy of separate study. On the other hand, if it is not destroyed by the equations of motion, then irrotationality would be a persistent condition in the fluid flow and would characterise an important subcategory of possible flows.

In this section we will show that

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Laplace's equation is

$$\nabla^2\phi = 0 \tag{10.2}$$

10.3 Flux of an Incompressible, Irrotational Fluid

Let ϕ be a solution of Laplace's equation. The vector field

$$\vec{v} = -\nabla\phi \tag{10.3}$$

has a flux across any surface S that is given by

$$\Phi_S = \int_S \vec{v}\cdot d\vec{S} = -\int_S \nabla\phi\cdot d\vec{S} \tag{10.4}$$

In fluid dynamics, $\vec{v}$ is the fluid velocity field, and Φ_S is equal to the total volume of the fluid crossing the surface S per unit time.

If the surface S is closed, and encloses the volume V then, by the divergence theorem, the total flux of $\vec{v}$ across S is given by

$$\Phi_S = -\oint_S \nabla\phi\cdot d\vec{S} = -\int_V \nabla^2\phi\; d^3\vec{x} \tag{10.5}$$

so that, if ϕ is a solution to Laplace's equation, the total flux of $\vec{v}$ across any closed surface S is

$$\Phi_S = -\int_V \nabla^2\phi\; d^3\vec{x} = 0 \tag{10.6}$$

This result has a simple and elegant interpretation: the nett volume of fluid exiting from the closed surface S is zero. Expressed differently, the total amount of fluid entering the closed surface S is equal to the total amount of fluid exiting from it. This result may be regarded as a conservation law. It states that the total volume occupied by a given set of fluid points is constant. The result is not unexpected. Since the fluid is incompressible, its density is constant, and so a fixed volume of fluid has a fixed mass. This is therefore just the mass conservation law for incompressible fluids.

10.4 Flux Tubes

Consider now the integral curves of the velocity field. Let S be any element of surface in the fluid that lies transverse to these integral curves at each of its points, and let C be the closed curve that is the boundary of S. Through each point of C there passes exactly one integral curve of the velocity field. These integral curves do not intersect and they therefore define a 'tube' in the fluid. A tube formed in this way is called a *tube of flow*, or a *flow tube*.

Consider now a second surface transverse to the flow tube defined by C. Its intersection with the flow tube defines a second element of surface S' with boundary C' that generates the *same* flow tube. Together with the tube wall S'', the elements S and S' define a closed cylindrical surface Σ. By the results of the previous section, the nett flux through Σ is zero. Also, the velocity field is tangent to the tube wall, so nett flux through the tube wall is zero. This means that the flux through S' is equal and opposite in sign to that through S. If we now choose the orientation of S' relative to the velocity field to be the same as that of S relative to it, we see that the flux through S' is identical to the flux through S. This result is true for arbitrary choices of S', and means that we can define unambiguously the strength σ of the flux tube to be

$$\sigma = \int_S \vec{v} \cdot d\vec{S} \tag{10.7}$$

We can guarantee σ to be positive if we make the convention that the orientation of its boundary C will always be determined by the velocity field using the right-hand rule. That is, if we point the thumb of the right-hand along the lines of $\vec{v}$, then the remaining fingers of the right hand will point in the direction assigned to C.

It is sometimes convenient to regard the entire space occupied by the fluid as made up of tubes of flow, where the size of each tube has been adjusted in such a way that the flux through each tube is the same. The flux across any surface is then proportional to the number of tubes which cross it. This picture is sometimes refined by replacing each flux tube by a central field line within it. The number of field lines crossing a given surface is then proportional to the flux across the surface, and the density of this reduced set of field lines is proportional to the strength of the velocity field.

The conservation theorem of the previous section, that the total flux through any closed surface is zero, can now be stated by saying that as many flux tubes as enter the surface must also leave it, or equivalently, as many field lines as enter the surface must also leave it.

It follows from this last result that no field line can either begin or end at a point inside the fluid. For, if they did, we would not have the same number of lines entering a closed surface surrounding that point as are leaving it, violating the above result. The field lines must thus either begin and end on the boundaries, or extend to and from infinity, or else form closed loops. We shall see in a later section that the field lines cannot form closed loops, leaving only the two former options.

10.5 Gauss' Mean Value Theorem

10.6 Uniqueness Theorems for Infinite Domains

Many problems of interest involve one or more obstacles placed in the fluid at finite positions in space and where the fluid is contained by very distant boundary walls. The distant boundary walls have little or no effect on the flow around the obstacles. The simplest way to solve these problems is to imagine that the container walls are infinitely removed from the obstacles of interest. In this idealisation, the fluid extends to infinity, but is bounded internally by one or more closed surfaces.

All of the preceding uniqueness theorems deal with fluids that are contained by boundary walls that are at finite distances from the obstacles. These do not include the case of flows that extend to infinity. We must therefore develop a new set of theorems to deal with these cases. Most of the theorems we need can be inferred by the methods of proof used when proving the preceding set of uniqueness theorems. One important theorem, however, cannot and requires a new set of considerations to establish it.

Theorem: If

- $\phi(\vec{x}) = C$ on each of the internal boundaries, where C is a given constant (that is, ϕ takes constant value on each of the internal boundaries, with that constant value being the same for all internal boundaries), and
- $\phi \to C$ as $|\vec{x}| = r \to \infty$ (that is, ϕ tends everywhere to the same constant value C at infinite distance from the internal boundary),

then

$$\phi(\vec{x}) = C$$

everywhere (that is, ϕ is constant everywhere with value equal to the value that it assumes on each of its boundaries, both finite and infinite).

Proof: Suppose ϕ is not constant. Then it must either rise or fall from value C as one moves away from the finite boundaries. It must then either fall or rise again to value C as we move to infinity in any direction. By continuity, ϕ must thus have a maximum or a minimum at some finite point within the region, which is impossible by a previous result. So ϕ must be constant throughout the region occupied by the fluid.

Theorem: If

- the value of ϕ is specified in any arbitrary way on each of the internal boundaries, and
- $\phi \to C$ as $|\vec{x}| = r \to \infty$, where C is any given constant value (that is, ϕ tends everywhere to the same constant value C at infinite distance from the internal boundary),

then the function ϕ, if it exists, is uniquely determined at each point $\vec{x}$ in the region occupied by the fluid.

Proof: Use the same method as was used in Article 40 on p 41-42 of Lamb.

The following important theorem cannot be proved by the methods used in the proof of the corresponding results for the case where the fluid is contained within a finite boundary and requires new methods to be introduced:

Theorem: If

- the normal component of the fluid velocity is zero at each of the internal boundaries, and
- the fluid is at rest at infinity,

then ϕ is everywhere constant, that is,

$$\phi(\vec{x}) = C$$

at each point in the region occupied by the fluid.

Proof: Cannot use the same method as was used in Article 40 on p 41-42 of Lamb. State why. Then write out the proof on p 42-43 of Lamb.

Lamb's Proof, (Lamb, 1932, p 42-43): "Since the velocity tends to the limit zero at an infinite distance from the internal boundary (S, say), it must be possible to draw a closed surface Σ completely enclosing S, beyond which the velocity is everywhere less than a certain value ϵ, which value may, by making Σ large enough, be made as small as we please. Now in any direction from S let us take a point P at such a distance beyond Σ that the solid angle which Σ subtends at it is infinitely small; and with P as centre let us describe two spheres, one just excluding, the other just including S. We shall prove that the mean value of ϕ over each of these spheres is, within an infinitely small amount, the same. For if Q, Q' be points of these spheres on a common radius PQQ', then if Q, Q' fall within Σ the corresponding values of ϕ may differ by a finite amount; but since the portion of either spherical surface which falls within Σ is an infinitely small fraction of the whole, no finite difference in the mean values can arise from this cause. On the other hand, when Q, Q' fall without Σ, the corresponding values of ϕ cannot differ by so much as $\epsilon.QQ'$, for ϵ is by definition a superior limit to the rate of variation of ϕ. Hence, the mean values of ϕ over the two spherical surfaces must differ by less than $\epsilon.QQ'$. Since QQ' is finite, whilst ϵ may by taking Σ large enough be made as small as we please, the difference of the mean values may, by taking P sufficiently distant, be made infinitely small.

"Now we have seen in Arts. 38,39 that the mean value of ϕ over the inner sphere is equal to its value at P, and that the mean value over the outer sphere is (since $M = 0$) equal to a constant quantity C. Hence, ultimately, the value of ϕ at infinity tends everywhere to the constant value C.

"The same result holds even if the normal velocity be not zero over the internal boundary; for in the theorem of Art. 39 M is divided by r, which is in our case infinite.

"It follows that if $\partial\phi/\partial n = 0$ at all points of the internal boundary, and if the fluid be at rest at infinity, it must be everywhere at rest. For no lines of motion can begin or end on the internal boundary. Hence such lines, if they existed, must come from an infinite distance, traverse the region occupied by the fluid, and pass off again to infinity; i.e. they must form infinitely long courses between places where ϕ has, with an infinitely small amount, the same value C, which is impossible."

My Version of Lamb's Proof:

Denote the internal boundaries by $S_i, i = 1, ..., N$. Since the fluid velocity tends to zero as $r \to \infty$, the further one moves away from the boundaries S_i, the smaller the fluid velocity becomes. It is possible therefore to draw a closed surface Σ that completely encloses the S_i and beyond which the velocity everywhere has magnitude less than a given value ϵ. By making Σ sufficiently large, ϵ can be made as small as we please.

Now consider a point P that lies outside the surface Σ. Choose P in any arbitrary direction relative to the internal boundaries, and sufficiently far away from Σ that the solid angle subtended by Σ at P is infinitesimally small. With centre P, draw two spheres S and S', one just excluding all of the S_i, the other just including them. We shall now prove that the average value of ϕ over these two spheres differs only by an infinitesimal amount.

Let Q and Q' be points on these spheres on a common radius PQQ'. Depending on the direction of this radius from P, the radius may be contained within the solid angle subtended by Σ at P, or it may fall outside of it. If within, the values $\phi(q)$ and $\phi(Q')$ could differ by a finite amount. However, since the portion of either spherical surface which falls within the solid angle subtended by Σ is an infinitely small fraction of the whole, no finite difference in the mean values can arise from this cause. If outside of it, the corresponding values of ϕ will differ by less than $\epsilon\, QQ'$

Theorem: If

- the normal component of the fluid velocity is arbitrarily specified at each point of the internal boundaries, that is, if we are give the value of

$$\vec{v}_n = -\vec{n} \cdot \nabla\phi = -\frac{\partial \phi}{\partial n}$$

- the fluid is at rest at infinity,

then the function ϕ, if it exists, is uniquely determined in the domain occupied by the fluid.

Proof: Use the same method as was used in Article 40 on p 41-42 of Lamb.

"The theorem that, if the fluid be at rest at infinity, the motion is determinate when the value of $-\partial\phi/\partial n$ is given over the internal boundary, follows by the same argument as in Art. 40."

11 Boundary Layers

11.1 The boundary layer concept

11.2 Governing equations for flow in the boundary layer

11.3 Steady flow along a flat plate

Consider the flow of a viscous fluid along an infinite flat plate. Choose axes such that the flat plate lies in the x, z plane, with the y axis perpendicular to it, and such that the flow is in the x direction only. We are interested in this section only in steady unidirectional flow, with flow in the direction of the x axis. We assume also that the flow has no z dependence.

In regions distant from the plate, the flow is unaware of the presence of the plate. The effects of viscosity are negligible and the fluid may be treated as if it were inviscid. In the absence of applied pressure differences, it will flow with uniform velocity $\vec{u}_0 = u_0 \boldsymbol{i}$, say.

Nearer to the plate, the effects of viscosity become significant. The fluid in contact with the plate remains at rest, and the velocity of the fluid increases steadily as we move away from the plate until it eventually attains the value $\vec{u}_0$.

In the boundary layer, the governing equations for the flow become

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2} \tag{11.1}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11.2}$$

The first is the Navier-Stokes equation for the boundary layer, while the second is the equation for mass conservation. The boundary values for this problem are

BC1. The non-slip condition: $u(x, 0) = 0$ and $v(x, 0) = 0$.
BC2. The asymptotic steady flow condition: $u(x, \infty) = u_0$.

In this problem, the equation of continuity is a constraint equation. We can guarantee that it is satisfied by introducing a stream function $\psi(x, y)$ and setting

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\,\frac{\partial \psi}{\partial x} \tag{11.3}$$

The system parameters that specify this problem are the kinematic viscosity $\vec{u}$ of the fluid, which features in the governing equations, and the asymptotic speed u_0, which features in the boundary conditions. The solution to the problem must depend of the coordinates x and y, and on the system parameters ν, u_0. We thus expect a solution of the form

$$\psi = \psi(x, y, \nu, u_0)$$

Non-dimensionalisation of the problem

The dimensions of the four governing parameters are

$$[x] = [y] = L, \quad [\nu] = L^2T^{-1}, \quad [u_0] = LT^{-1}$$

while that of the dependent parameter can be found from

$$LT^{-1} = [u] = \left[\frac{\partial \psi}{\partial y}\right] = \frac{[\psi]}{L}$$

giving

$$[\psi] = L^2T^{-1}$$

It is clear that this system has only two independent dimensions, L and T. There are therefore only two dimensionally independent governing parameters. We shall take these to be ν and u_0. To de-dimensionalise x and y, we consider the dimensional equation

$$L = [x] = [\nu]^\alpha [u_0]^\beta = (L^{2\alpha}T^{-\alpha})(L^\beta T^{-\beta}) = L^{2\alpha+\beta}T^{-(\alpha+\beta)}$$

which gives

$$1 = 2\alpha + \beta, \quad 0 = \alpha + \beta$$

which gives

$$\beta = -1, \quad \alpha = 1$$

and hence the variables

$$X = \frac{u_0}{\nu}\, x, \quad Y = \frac{u_0}{\nu}\, y$$

are dimensionless. Furthermore, since ψ has the same dimensions as ν, the dependent variable ψ, written in dimensionless form, becomes

$$\Psi = \frac{\psi}{\nu}$$

According to Buckingham's Π-Theorem, the solution to our problem must therefore have the form

$$\Psi = F(X, Y)$$

or

$$\psi = \nu\ F\left(\frac{u_0}{\nu}x, \frac{u_0}{\nu}y\right)$$

Now write the governing equations in non-dimensional form. First define the dimensionless velocities

$$U = \frac{u}{u_0}, \quad V = \frac{v}{u_0}$$

Then the Navier-Stokes equation for the boundary layer becomes,

$$(u_0\ U)\frac{u_0}{\nu}\ \frac{\partial U}{\partial X}\ u_0 + (u_0 V)\frac{u_0}{\nu}\ \frac{\partial U}{\partial Y}\ u_0 = \nu\left(\frac{u_0}{\nu}\right)^2 \frac{\partial^2 U}{\partial Y^2}\ u_0$$

or

$$U\ \frac{\partial U}{\partial X} + V\ \frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} \tag{11.4}$$

which is identical in form to the original equation, but with ν replaced by 1. Similarly, the mass conservation equation becomes

$$\frac{\partial U}{\partial X} + \frac{\partial U}{\partial Y} = 0 \tag{11.5}$$

and the equations relating the velocities to the stream function become

$$U = \frac{\partial \Psi}{\partial Y}, \quad V = -\ \frac{\partial \Psi}{\partial X} \tag{11.6}$$

The boundary conditions for these equations become, after rewriting them in terms of the dimensionless variables,

BC1. The non-slip condition: $U = V = 0$ when $Y = 0$.
BC2. The asymptotic steady flow condition: $U = 1$ when $Y = \infty$.

Similarity transformations of solutions

The non-dimensional version of the boundary layer equations is no easier to solve than the original equations. The only difference between them is that ν and u_0 have been replaced by 1 in the equations and in the boundary conditions respectively. However, the form of the equations to be solved for the dimensionless variables remains unchanged. We therefore search for a similarity transformation to simplify the equations. The solution has the form

$$\Psi = F(X, Y)$$

The graph space of this solution is three-dimensional, with variables X, Y, Ψ. We therefore search for a scaling transformation of the form

$$X^* = \alpha X, \quad Y^* = \beta Y, \quad \Psi^* = \gamma \Psi$$

Here, the asterisk does not denote complex conjugation but a new set of variables. The property that we demand of this transformation is that the new variables

X^*, Y^*, Ψ^* satisfy the same differential equations an boundary conditions as are satisfied by X, Y, Ψ. This means that the solution function F must be the same for both, and so the new variables will be related by

$$\Psi^* = F(X^*, Y^*)$$

or, substituting for the new variables in terms of the old,

$$\Psi = \frac{1}{\gamma}\, F(\alpha X, \beta Y)$$

First, we demand that the new variables satisfy the same differential equations as the old. Since we have eliminated the mass continuity equation by expressing the velocity field in terms of the stream function Ψ, we need consider only the Navier-Stokes equation for the boundary layer. We thus demand that the variables X^*, Y^*, Ψ^* satisfy the equation

$$U^*\,\frac{\partial U^*}{\partial X^*} + V^*\,\frac{\partial U^*}{\partial Y^*} = \frac{\partial^2 U^*}{\partial {Y^*}^2}$$

We need to discover how the velocities U and V transform under the similarity transformation. We demand that

$$U^* = \frac{\partial \Psi^*}{\partial Y^*}, \quad V^* = -\,\frac{\partial \Psi^*}{\partial X^*}$$

and so the transformation equations for U and V are,

$$U^* = \frac{\partial \Psi^*}{\partial Y^*} = \frac{\gamma}{\beta}\,\frac{\partial \Psi}{\partial Y} = \frac{\gamma}{\beta}\,U$$
$$V^* = -\,\frac{\partial \Psi^*}{\partial X^*} = -\frac{\gamma}{\alpha}\,\frac{\partial \Psi}{\partial X} = \frac{\gamma}{\alpha}\,V$$

The differential equation for X^*, Y^*, Ψ^* thus gives

$$\left(\frac{\gamma}{\beta}\,U\right)\frac{1}{\alpha}\,\frac{\partial}{\partial X}\left(\frac{\gamma}{\beta}U\right) + \left(\frac{\gamma}{\alpha}\,V\right)\frac{1}{\beta}\,\frac{\partial}{\partial Y}\left(\frac{\gamma}{\beta}U\right) = \frac{1}{\beta^2}\,\frac{\partial^2}{\partial Y^2}\left(\frac{\gamma}{\beta}U\right)$$

or

$$\frac{\gamma^2}{\beta^2\alpha}\left(U\,\frac{\partial U}{\partial X} + V\,\frac{\partial U}{\partial Y}\right) = \frac{\gamma}{\beta^3}\,\frac{\partial^2 U}{\partial Y^2}$$

This coincides with the governing equation for U, V, X, Y only if

$$\frac{\gamma^2}{\beta^2\alpha} = \frac{\gamma}{\beta^3}$$

or

$$\alpha = \beta\gamma$$

The mass conservation equation and the first boundary condition add no new information. The second boundary conditions however adds a further constraint. The new variables must satisfy boundary conditions identical to the old, so U^* must satisfy the asymptotic steady flow condition: $U^* = 1$ when $Y^* = \infty$.

Expressed in terms of U, V, X, Y, this means that $\gamma U/\beta = 1$ when $Y = \infty$. This coincides with the boundary condition for U if

$$\gamma = \beta$$

Hence,

$$\alpha = \beta^2$$

The similarity transformation thus contains only one independent parameter and the solutions to our equation have the property that

$$\Psi = \frac{1}{\beta}\, F(\beta^2 X, \beta Y) \tag{11.7}$$

Reduction of the solution to a function of one variable

The scaling property (11.7) has an interesting significance. Our proof above shows that, for each value of β, the function

$$F_\beta(X, Y) = \frac{1}{\beta}\, F(\beta^2 X, \beta Y)$$

is a solution to our governing equations with the given boundary conditions. It would thus appear that we have uncovered an infinity of solutions to the equations, each of which satisfies the same boundary conditions. However, this is impossible. Our problem is well posed and so, by the uniqueness theorems of partial differential equations, our equations can admit only one solution for the given boundary conditions. This means that all of the functions F_β, whatever choice we make for the value of β, represent one and the same solution.

The consequences of this last statement are far reaching. All of the functions F_β can coincide with the solution $F(X, Y)$ only if the scaling factor β cancels from the expression $F(\beta^2 X, \beta Y)/\beta$ to leave us with $F(X, Y)$. Thus, for every value of β, we must have

$$\frac{1}{\beta}\, F(\beta^2 X, \beta Y) = F(X, Y)$$

In turn, this is possible only if the function F does not depend on X and Y independently, but on some combination of them in which the scaling factor β cancels. The solution $F(X, Y)$ of our differential equations is thus not really a function of two variables but only of one. We should therefore be able to replace the governing partial differential equations of which F is a solution by ordinary differential equations whose independent variable is the as yet undiscovered combination of X and Y.

We now search for the combination of X and Y on which F depends. To discover this combination, we exploit the fact that any choice of the scaling factor β produces one and the same solution to the governing equations. First, note that if β, β' are any two values of the scaling parameter, we must have

$$\frac{1}{\beta}\ F(\beta^2 X, \beta Y) = \frac{1}{\beta'}\ F(\beta'^2 X, \beta' Y)$$

We will choose the value of β' in such a way as to reduce to 1 one of the arguments of F on the right hand side of this relation. For definiteness, we shall choose β' in such a way as to reduce the first argument to 1. This means that we must choose β' to satisfy

$$\beta'^2 X = 1$$

or

$$\beta' = \frac{1}{\sqrt{X}}$$

On the left hand side, we shall choose the value of β to be 1. This gives

$$F(X, Y) = \sqrt{X}\ F\left(1, \frac{Y}{\sqrt{X}}\right)$$

The function F on the right hand side now has the value 1 substituted for its first argument, and so is a function of its second argument alone. It is therefore a function of one variable only. We show this explicitly in the notation by introducing a new symbol for this function of a single variable. We define the function f of one variable by the prescription

$$f(\eta) = F(1, \eta)$$

In terms of this notation, we get

$$F(X, Y) = \sqrt{X}\ f\left(\frac{Y}{\sqrt{X}}\right)$$

showing that $F(X, Y)$ can be expressed in terms of an unknown function f of a single variable. To solution to our governing equations may therefore be expressed in the form

$$\Psi = \sqrt{X}\ f\left(\frac{Y}{\sqrt{X}}\right)$$

The dimensionless velocities are thus given by

$$U = \frac{\partial \Psi}{\partial Y} = f'\left(\frac{Y}{\sqrt{X}}\right)$$

and

$$V = -\frac{\partial \Psi}{\partial X} = -\frac{1}{2\sqrt{X}}\ f\left(\frac{Y}{\sqrt{X}}\right) + \frac{Y}{2X}\ f'\left(\frac{Y}{\sqrt{X}}\right)$$

or

$$V = \frac{1}{2\sqrt{X}}\left[\frac{Y}{\sqrt{X}}\ f'\left(\frac{Y}{\sqrt{X}}\right) - f\left(\frac{Y}{\sqrt{X}}\right)\right]$$

These results are more neatly expressed in terms of the variable η, defined by

$$\eta = \frac{Y}{\sqrt{X}}$$

Then

$$\begin{aligned}
\Psi &= \sqrt{X}\ f(\eta) \\
U &= f'(\eta) \\
V &= \frac{1}{2\sqrt{X}}\left[\eta f'(\eta) - f(\eta)\right]
\end{aligned}$$

This expresses the solution to our problem in terms of the dimensionless variables X, Y, U, V, and Ψ, and an unknown function f of a single variable.

In terms of the original dimensioned variables, η becomes

$$\eta = \frac{Y}{\sqrt{X}} = \frac{u_0 y/\nu}{\sqrt{u_0 x/\nu}} = \sqrt{\frac{u_0}{\nu}}\frac{y}{\sqrt{x}}$$

and the solution is given

$$\begin{aligned}
\psi &= \sqrt{u_0 \nu x}\ f\left(\sqrt{\frac{u_0}{\nu}}\frac{y}{\sqrt{x}}\right) \\
u &= u_0 f'\left(\sqrt{\frac{u_0}{\nu}}\frac{y}{\sqrt{x}}\right) \\
v &= \frac{1}{2}\sqrt{\frac{u_0 \nu}{x}}\left[\sqrt{\frac{u_0}{\nu}}\frac{y}{\sqrt{x}}\ f'\left(\sqrt{\frac{u_0}{\nu}}\frac{y}{\sqrt{x}}\right) - f\left(\sqrt{\frac{u_0}{\nu}}\frac{y}{\sqrt{x}}\right)\right]
\end{aligned}$$

or, in terms of η,

$$\begin{aligned}
\psi &= \sqrt{u_0 \nu x}\ f(\eta) \\
u &= u_0 f'(\eta) \\
v &= \frac{1}{2}\sqrt{\frac{u_0 \nu}{x}}\left[\eta f'(\eta) - f(\eta)\right]
\end{aligned}$$

Reduction of the governing equations

We now use the governing equations to deduce the ordinary differential equation that is satisfied by the function f. Equation (11.4) is

$$U\ \frac{\partial U}{\partial X} + V\ \frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}$$

Using the results obtained above, this gives

$$f'(\eta)\ \frac{\partial}{\partial X}\left[f'(\eta)\right] + \frac{1}{2\sqrt{X}}\left[\eta f'(\eta) - f(\eta)\right]\frac{\partial}{\partial Y}\left[f'(\eta)\right] = \frac{\partial^2}{\partial Y^2}\left[f'(\eta)\right]$$

or

$$f'(\eta)\ f''(\eta)\ \frac{\partial \eta}{\partial X} + \frac{1}{2\sqrt{X}}\left[\eta f'(\eta) - f(\eta)\right] f''(\eta)\ \frac{\partial \eta}{\partial Y} = f'''(\eta)\left(\frac{\partial \eta}{\partial Y}\right)^2 + f''(\eta)\underbrace{\frac{\partial^2 \eta}{\partial Y^2}}_{=0}$$

which gives

$$-\frac{Y}{2X^{3/2}}\ f'(\eta)\ f''(\eta) + \frac{1}{2X}\left[\eta f'(\eta) - f(\eta)\right] f''(\eta) = \frac{1}{X}\ f'''(\eta)$$

and hence

$$-\underbrace{\frac{Y}{X^{1/2}}}_{=\eta}\ f'(\eta)\ f''(\eta) + \left[\eta f'(\eta) - f(\eta)\right] f''(\eta) = 2f'''(\eta)$$

leaving, finally,

$$2f'''(\eta) + f(\eta)f''(\eta) = 0 \tag{11.8}$$

This is the ordinary differential equation satisfied by the unknown function f. The boundary conditions, expressed in terms of f, become

BC1. The non-slip condition: $U = V = 0$ when $Y = 0$ means that, when $\eta = Y/\sqrt{X} = 0$,

$$U = f'(\eta) = 0$$
$$\text{and}\ \ V = \frac{1}{2\sqrt{X}}\left[\eta f'(\eta) - f(\eta)\right] = -\frac{1}{2\sqrt{X}} f(\eta) = 0$$

Hence,

$$f'(0) = 0 \ \ \text{and} \ \ f(0) = 0$$

BC2. The asymptotic steady flow condition: $U = 1$ when $Y = \infty$ means that when $\eta = Y/\sqrt{X} = \infty$, we have

$$U = f'(\eta) = 1$$

Hence,

$$f'(\infty) = 1$$

Solution

There is no known analytic solution for the equation

$$2f'''(\eta) + f(\eta)f''(\eta) = 0$$

It must therefore be solved numerically.

*** Complete ******

12 Free Convection

In free convection, the cause of motion is the action of the gravitational field on density variations in a fluid caused by changes of temperature. Cold fluid is denser than hot fluid, and so it sinks. Similarly, hotter fluid is less dense and so rises.

12.1 Convection in Horizontal Layers

Consider a fluid of depth d contained between two horizontal plates. The plates are kept at different temperatures, with the temperature of each plate held constant. Denote the temperature of the upper plate by T_U and that of the lower one by T_L. The fluid is initially at rest.

No interesting motion arises when $T_U \geq T_L$. The high temperature plate will cause heat to be conducted into the layer of fluid immediately in contact with it, from where it will be conducted down wards through successive layers until it reaches the cold plate. From there, it is conducted out through the cold plate. In this configuration, the warmer, less dense fluid resides above the colder, more dense fluid, so the fluid layers are each in stable equilibrium, and the fluid remains at rest.[1] A steady state temperature distribution will establish itself in the fluid in which the temperature changes continuously with depth. When the steady state has been reached, heat will be conducted through the fluid at a rate given by

$$\frac{Q}{t} = kA\,\frac{T_L - T_U}{d} \tag{12.1}$$

If, however, the lower plate is held at a higher temperature than the upper plate, then an interesting situation results. The hot plate at the base of the fluid supplies heat to the layer of fluid in contact with it. If the adjacent fluid is not able to conduct away the heat supplied to it at a rate equal to or greater than that at which the heat is supplied, then the temperature of that layer rises and the fluid expands. Its density will therefore drop to a value lower than that of the fluid immediately above it. If the fluid remains in equilibrium, then the equilibrium is no longer stable, as it was before, but unstable. In this condition,

[1] This assumes that the fluid density does not change anomalously with temperature.

the slightest inhomogeneity in the fluid density, or the slightest disturbance of the fluid equilibrium, will set the fluid into motion. Once the motion has begun, a current is established in which the hot fluid rises to the top of the fluid column, where it is in contact with the colder plate and so it yields up its heat to it and cools, while the cooler fluid from the top falls to the bottom plate where it is heated and the cycle is complete. The circulatory motion that results in this situation is commonly called *Bénard convection*.[2]

Onset of Bènard Convection

A fluid at rest in the Bènard configuration is in unstable equilibrium: heavy cold fluid is at rest above the lighter hot fluid. If the cold fluid moves downward and hot fluid moves up, potential energy is released which could then be available for conversion to kinetic energy. If the fluid is viscous, some of this potential energy may be converted into internal energy of the fluid. The instability of the equilibrium is thus responsible for Bènard convection.

Under what conditions can we expect the flow to occur? We clearly need $T_L > T_U$, so that the heavier fluid lies above the lighter one. However, this is not sufficient. The flow is inhibited by the action of viscosity, which is a form of friction and thus helps to maintain the equilibrium, though unstable. The onset of motion is also opposed by the action of thermal conductivity: heat conduction in the fluid tends to remove heat from the lighter underlying fluid and so inhibits its further expansion and associated fall in density. Convection only occurs when the destabilising action of temperature difference is strong enough to overcome the combined action of viscosity and thermal conductivity.

A good measure of when the unstable equilibrium breaks down is provided by the Rayleigh number,

$$\mathrm{Ra} = \frac{g\alpha(T_U - T_L)d^3}{\nu k} \tag{12.2}$$

where α is the coefficient of thermal expansion of the fluid, κ its coefficient of thermal conductivity, and ν its kinematic viscosity. Note that the factors that drive the convection are in the numerator, while those that oppose it appear in the denominator. So the larger Ra, the more likely it is that convection will occur.

In general, convection appears to occur when Ra exceeds a critical value of about 1700. Below this,. the fluid remains at rest, above it, the convection sets in. The resultant motion consists of rising hot currents accompanied by falling cold currents, with the fluid moving horizontally at the top and bottom of the

[2] Tritton, p 30, states that, “Historically, the name is inaccurate; Bénard’s pioneering observations, although for long believed to relate to this configuration, were actually mostly of another phenomenon ... that gives rise to similar effects. However, the name is so well established that its use in this way causes no confusion.” Tritton discusses Bénard’s original configuration in his Section 17.4.

fluid layer, where heat exchange with the plates takes place. The system is essentially an heat engine with the fluid as the working substance. Potential energy is supplied by the heating at the bottom of the fluid and the cooling at the top.

The onset of convection is most easily detected by the increased rate of heat transfer from the lower plate to the upper one. In the absence of convection, the heat transfer takes place by conduction. Radiative transfer through the fluid is generally negligible in fluids, but not in gases where it must be taken into account. Thus if we measure the actual heat transfer rate and compare it with what this rate ought to be if due to conduction alone, the presence of convection raises this ratio to a value greater than 1. The ratio is called the *Nusselt number*, and is given by

$$\mathrm{Nu} = \frac{(Q/t)_{\mathrm{actual}}}{(Q/t)_{\mathrm{conduction}}} = \frac{d\ (Q/t)_{\mathrm{actual}}}{\kappa(T_U - T_L)} \tag{12.3}$$

A typical plot of Ra vs. Nu is shown in Figure 1 p 32 Tritton.

12.2 Convective Flux

Consider an element of fluid in a convective current. This is probably what texts mean by a “bubble”. The element begins its upward motion at the base of the convection layer and rises rapidly through the fluid until it disperses. During this motion, it moves too rapidly for heat to be conducted out of it. It thus retains all of its internal energy until it reaches the top of its convective motion and disperses. At the top of its motion, it rams into colder stationary fluid where it slows and disperses. It is now able to give up its heat to its surroundings by conduction. It thus cools to ambient temperature.

Denote the temperature difference between the convecting element and its surroundings in the dispersion layer by δT. Denote also the specific heat per unit mass of the convecting material by c_p.[3] Then the total heat lost to the surroundings by the element is

$$\delta Q = \delta m\ c_p\ \delta T \tag{12.4}$$

Now, in time δt, the total amount of mass flowing into the dispersion layer is

$$\delta m = \rho\ \delta V = \rho\ \delta A\ (v_c\ \delta t) \tag{12.5}$$

where v_c is the average speed of the convection current immediately before ramming into the dispersion layer, and δA is the cross-sectional area of the current. Thus

$$\delta Q = \rho\ v_c\ c_p\ \delta T\ \delta A\ \delta t \tag{12.6}$$

[3] We use c_p here because, once stationary, the element is at the same pressure as the dispersion layer. In a static star, the pressure of this layer is constant, so that the conduction process takes place at constant pressure.

The convective flux is the rate of heat flow per unit area into the dispersion layer, and so is given by

$$F_{\text{conv}} = \frac{\delta Q}{\delta A\ \delta t} = \rho\ v_c\ c_p\ \delta T \tag{12.7}$$

Note that ρv_c is the magnitude of the mass current-density of the convecting fluid. This provides us with an alternative derivation of this result. The rate of mass flow across a surface S is given by

$$\frac{dm}{dt} = \int_S \vec{J}_m \cdot d\vec{S} \tag{12.8}$$

In the case of our convective current, we take the surface S to be a sphere at the boundary of the dispersion layer. Since the convective flow is perpendicular to this surface, and taking $\vec{J}$ to be the average convective current across it, we have

$$\frac{dm}{dt} = J_m\ A = \rho v_c A \tag{12.9}$$

so that the rate of heat flux into the dispersion layer through the surface S is

$$\dot{Q} = \rho v_c A\ c_p\ \delta T \tag{12.10}$$

and hence the flux into the dispersion layer is

$$F_{\text{conv}} = \frac{\dot{Q}}{A} = \rho v_c c_p\ \delta T \tag{12.11}$$

as before.

To use this formula, we need to calculate first the difference of temperature δT between the convected fluid and its surroundings in the dispersion layer, and also the average speed of convection, v_c.

The Temperature Difference δT:

The Average Speed of Convection v_c:

12.3 Free Convection from a Vertical Plate

The system considered in this section consists of a semi-infinite vertical plate, held at fixed temperature T_1, surrounded by fluid, originally at temperature T_0, where $T_0 < T_1$. The fluid next to the plate will be heated by conduction in the boundary layer. When the temperature gradient exceeds the adiabatic gradient, convection currents will be set up and the heat from the plate will be carried away by these currents. The fluid that is sufficiently distant from the plate will remain at temperature T_0.

Choose coordinates with the x-axis vertically upwards, and origin at the base of the plate, with z-axis running along the bottom edge of the plate. Thus the coordinate y is measure in the direction perpendicular to the plate.

In this problem, we are interested only in the steady state that is set up after a sufficient length of time has passed. Since the fluid is initially at temperature

T_0 and since heating can only elevate the temperature, it is convenient to work with the temperature difference $T - T_0$ rather than directly with the temperature itself. Furthermore, since the temperature of the plate is held at the fixed value T_1 and the fluid distant from the plate remains at temperature T_0, the system has a well defined fixed reference temperature difference $T_1 - T_0$. We may therefore define a dimensionless temperature difference, denoted by Θ, and given by

$$\Theta = \frac{T - T_0}{T_1 - T_0}$$

This variable will therefore satisfy the boundary conditions

$$\Theta = 1 \text{ at } y = 0, \quad \text{and} \quad \Theta = 0 \text{ at } y = \infty$$

Since both T_1 and T_1 are constant, all derivatives of Θ will differ from the corresponding derivatives of T by a constant scalar factor. If the governing equations are linear in T, the same governing equations will hold for Θ.

Assumptions and governing equations

For the purpose of this calculation, we shall accept the following simplifying assumptions. Firstly, we assume that the flow has translational symmetry along the z-axis. The means that no fluid parameter or variable has any z dependence. The problem is thus essentially two dimensional.

Secondly, we adopt the Boussinesq approximation. In this approximation, we assume that the changes of density of the fluid are so small that they may be neglected everywhere except in the gravitational term of the equation of motion where, though small, they drive the convective motion of the fluid. We thus neglect the change in density in all terms where it is used to convert densities peer unit mass into densities per unit volume, except the gravitational. In these, we use the value ρ_0, which is the density of the fluid at the reference temperature T_0. Thus, we use the value ρ_0 for ρ in the acceleration term to relate it to the total force per unit volume on a fluid element, in the internal energy term where we need to convert specific heat per unit mass to specific heat per unit volume, and in the mass conservation equation. The rationale underlying this approximation is that the effect of the slight change of density experienced by the fluid on heating is negligible when compared with the corresponding total quantities involved. For example, in the energy equation, the total heat influx into a fluid element is so huge that difference in the heat influx per unit volume calculated from ρ_0 as opposed to that calculated from ρ is utterly insignificant in comparison with its value. Similarly, in the mass conservation equation, the time rate of change of density per unit density experience by the fluid as a result of heating is so small that we can regard the flow as incompressible.

The Boussinesq assumptions, together with the assumption of translational symmetry in the z-direction, allow us to write the mass conservation equation as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and the Navier-Stokes equation, using the fact that $\nabla \cdot \vec{v} = 0$ and the fact that we are interested only in the steady state, as

$$\rho_0 \vec{v} \cdot \nabla \vec{v} = -\nabla p - \rho g \boldsymbol{i} + \mu \nabla^2 \vec{v}$$

We now make some further assumptions to simplify this equation.

In the absence of the plate, the fluid would be in equilibrium at temperature T_0 in the gravitational field. Its equation of motion would therefore be

$$0 = -\nabla p - \rho_0 g \boldsymbol{i}$$

In general, this equation would need to be solved in conjunction with the equation of state for the fluid to find p and ρ as functions of x. We shall assume that the vertical extent of the fluid is not sufficiently large to lead to any appreciable change of density as we rise or fall in the fluid. That is, we assume that ρ_0 is independent of x. We also assume that the fluid is homogeneous, so that ρ_0 is also independent of y and z, and so has a constant value.

Fourthly, we assume that the convection currents that are set up in the fluid by the hot plate are not sufficiently rapid for the static pressure profile to be changed significantly. We thus assume that the above equation for ∇p continues to be satisfied in the convecting fluid.

Sixthly, we assume that the temperature changes that occur in the fluid as a result of the heating are sufficiently small for the density of the fluid to be well approximated as a linear function of the temperature change. In general, we would have

$$\rho = \rho(p, T)$$

For small changes in p and T, this can be written approximately as

$$\rho = \rho(p_0, T_0) + \frac{\partial \rho}{\partial p}(p_0, T_0)\ (p - p_0) + \frac{\partial \rho}{\partial T}(p_0, T_0)\ (T - T_0) + \cdots$$

We have already assumed that, at constant temperature, the change in density as p changes is negligible, so the first term on the right hand side can be omitted. This equation thus be written as

$$\rho = \rho_0 - \rho_0 \alpha (T - T_0)$$

where we have put $\rho_0 = \rho(p_0, T_0)$ and

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

is the expansivity of the fluid. In theory, $\alpha = \alpha(p, T)$. In practice, α is constant for liquids over very wide ranges of p and T. For ideal gases, $\alpha = 1/T$. For our purposes, and in the spirit of the Boussinesq approximation which neglects all changes due to temperature variation except in the gravitational term, we can take $\alpha \approx 1/T_0$. This assumes that the heat is carried away by convection currents sufficiently rapidly for us to assume that $T_0 + \delta T \approx T_0$. The gravitational term in the Navier-Stokes equation thus becomes

$$-\rho g \boldsymbol{i} = -\rho_0 g \boldsymbol{i} + \rho_0 g \alpha (T - T_0) \boldsymbol{i}$$

Seventhly, we add the boundary layer approximation in which we assume that conditions along the boundary layer in the direction of flow do not change substantially. Thus, in the viscous terms, we have

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}$$

With these many assumptions, the Navier-Stokes equation becomes

$$\rho_0 \vec{v} \cdot \nabla \vec{v} = -\nabla p - \rho_0 g \boldsymbol{i} + \rho_0 g \alpha (T - T_0) \boldsymbol{i} + \mu \nabla^2 \vec{v} = \rho_0 g \alpha (T - T_0) \boldsymbol{i} + \mu \nabla^2 \vec{v}$$

The x component of this vector equation then becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \alpha (T - T_0) + \nu \frac{\partial^2 u}{\partial y^2}$$

where $\nu = \mu/\rho_0$. In terms of the dimensionless temperature difference Θ, this becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \alpha (T_1 - T_0) \Theta + \nu \frac{\partial^2 u}{\partial y^2}$$

The above considerations have supplied two equations for three fluid variables u, v, Θ. We need one more equation for the system to be determined. The remaining two Navier-Stokes equations do not contain Θ and so are not likely to be of any use. (However, their consistency with the above equations should nevertheless be checked, which I don't do in these notes. I simply assume that the equations, after application of the Boussinesq approximation, are consistent.) We therefore consider the energy equation.

The energy equation can be written in the form,

$$\rho c_v \frac{DT}{Dt} = p \ \nabla \cdot \vec{v} + \nabla \cdot (k \nabla T) + \Phi$$

where c_v is the specific heat at constant volume, k the thermal conductivity of the fluid, and Φ is the dissipation function for the viscous fluid. Using the Boussinesq assumptions, we take $\rho = \rho_0$ and regard k and c_v as constants, and $\nabla \cdot \vec{v} = 0$. So, for a steady state, two dimensional flow, we get

$$\rho_0 c_v u \left(\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \nabla^2 T + \Phi$$

We now introduce our eighth and last assumption: the amount of kinetic and potential energy lost from the fluid and converted into internal energy by viscous dissipation in the fluid is negligible when compared with the total amount of energy conducted into the fluid as heat in the moving layers adjacent to the hot plate. We thus neglect the term Φ in comparison with the heat conduction term $k\nabla^2 T$. The equation can therefore be written as

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \kappa \nabla^2 T$$

where $\kappa = k/\rho_0 c_v$ is the thermal diffusivity of the fluid. In terms of Θ, this equation becomes

$$u\,\frac{\partial \Theta}{\partial x} + v\,\frac{\partial \Theta}{\partial y} = \kappa \nabla^2 \Theta$$

Its form remains the same as that for T because the original was linear in T. Finally, the derivative on the right hand side may be simplified by the boundary layer assumption that rates of change along the boundary are negligible when compared to those in the direction perpendicular to the boundary. Noting further that this is a two dimensional problem with no dependence on z, we get

$$\nabla^2 \Theta = \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} \approx \frac{\partial^2 \Theta}{\partial y^2}$$

so that

$$u\,\frac{\partial \Theta}{\partial x} + v\,\frac{\partial \Theta}{\partial y} = \kappa\,\frac{\partial^2 \Theta}{\partial y^2}$$

This gives, as the governing equations for this problem, the three partial differential equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{12.12}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = g\alpha(T_1 - T_0)\Theta + \nu\,\frac{\partial^2 u}{\partial y^2} \tag{12.13}$$

$$u\,\frac{\partial \Theta}{\partial x} + v\,\frac{\partial \Theta}{\partial y} = \kappa\,\frac{\partial^2 \Theta}{\partial y^2} \tag{12.14}$$

The boundary conditions for these equations are,

BC1. Non-slip conditions at the hot plate: $u = v = 0$ when $y = 0$.
BC2. Flow unaffected at large distances from the plate: $u = v = 0$ at $y = \infty$.
BC3. Conditions on the fluid temperature: $\Theta = 1$ when $y = 0$ and $\Theta = 0$ when $y = \infty$.

Dimensional analysis of governing equations

The form of the mass conservation equation (12.12) allows us to look for a solution in terms of a stream function ψ in terms of which the velocities can be calculated using the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

We are thus searching for two solution functions ψ and Θ of the governing parameters. The governing parameters for this problem, by inspection of the governing equations are the independent variables x and y, and the system parameters $\beta = g\alpha(T_1 - T_0)$ which determines the fluid buoyancy, the kinematic viscosity ν, and the thermal diffusivity κ of the fluid. We therefore expect the solution functions of this problem to be

$$\psi = \psi(x, y, \beta, \nu, \kappa), \quad \text{and} \quad \Theta = \Theta(x, y, \beta, \nu, \kappa)$$

The dimensions of the governing parameters are given by

$$[x] = [y] = L, \quad [\beta] = LT^{-2}, \quad [\nu] = L^2T^{-1}, \quad [\kappa] = L^2T^{-1}$$

This is an LT system, and so there can be only two dimensionally independent quantities among the governing parameters. We shall choose $\beta = g\alpha(T_1 - T_0)$ and ν as the dimensionally independent parameters and use these to define dimensionless versions of the system parameters. Since ν and κ have the same dimensions, it is clear that the ratio

$$K = \frac{\kappa}{\nu}$$

is dimensionless. The non-dimensional version of x, y can be found by noting that

$$L = [x] = [\beta]^a[\nu]^b = (L^aT^{-2a})(L^{2b}T^{-1b}) = L^{a+2b}T^{-(2a+b)}$$

so that

$$1 = a + 2b \quad \text{and} \quad 0 = 2a + b$$

which solves to $a = -1/3$ and $b = 2/3$. Thus x and y have the same dimensions as $\nu^{2/3}/\beta^{1/3}$. We define the dimensionless coordinates X, Y by

$$X = \frac{\beta^{1/3}}{\nu^{2/3}}\, x = \frac{[g\alpha(T_1 - T_0)]^{1/3}}{\nu^{2/3}}\, x$$

$$Y = \frac{\beta^{1/3}}{\nu^{2/3}}\, y = \frac{[g\alpha(T_1 - T_0)]^{1/3}}{\nu^{2/3}}\, y$$

From the equations relating u, v to ψ, we see that

$$[\psi] = L^2T^{-1} = [\nu]$$

so that the ratio

$$\Psi = \frac{\psi}{\nu}$$

is dimensionless. Since Θ is already dimensionless, there is nothing more to consider. By Buckingham's Π-theorem, we expect the solutions of the governing equations to have the form

$$\Psi = F(X, Y, K)$$
$$\Theta = G(X, Y, K)$$

We can calculate the velocities u and v from ψ according to

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial Y}(\nu \Psi)\ \frac{\partial Y}{\partial y} = \nu\ \frac{\beta^{1/3}}{\nu^{2/3}} \frac{\partial \Psi}{\partial Y} = (\nu\beta)^{1/3} \frac{\partial \Psi}{\partial Y}$$

Since the derivative on the right hand side is dimensionless, the factor $(\nu\beta)^{1/3}$ must have the dimensions of velocity. We can thus define the dimensionless velocities U and V to be,

$$U = (\nu\beta)^{-1/3} u = [\nu g \alpha (T_1 - T_0)]^{-1/3} u$$
$$V = (\nu\beta)^{-1/3} v = [\nu g \alpha (T_1 - T_0)]^{-1/3} v$$

which, according to the derivation above, are related to Ψ according to

$$U = \frac{\partial \Psi}{\partial Y} \quad \text{and} \quad V = -\frac{\partial \Psi}{\partial X}$$

We now find the dimensionless form of the governing equations. Since the derivative in each term of the mass conservation equation has the same dimensions, the equation can be brought into dimensionless form by multiplication with a single common factor. So it becomes

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

The x-component of the Navier-Stokes equation becomes

$$(\nu\beta)^{1/3} U\ \frac{\partial}{\partial X}[(\nu\beta)^{1/3} U] \frac{\partial X}{\partial x} + (\nu\beta)^{1/3} V\ \frac{\partial}{\partial Y}[(\nu\beta)^{1/3} U] \frac{\partial Y}{\partial y}$$
$$= \beta\Theta + \nu\ \frac{\partial^2}{\partial Y^2}[(\nu\beta)^{1/3} U] \left(\frac{\partial Y}{\partial y}\right)^2$$

But

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = \frac{\beta^{1/3}}{\nu^{2/3}}$$

so

$$(\nu\beta)^{2/3} \frac{\beta^{1/3}}{\nu^{2/3}} \left(U\ \frac{\partial U}{\partial X} + V\ \frac{\partial U}{\partial Y} \right) = \beta\Theta + \nu (\nu\beta)^{1/3}\ \frac{\beta^{2/3}}{\nu^{4/3}} \frac{\partial^2 U}{\partial Y^2}$$

or

$$U\ \frac{\partial U}{\partial X} + V\ \frac{\partial U}{\partial Y} = \Theta + \frac{\partial^2 U}{\partial Y^2}$$

Finally, the energy equation becomes

$$(\nu\beta)^{1/3}U\ \frac{\partial\Theta}{\partial X}\frac{\partial X}{\partial x} + (\nu\beta)^{1/3}V\ \frac{\partial\Theta}{\partial Y}\frac{\partial Y}{\partial y} = \kappa\ \frac{\partial^2\Theta}{\partial y^2}\left(\frac{\partial Y}{\partial y}\right)^2$$

and hence, using the definition $K = \kappa/nu$, we get

$$U\ \frac{\partial\Theta}{\partial X} + V\ \frac{\partial\Theta}{\partial Y} = K\ \frac{\partial^2\Theta}{\partial y^2}$$

The governing equations, in dimensionless form, are thus

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{12.15}$$

$$U\ \frac{\partial U}{\partial X} + V\ \frac{\partial U}{\partial Y} = \Theta + \frac{\partial^2 U}{\partial Y^2} \tag{12.16}$$

$$U\ \frac{\partial\Theta}{\partial X} + V\ \frac{\partial\Theta}{\partial Y} = K\ \frac{\partial^2\Theta}{\partial Y^2} \tag{12.17}$$

In dimensionless form, the boundary conditions for these equations are,

BC1. Non-slip conditions at the hot plate: $U = V = 0$ when $Y = 0$.
BC2. Flow unaffected at large distances from the plate: $U = V = 0$ at $Y = \infty$.
BC3. Conditions on the fluid temperature: $\Theta = 1$ when $Y = 0$ and $\Theta = 0$ when $Y = \infty$.

Scaling transformation

We now look for a scaling transformation of the dimensionless variables X, Y, Ψ, Θ that leave the governing equations and the boundary condition unchanged. Suppose

$$X^* = \alpha X, \quad Y^* = \beta Y, \quad \Psi^* = \gamma\Psi \quad \Theta^* = \lambda\Theta$$

Now,

$$U^* = \frac{\partial\Psi^*}{\partial Y^*} = \frac{\gamma}{\beta}\ \frac{\partial\Psi}{\partial Y} = \frac{\gamma}{\beta}\ U$$

and

$$V^* = -\frac{\partial\Psi^*}{\partial X^*} = -\frac{\gamma}{\alpha}\ \frac{\partial\Psi}{\partial X} = -\frac{\gamma}{\alpha}\ V$$

so if we demand that equation (12.16) holds for the scaled variables, we have

$$U^*\ \frac{\partial U^*}{\partial X^*} + V^*\ \frac{\partial U^*}{\partial Y^*} = \Theta^* + \frac{\partial^2 U^*}{\partial {Y^*}^2}$$

which gives

$$\frac{\gamma^2}{\beta^2\alpha}\left(U\,\frac{\partial U}{\partial X}+V\,\frac{\partial U}{\partial Y}\right)=\lambda\Theta+\frac{\gamma}{\beta^3}\frac{\partial^2 U}{\partial Y^2}$$

and hence

$$U\,\frac{\partial U}{\partial X}+V\,\frac{\partial U}{\partial Y}=\frac{\beta^2\alpha}{\gamma^2}\,\lambda\Theta+\frac{\alpha}{\gamma\beta}\,\frac{\partial^2 U}{\partial Y^2}$$

This equation does not coincide with that satisfied by the unscaled variables unless we have

$$\frac{\beta^2\alpha}{\gamma^2}\,\lambda=1\quad\text{and}\quad\frac{\alpha}{\gamma\beta}=1$$

that is,

$$\alpha=\gamma\beta\quad\text{and}\quad\lambda=\frac{\gamma}{\beta^3}$$

Furthermore, demanding that equation (12.17) holds for the scaled variables gives

$$U^*\,\frac{\partial\Theta^*}{\partial X^*}+V^*\,\frac{\partial\Theta^*}{\partial Y^*}=K\,\frac{\partial^2\Theta^*}{\partial {Y^*}^2}$$

so that

$$\frac{\gamma\lambda}{\alpha\beta}\left(U\,\frac{\partial\Theta}{\partial X}+V\,\frac{\partial\Theta}{\partial Y}\right)=K\,\frac{\lambda}{\beta^2}\frac{\partial^2\Theta}{\partial Y^2}$$

This equation coincides with that satisfied by the unscaled variables only if

$$\beta\gamma=\alpha$$

which coincides with one of the equations previously obtained.

We must now consider the boundary conditions for the scaled variables. By inspection of these conditions, we see that the only boundary condition that changes on account of the scaling is that for Θ when $Y=0$. If we demand that

$$\Theta^*=1\quad\text{when}\quad Y=0$$

this is equivalent to demanding

$$\Theta=\frac{1}{\lambda}\quad\text{when}\quad Y=0$$

which coincides with the boundary condition for Θ only if $\lambda=1$. We thus have three conditions on the four parameters $\alpha,\beta,\gamma,\lambda$, leaving us with a one parameter family of scaling transformations. These are determined by the relations

$$\alpha=\gamma\beta\quad\text{and}\quad\gamma=\beta^3$$

leading to the transformations

$$X^*=\beta^4 X,\quad Y^*=\beta Y,\quad \Psi^*=\beta^3\Psi\quad \Theta^*=\Theta$$

For each value of β, the variables $X^*, Y^*, \Psi^*, \Theta^*$ satisfy the governing equations with the stated boundary conditions. Since the solution to these equations is unique, we must have

$$\Psi^* = F(X^*, Y^*, K)$$
$$\Theta^* = G(X^*, Y^*, K)$$

and hence

$$\Psi = \frac{1}{\beta^3}\ F(\beta^4 X, \beta Y, K)$$
$$\Theta = G(\beta^4 X, \beta Y, K)$$

Since the solution to the governing equations with the given boundary conditions are unique, this means that we must have for all values of β

$$F(X, Y, K) = \frac{1}{\beta^3}\ F(\beta^4 X, \beta Y, K)$$
$$G(X, Y, K) = G(\beta^4 X, \beta Y, K)$$

This means that X and Y do not appear in these functions independently, but in such a combination that the parameter β cancels from the combination. We shall now discover that combination. Choose the value of β in such a way that the first argument in each function becomes 1. That is, choose β such that

$$\beta^4 X = 1$$

We then have

$$F(X, Y, K) = X^{3/4}\ F\left(1, \frac{Y}{X^{1/4}}, K\right)$$
$$G(X, Y, K) = G\left(1, \frac{Y}{X^{1/4}}, K\right)$$

Define the functions f and g by

$$f(\eta, K) = F\,(1, \eta, K)$$
$$g(\eta, K) = G\,(1, \eta, K)$$

The dimensionless solution can therefore be written in the form

$$\Psi = X^{3/4}\ f\left(\frac{Y}{X^{1/4}}, K\right)$$
$$\Theta = g\left(\frac{Y}{X^{1/4}}, K\right)$$

where f and g are unknown functions, still to be determined, of the single variable $\eta = Y/X^{1/4}$ and of one parameter K.

Expressed in terms of the original dimensioned variables, we have

$$\eta = \frac{Y}{X^{1/4}} = \left[\frac{g\alpha(T_1 - T_0)}{\nu^2}\right]^{1/4}\ \frac{y}{x^{1/4}}$$

and hence

$$\psi = \left[\nu^2 g\alpha(T_1 - T_0)\right]^{1/4} x^{3/4} \; f\left(\left[\frac{g\alpha(T_1 - T_0)}{\nu^2}\right]^{1/4} \; \frac{y}{x^{1/4}}, K\right)$$

$$T = T_0 + (T_1 - T_0) g\left(\left[\frac{g\alpha(T_1 - T_0)}{\nu^2}\right]^{1/4} \; \frac{y}{x^{1/4}}, K\right)$$

13 Reynolds' Number

Consider a fluid that is incompressible, subject to no body forces, and moves under steady imposed conditions. The equations for such a fluid are

$$\nabla \cdot \vec{v} = 0 \tag{13.1}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\ \nabla p + \nu \nabla^2 \vec{v} \tag{13.2}$$

13.1 Dimensionless Equations

Physics deals with variables that have physical dimensions. On the other hand, mathematics deals only with pure numbers. Physical dimensions are irrelevant to mathematics. Indeed, substitution of dimensioned quantities into mathematical functions results in outputs that contain, in general, mixtures of dimensions and are therefore physically absurd. The remedy for this is the removal of all physical dimensions before substitution into mathematical formulae. This is accomplished by defining a new variable that has no dimensions obtained by dividing the dimensioned variable by a standard reference constant with the same dimensions. This, in effect, means that we are measuring the dimensioned variable in units of the reference constant.

To convert the equations of fluid flow to dimensionless form, we need to identify natural scales for the problem considered from which all other scales of interest in the problem can be constructed. This takes some skill, as well as a thorough understanding of the physics of the problem considered.

A fluid system described by equations (13.1) and (13.2) above may have boundary conditions imposed on it that define in a natural way a length scale for the flow. Denote the length scale for the problem by L. The problem will also involve some typical fluid speeds that define a velocity scale. Denote a typical speed for the fluid of the system by U.

The choice of a particular value of L and of U is arbitrary. Our only requirement is that they reflect typical scales for the problem considered. L and U are therefore nothing more than typical measures of the size of the apparatus containing the fluid or of obstacles in the path of the fluid, and of the rate at which the fluid flows.

Note that we do not introduce density, pressure, or time scales. The reasons are as follows. Since the fluid is incompressible, the fluid has a constant density. The density therefore is a property of the fluid itself, and not a characteristic of the flow. There is therefore no density scale needed.

As regards the pressure, pressure differences in the fluid are determined by the flow. The are therefore not independent of the velocity field. We could begin with a pressure difference scale rather than a velocity scale. We would then determine the velocity scale from it. This approach would not change the conclusions. I would only change the form terms of which they are expressed.

Lastly, no time scale is introduced. We have restricted ourselves to considering flows that take place under steady imposed conditions. This does not mean that the flow will be steady. Unsteadiness can arise spontaneously in a fluid flow. But even when it does, it does not provide a time scale for the flow. The rapidity of the fluctuations is related to the length and velocity scales for the problem. Only if the imposed external conditions vary with a typical timescale will there be an independent time scale in the problem. The following analysis does not cover this case.

We now use the natural scales L and U to define dimensionless variables as follows. The most obvious dimensionless variables to define are

$$\xi^i = \frac{x^i}{L}, \quad u^i = \frac{v^i}{U}$$

The quantity L/U clearly defines a characteristic time for the flow. It is the time taken for the fluid to travel the characteristic length L. This enables us to define a dimensionless time by

$$\tau = \frac{t}{L/U} = \frac{Ut}{L}$$

To define a dimensionless pressure, first consider the units of pressure. They are

$$\frac{N}{m^2} = \frac{kg.m.s^{-2}}{m^2}$$

To introduce the velocity U, note that these units can be written in the form

$$\frac{kg.m^2.s^{-2}}{m^3} = \frac{kg}{m^3}\ m^2.s^{-2}$$

These are the units of density multiplied by those of velocity. Since the density of the fluid considered is constant, we can define a characteristic pressure π by setting

$$\pi = \rho U^2$$

14 Lagrangian Formulation

14.1 Alternative Formulation of Fluid Theory

historical

14.2 Fluid Trajectories

The Lagrangian description is based on the concept of fluid trajectories. The fluid continuum is imagined to be made up of a set of points, called the fluid points, that move through space in a manner governed by the fluid equations of motion. Consider the configuration of the fluid at a chosen reference time $t = 0$. At this time, the fluid points each occupy a position $\vec{a}$. As time advances, the fluid point originally at position $\vec{a}$ will occupy a new position at each time t. So, each fluid point has a trajectory through space. Since the trajectory depends on which point we are considering, we can express it in the form

$$x^i = x^i(\vec{a}, t) \tag{14.1}$$

The initial position $\vec{a}$ of each particle can thus be used as a permanent 'marker', or *intrinsic* or *Lagrangian* of *comoving coordinate*, to identify each fluid point.

Eulerian and Lagrangian descriptions of the fluid differ in which coordinates that are regarded as the independent variables of the theory. In the Eulerian description, the present position $\vec{x}$ of the fluid points are regarded as the independent variables. For this description, the Lagrangian coordinates $\vec{a}$ are eliminated in favour of $\vec{x}$ and all quantities are then expressed in terms of these as the independent variables. In the Lagrangian description, we regard the $\vec{a}$ as the independent variables, and treat the $\vec{x}$ as dependent variables that are to be eliminated in favour of the $\vec{a}$. The difference between Eulerian and Lagrangian theories therefore is not dynamical, but kinematical. The dynamical equations used in both are the same. The form of the kinematical description of the fluid flow, however, is different.

In the Lagrangian description, the velocity of the fluid particle is obtained by the standard Newtonian prescription: we differentiate the fluid point trajectory with respect to time. The parameters $\vec{a}$ are held constant in this differentiation, since we are interested in the velocity of a single fluid point. We will denote the velocity obtained in this way by $\dot{x}$. Thus,

$$\dot{x}^i = \frac{\partial x^i}{\partial t}(\vec{a}, t) = \dot{x}^i(\vec{a}, t) \tag{14.2}$$

This is the Lagrangian velocity field. The fluid velocity is given as a function of the Lagrangian coordinates $\vec{a}$ at each time t. Similarly, the fluid-point acceleration is given in the Lagrangian description by

$$\ddot{x}^i = \frac{\partial^2 x^i}{\partial t^2}(\vec{a}, t) = \ddot{x}^i(\vec{a}, t) \tag{14.3}$$

This too is given as a function of the Lagrangian coordinates $\vec{a}$ at each time t.

The partial derivatives of $x^i(\vec{a}, t)$ with respect to the a^α also have a physical meaning and importance that will become evident is later sections. For the moment, note that, by definition, we must have

$$x^i(\vec{a}, 0) = a^i \tag{14.4}$$

Differentiating this expression, we get

$$\frac{\partial x^i}{\partial a^\alpha}(\vec{a}, 0) = \delta^i_\alpha \tag{14.5}$$

Since the a^α and t are independent variables, it makes no difference if we first find the partial derivatives with respect to the a^α and then evaluate these at $t = 0$, of if we first evaluate the functions $x^i(\vec{a}, t)$ at $t = 0$, and then form the partial derivatives. Thus

$$\left[\frac{\partial x^i}{\partial a^\alpha}(\vec{a}, t)\right]_{t=0} = \frac{\partial}{\partial a^\alpha}\left[x^i(\vec{a}, 0)\right] = \delta^i_\alpha \tag{14.6}$$

There is no ambiguity therefore when using the above notation.

14.3 Relation to the Eulerian Description

The Lagrangian velocity field $\dot{x}(\vec{a}, t)$ is easily related to the Eulerian velocity field $\vec{v}(\vec{x}, t)$. In a continuum, each fluid point retains its identity. The trajectories of distinct fluid points cannot cross, nor can they break up into one or more branches at any time. There is therefore a one-to-one correspondence between the fluid points at any time t and ant the initial time $t = 0$. This means that, for fixed t, the equations

$$x^i = x^i(\vec{a}, t) \tag{14.7}$$

can be solved uniquely for the a^α in terms of the x^i. The time t plays the role of a parameter in this inversion. We thus have

$$a^\alpha = a^\alpha(\vec{x}, t) \tag{14.8}$$

By definition of inversion of equations, substitution of one into the other must yield an identity in the independent variables. Thus,

$$x^i = x^i(\vec{a}(\vec{x},t),t) \tag{14.9}$$

are identities in $\vec{x}$ and t, while

$$a^\alpha = a^\alpha(\vec{x}(\vec{a},t),t) \tag{14.10}$$

are identities in $\vec{a}$ and t.

To obtain the Eulerian velocity field from the Lagrangian one, we use the inverted equations to eliminate the $\vec{a}$ in favour of the $\vec{x}$. This gives,

$$\dot{x}^i(\vec{a}(\vec{x},t),t) = \frac{\partial x^i}{\partial t}(\vec{a}(\vec{x},t),t) = v^i(\vec{x},t) \tag{14.11}$$

We will always distinguish Eulerian and Lagrangian velocity fields by writing the former as $\vec{v}(\vec{x},t)$ and the latter as $\dot{\vec{x}}(\vec{a},t)$. Both fields describe the same physical fluid parameter. They differ only in the variables that they regard as independent.

This procedure can be reversed. We can obtain the Lagrangian velocity field from the Eulerian as follows. Given $\vec{v}(\vec{x},t)$, we have identically,

$$\frac{\partial x^i}{\partial t}(\vec{a},t) = v^i(\vec{x}(\vec{a},t),t) \tag{14.12}$$

The functions $x^i(\vec{a},t)$ are therefore those solutions of the partial differential equations

$$\frac{\partial x^i}{\partial t} = v^i(\vec{x},t) \tag{14.13}$$

that satisfy the initial conditions

$$x^i(\vec{a},0) = a^i \tag{14.14}$$

Once these solutions are obtained, they can be differentiated partially with respect to t to obtain the Lagrangian velocity field.

Appendix A Glossary of Terms

Adiabatic Flow: A flow is said to be *adiabatic* if there is no heat exchanged between its different parts. Since, in general, the different elements of the fluid will be at different temperatures, an adiabatic flow will in general only be possible if the thermal conductivity of the fluid is zero.

Since dissipation of energy can be regarded as equivalent to the internal generation of heat, some authors like Landau and Lifschitz, p 3, insist also on the flow also being non-dissipative. However, this does not appear to be necessary. Adiabatic can thus be taken to mean that there is no heat flow into or out of any given fluid element. If so, then the internal dissipation will increase the temperature of each fluid element, but the fact that the conductivity of the fluid is zero means that this cannot give rise to heat exchanges between the different parts of the fluid.

Baroclinic Flow: A *baroclinic flow* is one in which the *surfaces of constant pressure intersect those of constant density.* See *Barotropic Flow* for a detailed discussion. (Tritton, 1977, p 140-1.)

Barotropic Flow: A flow is said to be *barotropic* if the surfaces of constant pressure coincide with those of constant density (Shore, 1992, p 67-68). Flows in which this condition is not met are called *baroclinic* (Shore, 1992, p 87-89). This classification arises because certain equations (like the vorticity equation) contain terms with factor $\nabla\rho \times \nabla p$. One way of simplifying their solution is by assuming that

$$\nabla\rho \times \nabla p = 0$$

This is the *barotropic condition.* Its physical interpretation is as follows. $\nabla\rho$ is a vector field normal to the constant density surfaces, also called the level-surfaces for the density. Similarly, ∇p is a vector field normal to the surfaces of constant pressure. The barotropic condition means that these two normal fields have the same direction at all points. It can only be satisfied, therefore, if and only if the corresponding surfaces coincide everywhere. Violation of this condition means that, in general, the surfaces intersect.

A simple consequence of the barotropic condition is that both pressure and density are constant on the same surfaces at each time t. If we think of this common set of surfaces as a one parameter family, then the corresponding values of p and ρ constitute different parameterisations of one and the same one-parameter family. We may therefore regard p as a reparameterisation of the ρ-surfaces, or ρ as a reparameterisation of the p-surfaces. If this reparameterisation is the same for all time, then there is a relation of the form $p = p(\rho)$ or $\rho = \rho(p)$ between p and ρ in this flow. If the reparameterisation is different for different times, then this relationship becomes $p = p(\rho, t)$ or $\rho = \rho(p, t)$. This explains Yih's definition of a barotropic flow. He says that it is one for which a relationship of the form $F(\rho, p, t) = 0$ holds (Yih, p 61). In other words, ρ is a function of p alone (or, more generally, of p and t only). (Yih, p 63.)

The terms *barotropic* and *baroclinic* are descriptions of types of flows, and are not properties of fluids, in spite of the fact that some books speak of barotropic or baroclinic fluids. As flows, they are mutually exclusive. The general flow is baroclinic. A barotropic flow is a very special case and represents a "thin set" in the set of all possible flows. A *barotropic flow* is one in which the *surfaces of constant pressure coincide with those of constant density*. A *baroclinic flow* is one in which the *surfaces of constant pressure intersect those of constant density*. So, every flow that is not barotropic is baroclinic. (Tritton, 1977, p 140-1.)

The Greek word *baros* means *heaviness, weight* or *pressure*. *Trope* means *turn* or *convert*, but is sometimes used in science to refer to power-law dependence. Conjecture: since in many applications it is assumed that the density-pressure relation is of the form$\rho = p^\beta$, barotropic would mean that density is given by a pressure power-law. *Kline* means *angle*, so baroclinic means that the constant pressure surfaces are inclined to those of constant density.

Eckman Number: Tassoul (2000), p 30: The Eckman number is a dimensionless number that measures the relative importance of the viscous force to the Coriolis force. It is defined to be the ratio

$$\mathrm{Ek} = \frac{|\nu \nabla^2 \vec{u}|}{|\vec{\Omega} \times \vec{u}|} = \frac{\nu U / L^2}{\Omega U} = \frac{\nu}{\Omega L^2}$$

Homogeneous: *Homogeneous* means independent of position. Thus a solid or a fluid is said to be homogeneous if its density is the same at all points inside it. That is, its density ρ is independent of where you are inside the body. Similarly, any physical quantity or property is said to be *homogeneous* if its values at any given time do not depend the position in space where they are measured. For example, an homogeneous stress is one where the stress has the same value at all points of space.

Ideal Fluid, of Perfect Fluid: Some authors define an *ideal fluid*, also called a *perfect fluid*, to be one which obeys Euler's equations and the equation of continuity. However, this definition is incomplete. This is analogous to defining an ideal gas as one that obeys the ideal gas equation - this leaves the heat equation of the gas unspecified and admits very many different types of ideal gas. Similarly here. The requirement that the fluid obeys Euler's equation and the equation of continuity is incomplete, and leaves open many possibilities.

For Lamb (1945) does not appear to offer an explicit definition of an ideal fluid. The term seems to appear for the first time on p 17, where he refers to a *perfect fluid*. Implicitly, he assumes that a perfect fluid is one that obeys Euler's equation. He seems to require no further defining conditions.

Landau and Lifschitz (2003) define the ideal fluid as follows: p 3, "In deriving the equations of motion we have taken no account of the process of energy dissipation, which may occur in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it. ... motions of fluids in which thermal conductivity and viscosity are unimportant ... are said to be ideal." Euler's equation explicitly excludes fluid viscosity. However, it says nothing about the conduction of heat through the fluid. It is therefore not correct to state that an ideal fluid is one that obeys Euler's equation. We need to add also the condition that the heat conductivity of the fluid is zero. This last condition can be restated as follows: Landau and Lifschitz, p 3, "The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. The motion of an ideal fluid must necessarily be supposed adiabatic. In adiabatic motion the entropy of any particle of fluid remains constant as the particle moves about in space."

According to the Landau and Lifschitz, therefore, an *ideal fluid* is one that obeys Euler's equations, the equation of continuity, and in which the entropy of every fluid element remains constant during the flow, that is, the entropy per unit mass of the fluid is conserved. This last condition can be expressed in terms of a conservation law for entropy (Landau and Lifschitz, p 4),

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

Using the continuity equation, this can also be rewritten in the form of a continuity equation for entropy,

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s \vec{v}) = 0$$

where ρs is the entropy per unit volume of the fluid, and $\rho s \vec{v}$ is the entropy flux density.

Landau and Lifschitz (2003), p 4, "The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout the volume of the fluid at some initial instant, it retains everywhere that same constant value at all times and for any subsequent motion of the fluid. In this case we write the adiabatic equation simply as

$$s = \text{constant}$$

... Such a motion is said to be *isentropic*.

Incompressible Flow, Incompressible Fluid: A fluid, or a flow, is said to be incompressible if the volume of every infinitesimal element of the fluid remains constant as the element convects. Incompressibility thus means that the velocity field as the property

$$\nabla \cdot \vec{v} = 0$$

Note that this does *not* mean necessarily that the density ρ of the fluid is a constant. The equation of continuity becomes, for incompressible fluids,

$$0 = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = \frac{d\rho}{dt}$$

which means that the density of each given element of fluid remains constant as it convects. The densities of different fluid elements however need not be the same. The fluid may be "lumpy" or *inhomogeneous* in constitution, in which case its lumpiness persists in incompressible flow, with the density of each of its parts remaining fixed as they convect.

The case of *homogeneous* incompressible fluids, or flows, is important. The fluid is said to be *homogeneous* if its density at any given time is constant. This means that ρ is independent of position, $\vec{x}$, so that $\nabla\rho = 0$. If the fluid, or flow, is both homogeneous and incompressible then we have $\nabla\rho = 0$ and $\nabla \cdot \vec{v} = 0$, so that $\nabla \cdot (\rho\vec{v} = 0$. In this case, the continuity equation yields

$$\frac{\partial \rho}{\partial t} = 0$$

and so ρ is a constant.

Inviscid Fluid: A fluid with no viscosity. For a gas, inviscid means means that both viscosity and volume viscosity are zero. (Yih, p 60.)

Irrotational Flow: "A flow in which the fluid does not rotate at every point in space is called "irrotational" or "potential" flow. In general, there is no potential flow in nature, but many flow regions can be considered irrotational in an approximate way." (Lugt, 1983, p 18.) That is, a flow is said to be irrotational flow in some domain $\mathcal{D}$ if $\nabla \times \vec{v} = 0$ everywhere in $\mathcal{D}$. If $\mathcal{D}$ is simply connected, $\vec{v}$ can be written in terms of a potential ϕ, $\vec{v} = -\nabla\phi$. Hence the synonym "potential flow". If $\mathcal{D}$ is not simply connected, there is no global potential, but local potentials exist that can be used to express the flow, with potential transformation laws relating the local potentials in regions of overlap.

Incompressible Flow: An *incompressible flow* is a flow with the property that $\nabla \cdot \vec{v} = 0$. This means that the density of any fluid element remains constant as it moves. It does *not* mean that the density of the fluid is constant. Constant density is a stronger condition, and is a special case of incompressible flow.

Isotropic: *Isotropic* means independent of direction. This term is used to describe quantities

Isentropic Flows: Landau and Lifschitz (2003), p 4: An *isentropic flow* is one in which the entropy of the fluid is constant throughout its volume at each instant of time,

$$s = \text{constant}$$

In such a flow, there are clearly no dissipative processes. This means that the effects of viscosity in the fluid are negligible or absent, and that the fluid admits negligible or no heat conduction. Landau and Lifschitz take these conditions as the definition of an ideal fluid. (See above for a definition of an ideal fluid.) Only and ideal fluid, therefore, can admit an isentropic flow. Note, however, not every flow admitted by an ideal fluid need be isentropic. Isentropic is a special case of ideal fluid flow. (See above definition and discussion of an ideal fluid.)

Mechanical Equilibrium: A fluid is said to be in *mechanical equilibrium* if it exhibits no macroscopic motions. This means that the velocity field is zero everywhere and at all times,

$$\vec{v}(\vec{x}, t)$$

Mechanical equilibrium does not mean that the fluid is in thermal or chemical equilibrium. It is possible for the mechanical equilibrium to be stable without it being in equilibrium with respect to the other thermodynamic variables.

Perfect Fluid. See *ideal fluid*.

Potential (or, Irrotational) Flow: Landau and Lifschitz (2003), p 14: "A flow for which $\mathbf{curl}\vec{v} = 0$ in all space is called a *potential flow* or *irrotational flow*, as opposed to a *rotational flow*, in which the curl of the velocity is not everywhere zero." Some authors require only that $\nabla\vec{v} = 0$ in some smaller domain in which the flow takes place. Note however that the presence of a solid boundary requires special care when the flow is treated as potential flow. See the very illuminating discussion in Landau and Lifschitz (2003), p 14-16.

Some authors (Lamb, p 17, 1945) restrict the term "potential flow" to those special irrotational flows that possess a single, global potential function. Thus, for Lamb, a potential flow is one whose velocity field can be written in the form

$$\vec{v} = -\nabla\phi$$

This is not guaranteed in general by the irrotational condition only, except in the special case when the domain is simply connected. Lamb calls irrotational flows which admit only local potentials, but not a global potential, flows with *cyclic velocity potentials* flows (p 17, footnote *).

See also entry for *Irrotational Flow*.

Rossby Number: Tassoul (2000), p 30: The Rossby number is a dimensionless number that measures the relative importance of the advective term to the Coriolis force. It is defined to be the ratio

$$\mathrm{Ro} = \frac{|\vec{u}\cdot\vec{u}|}{|\vec{\Omega}\times\vec{u}|} = \frac{\nu U^2/L}{\Omega U} = \frac{U}{\Omega L}$$

Steady Flow: A flow is steady if all associated dependent variables, such as velocity, acceleration, density, temperature, and pressure, are independent of time at any fixed point (Yih, 1977, p 5). Thus, velocity, acceleration, density, pressure, and temperature, etc., all have partial derivative with respect to t equal to zero.

Streamlines: Landau and Lifschitz, p 8, define streamlines are curves whose tangent at any point gives the direction of the velocity at that point. The equations defining streamlines are

$$\frac{dx}{v_x(\vec{x},t)} = \frac{dy}{v_y(\vec{x},t)} = \frac{dz}{v_z(\vec{x},t)}$$

Note that t plays the role of a fixed parameter in this definition. There is therefore a family of streamlines at each instant in time. In general, the curves do not coincide from instant to instant, but the flow pattern changes.

This definition is weaker than it need be. It defines the streamlines as *unparameterised* curves. It is thus only their *direction* at a point that is important. Landau and Lifschitz, accordingly, parameterise them by their arc-length when they need to work with them.

A stronger definition requires the streamlines to be curves whose tangent at any point is the velocity of the fluid at that point at any given time. According to this definition, the equations for the streamlines are

$$\frac{dx}{d\lambda} = v_x(\vec{x},t)$$
$$\frac{dy}{d\lambda} = v_y(\vec{x},t)$$
$$\frac{dz}{d\lambda} = v_z(\vec{x},t)$$

These are ODEs for a family of parameterised curves, whose tangents are the velocity vectors for the flow. Again, t is a fixed parameter in these equations and is not involved in the integration, so we get a family of curves for each time t. This definition makes the streamlines the *integral curves* of the velocity field at time t.

If the flow is steady (see above), the velocity field does not depend on time, and so the streamlines will not change with time. Furthermore, the streamlines also coincide with the paths of the fluid points.

If the flow is not steady, the streamlines do not coincide with the paths of the fluid points. Then the tangents to the streamlines give the directions of the velocity of the fluid points at the various points in space at a fixed time, whereas the tangents to the paths give the directions of the velocities of given fluid particles at various times (Landau and Lifschitz, 2003), p 8).

Steady Flow: By steady flow, we mean a flow in which the velocity of the flow is constant in time at any point occupied by the fluid. That is, $\vec{v} = \vec{v}(\vec{x})$ is a function of position alone, and does not depend on time, so that

$$\frac{\partial \vec{v}}{\partial t} = 0$$

(Landau and Lifschitz, 2003, p 8.)

Thermal Equilibrium: Landau and Lifschitz, p 7, appear to say that a fluid is in *thermal equilibrium* when the temperature of the fluid is constant. Presumably, this means no thermal flows. If the conductivity of the fluid is zero, however, this also results in no thermal flows. Is this classified as a thermal equilibrium? Perhaps, there is no real fluid in which the thermal conductivity is zero, so this case may be of academic interest only and should be excluded from the definition of thermal equilibrium.

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Appendix A Stress

A.1 Definition of Stress

When a force is applied to the surface of an object, it is not usually applied at a single point only, but over an entire area of the boundary surface. This situation arises in particular when the force is applied to our object by *contact* with another object. The contact does not ordinarily occur at a single point, but over an entire area. Force applied in this way are said to be *distributed*.

Elementary mechanics does not deal with distributed force. There, all forces are assumed to be applied at a single point. To deal with distributed forces, we need to introduce new concepts. In particular, we need to consider the concept of *stress*.

Let an object have surface S, and let $\vec{x}$ be any point on its surface. Let δS be a small segment of its surface about the point $\vec{x}$. If a distributed force is applied to the object, the applied force will be distributed over some finite portion of its surface area. In particular, some of the applied force will be due to contact with the segment δS of the surface. Denote the total force applied through δS by $\delta\vec{F}$. We define the *average stress at $\vec{x}$ over the area δS* to be

$$\vec{\Sigma}_{\mathrm{av}} = \frac{\delta\vec{F}}{\delta S} \tag{A.1}$$

In general, the applied force will be unevenly distributed over the surface of the object. The value obtained for the average stress at $\vec{x}$ will then depend on both the shape and size of the segment δS the surface chosen for measuring the average stress at $\vec{x}$. This means that the average stress is not a suitable quantity for describing the way that the force is distributed over the surface S.

We eliminate the dependence of the average stress on the shape and size of the element considered by considering its limiting value as the size of the surface element is reduced to zero. We thus define the *stress $\vec{\Sigma}$ at the point $\vec{x}$* to be

$$\vec{\Sigma} = \lim_{\delta S\to 0}\frac{\delta\vec{F}}{\delta S} \tag{A.2}$$

The concept of stress enables us to model the effect of distributed forces on an object. The modelling procedure consists of choosing, or predicting from deeper principles, the function $\vec{\Sigma}(\vec{x})$ at each surface point $\vec{x}$ on the surface. The force acting on any arbitrarily chosen surface element δS of the surface is then given by

$$\delta\vec{F} = \vec{\Sigma}\ \delta S \tag{A.3}$$

and the total force acting on the object is given by the surface integral

$$\vec{F} = \oint_S \vec{\Sigma}\ dS \tag{A.4}$$

A.2 Internal Stress

In continuous media, each portion of the medium is in direct contact with the material surrounding it. We expect this contact to give rise to a force on the portion of material considered. Furthermore, since this force arises from contact, we also expect it to be distributed over the surface of contact. The correct description of this force will therefore be in terms of stress. Unlike the case previously considered, where the force was applied to the object by contact with an external agent, the force applied to the elements of a continuous medium is by contact with the surrounding medium. In this sense, the stress considered is an ‘internal matter’ that does not involve any outside agent. We refer to stresses that arise in this way as *internal stresses*. Stresses arising by contact with external agents are sometimes called *external stresses* or, more commonly, *boundary stresses*.

The concept of internal stress is more complicated than that of boundary stress. In the case of boundary stresses, the surface over which the distributed force is applied is well defined. In the case of internal stress, there are no naturally defined surfaces to consider. Any surface introduced for consideration is a mental construct. Accordingly, the mathematical description of internal stresses is more complicated that that needed for boundary stresses.

Let $\vec{x}$ be a point in the interior of the continuous medium, and Π a segment of a plane through $\vec{x}$ with area S. The plane segment has two sides: a “top” side and a “bottom”. We will distinguish the top side by means of an unit normal $\hat{n}$ as follows: attach the vector $\hat{n}$ by its tail to the point $\vec{x}$. Its tip then points away from the plane. The side of the plane on which the vector $\hat{n}$ is found will be considered as the “top side” of the plane. We build this definition of a ‘top-side’ of the plane segment into its mathematical description by means of the *vector area*, or the *oriented area*, defined by

$$\delta\vec{S} = \hat{n}\ \delta S \tag{A.5}$$

The direction $\hat{n}$ of this oriented area in space is called the *attitude* of the plane segment. Essentially, the attitude of a plane segment is the direction in space in which its top-side faces.

Consider now the force exerted *by* the material above the top-side of the plane segment *on* the material below it. Denote it by $\delta\vec{F}$. This idea may seem somewhat abstract. It may be thought of more concretely as follows. Imagine that we remove the material above the plane segment. The material previously above the plane segment exerted forces on the material that remains by intermolecular bonding. Now that the upper material has has been removed, those bonding forces are now absent, and there is an imbalance of forces on the remaining material pulling it away from the plane segment. If we do not apply forces to the remaining material, it will collapse. The total force that we need to apply over the plane segment to stop this collapse is $\delta\vec{F}$.

Note that $\delta\vec{F}$ is not in general necessarily perpendicular to the plane segment. It could, in principle, act at any angle to it. In the case of an isotropic medium, there is nothing to distinguish one direction above another, so the intermolecular forces are expected to be the same in all directions. The nett force exerted by the material above the plane segment on that below is then expected to have a resultant in the direction perpendicular to the segment. However, it the material is not isotropic, as is the case in a crystalline solid, the is no reason to expect the resultant force to be perpendicular to the segment. It could be directed at any angle to the normal.

With this concept of the force exerted by the material above the plane segment on that below it, we can now define the concept of an *internal stress*. The *stress* in the material at $\vec{x}$ for a surface of attitude $\hat{n}$ is defined to be

$$\vec{\Sigma} = \lim_{\delta S \to 0} \frac{\delta\vec{F}}{\delta S} \tag{A.6}$$

Note that this concept of stress is a little more abstract than that previously defined. In the case of boundary stress, the element of surface at $\vec{x}$ is defined by the physical boundary of the material. For internal stress, on the other hand, we must introduce artificially an element of surface with a given attitude. Since this attitude could have been chosen in an infinite number of ways, we have potentially an infinite number of actual stresses embodied into the notion of internal stress.

A.3 The Stress Tensor

Consider now a finite plane segment of area S and with attitude $\hat{n}$ at position $\vec{x}$ in a continuous medium. Denote by $\vec{F}$ the force exerted by the material above the plane segment on that below it. In general, we expect $\vec{F}$ to depend on the attitude $\hat{n}$ of the pane segment, as well as on its surface area S and its position $\vec{x}$ in the medium. This means that $\vec{F}$ must be a function of the vector area $\vec{S}$ of the segment, and of its position $\vec{x}$,

$$\vec{F} = \vec{F}(\vec{x}, \vec{S}) \tag{A.7}$$

We now examine the dependence of $\vec{F}$ on the variables $S^i = Sn^i$ for given attitude n^i. Expanding the functions $F^i(\vec{x}\vec{S})$ in a Taylor series about the values $S^i = 0$, we get

$$F^i(\vec{x}, \vec{S}) = F^i(\vec{x}, \vec{0}) + \frac{\partial F^i}{\partial x^j}(\vec{x}, \vec{0})\ S^j + \frac{1}{2}\ \frac{\partial^2 F^i}{\partial x^j \partial x^k}(\vec{x}, \vec{0})\ S^j S^k + \cdots \tag{A.8}$$

Since n^i are the components of a given non-zero vector, $S^i = 0$ means that we have put $S = 0$. This corresponds to a plane segment of zero area. Since the force $\vec{F}$ is finite, and is distributed over a finite area S, the total force acting on a segment of zero area must be zero. Thus, $F^i(\vec{x}, \vec{0}) = 0$. Further, the derivatives in the expansion are all evaluated at $S{=}0$, so they can depend at most on the position $\vec{x}$. Denote them by,

$$\sigma^i{}_j(\vec{x}) = \frac{\partial F^i}{\partial x^j}(\vec{x}, \vec{0}), \quad \sigma^i{}_{jk}(\vec{x}) = \frac{\partial^2 F^i}{\partial x^j \partial x^k}(\vec{x}, \vec{0}), \quad \text{etc.} \tag{A.9}$$

Then F^i becomes,

$$F^i(\vec{x}, \vec{S}) = \sigma^i{}_j(\vec{x}) S^j + \frac{1}{2}\, \sigma^i{}_{jk}(\vec{x}) S^j S^k + \cdots \tag{A.10}$$

The stress at $\vec{x}$ on the plane segment of attitude $\hat{n}$ is then given by

$$\begin{aligned} \Sigma^i(\vec{x}, \vec{n}) &= \lim_{S \to 0} \frac{F^i}{S} \\ &= \lim_{S \to 0} \frac{\sigma^i{}_j(\vec{x}) S^j + \frac{1}{2}\, \sigma^i{}_{jk}(\vec{x}) S^j S^k + \cdots}{S} \\ &= \lim_{S \to 0} \left(\sigma^i{}_j(\vec{x})\, n^j + \frac{1}{2}\, S \sigma^i{}_{jk}(\vec{x})\, n^j n^k + \cdots \right) \end{aligned}$$

or,

$$\Sigma^i(\vec{x}, \vec{n}) = \sigma^i{}_j(\vec{x})\, n^j \tag{A.11}$$

This means that the stress on any surface of attitude $\hat{n}$ at position $\vec{x}$ of the medium is completely determined by the nine fields $\sigma^i{}_j(\vec{x})$. These fields are the components of the *stress tensor* for the material. To model the internal stresses in a continuous medium, therefore, we need to specify or calculate the functions $\sigma^i{}_j(\vec{x})$. These functions together contain all information about the internal stresses of the material medium.

It is often convenient to work with a contravariant description of the stress rather than a mixed tensor. We then write the stress tensor as σ^{ij} rather than as $\sigma^i{}_j$, and the formula for the stress is then accordingly changed to

$$\Sigma^i(\vec{x}, \vec{n}) = \sigma^{ij}(\vec{x})\, n_j \tag{A.12}$$

A.4 Isotropic Stress

We say that the internal stress in a continuous medium is *isotropic* if the stress tensor has the special form,

$$\sigma^i{}_j(\vec{x}) = \lambda(\vec{x})\, \delta^i_j \tag{A.13}$$

where δ^i_j is the Kronecker delta, and $\lambda(\vec{x})$ is a function. This definition requires a substantial amount of theory for its justification which will not be covered here. Essentially, in the context of the stress tensor, isotropy means that the stress in the material at each point $\vec{x}$ is the same for all possible attitudes $\hat{n}$ of the surface used to measure it.

An immediate consequence of this definition is that the force exerted by the material above a surface element $\delta\vec{S}$ on that below it is given by

$$\delta F^i = \sigma^i{}_j(\vec{x})\ \delta S^j = \lambda(\vec{x})\ \delta^i_j \delta S^j = \lambda(\vec{x})\ \delta S^i \tag{A.14}$$

or

$$\delta\vec{F} = \lambda(\vec{x})\ \delta\vec{S} \tag{A.15}$$

Thus, if the stress is isotropic, the force exerted on the material below the element of surface by that above it is always *perpendicular* to the surface element, and its magnitude is the same for all possible orientations of the surface element at $\vec{x}$.

Conversely, if the force is always perpendicular to the surface element, and its magnitude is independent of the attitude of the element at $\vec{x}$, then

$$\delta\vec{F} = \lambda(\vec{x})\ \delta\vec{S} \tag{A.16}$$

or

$$\delta F^i = \lambda(\vec{x})\ \delta^i_j \delta S^j \tag{A.17}$$

so that

$$\sigma^i{}_j(\vec{x}) = \lambda(\vec{x})\ \delta^i_j \tag{A.18}$$

and the stress is isotropic.

A.5 Force Exerted on an Element of Continuous Medium by Contact with the Surrounding Material

Consider a finite element of fluid with volume V and boundary surface S. This element is in contact with the surrounding fluid at each point of its surface. It thus experiences a force due to this contact. We now calculate this force.

Consider a small segment $\delta\vec{S}$ of the boundary surface of this element. We are interested in the force exerted by the surrounding fluid on the element, so we choose the normal $\hat{n}$ to the surface to point outwards at each of its points. The force exerted by the surrounding material on the element through segment $\delta\vec{S}$ is then

$$\delta F^i = \sigma^{ij}\ \delta S_j \tag{A.19}$$

The total force exerted on the element is therefore

$$F^i = \oint_S \sigma^{ij}\ dS_j \tag{A.20}$$

We can transform this surface integral into a volume integral by means of the divergence theorem. This gives,

$$F^i = \int_V \frac{\partial \sigma^{ij}}{\partial x^j}\, dV \tag{A.21}$$

Abbreviating the derivative $\partial/\partial x^j$ bt the symbol ∂_j, this becomes,

$$F^i = \int_V \partial_j \sigma^{ij}\, dV \tag{A.22}$$

The divergence of the stress tensor, $\partial_j \sigma^{ij}$ is clearly a force density per unit volume,

$$f^i = \partial_j \sigma^{ij} \tag{A.23}$$

It is not difficult to interpret this force density. Suppose we make the element of material infinitesimally small so that its volume is δV. Then the integral is approximately the sum over a single volume element and gives,

$$F^i = \int_V \partial_j \sigma^{ij}\, dV \approx \partial_j \sigma^{ij}\, \delta V \tag{A.24}$$

Thus

$$\partial_j \sigma^{ij}\, \delta V = \begin{cases} \text{force exerted on an infinitesimal element of material} \\ \text{by contact at its surface with the surrounding fluid} \end{cases} \tag{A.25}$$

This result can also be demonstrated more naively as follows.

***** Naive proof *****

At first sight, it may seem strange that the force exerted on the finite element of volume V by contact with the surrounding fluid can be calculated by an integral that extends through the entire volume of the element. After all, there is no direct contact between the interior of the element and the surrounding fluid. On reflection, however, this result is not difficult to understand. Divide the volume of the fluid element into a large number of infinitesimal volume elements. Then, each of these elements experiences a force due to its contact with those infinitesimal elements contiguous with it. The volume integral calculates the *resultant* of all these forces. If two infinitesimal elements of volume are in direct contact with each other, the force exerted by one on the other is equal and opposite to that exerted by the other on the first. When summing the forces, therefore, these action and reaction pairs will cancel each other. Thus all of the forces due to the contact of infinitesimal elements within the volume V with each other will sum to zero, leaving only those forces due to contact with the surrounding fluid at the boundary surface S. The sum of all of the forces acting on all of the infinitesimal volume elements must thus equal the sum of all of the forces exerted on the boundary S of the finite volume element.

A.6 Hydrostatic Pressure

According to the definition given above, stress is regarded as a tensile force: the material above the surface exerts a force that pulls on the material below the surface. In many cases of interest, the stress is not tensile but compressive: the force exerted by the material above the surface *pushes* on the material below it. Whether one chooses to regard the contact force as pulling or pushing on the material below the surface is a convention. The convention we have chosen to use in the above definition is the one that regards stress as a tensile force on the element considered.

Compressive stresses are called *pressures*. To obtain a description of compressive stress, all we need do is reverse the sign of the stress as defined above. This leads to the concept of a *pressure tensor*. The pressure tensor is thus the negative of the stress tensor,

$$P^{ij} = -\sigma^{ij} \tag{A.26}$$

It is customary to use the concept of pressure in situations in which the stress is everywhere compressive. This is the case in all fluids, whether gaseous of liquid. In the case of a gas, the stress is always compressive due to the continual bombardment of the element by the surrounding gas molecules. Stresses in a gas can therefore never be tensile. In the case of a liquid, the situation is slightly different. In a liquid, the molecules are bonded, but the bonds are tenuous and weak. So, unlike solids, liquids do not have the ability to sustain substantial tensile stresses. In fact, even an extremely small tensile stress applied to a liquid will cause the tenuous bonds to break, with the result that the liquid cavitates. The liquid literally tears apart, leaving a void that is free from the liquid. So, to good approximation, we can regard the stresses in a fluid always as compressive rather than tensile. This makes it more convenient (though not essential) to use a pressure tensor rather than a stress tensor when discussing fluids.

Pascal identified three laws of fluid pressure that apply to fluids at rest in an uniform gravitational field. These are,

1. The pressure at a point in a fluid is the same for all orientations of the surface on which it is measured, and the force exerted on the surface is always perpendicular to it.
2. The pressure in a fluid is the same at all points at the same horizontal level.
3. The pressure in a fluid increases with depth.

The first law tells us that the stress tensor in a fluid is isotropic. Writing the stress tensor in terms of compressive stress rather than tensile stress, this means that it is given by

$$\sigma^i{}_j(\vec{x}) = P(\vec{x})\ \delta^i_j \tag{A.27}$$

The second law tells us that the function $P(\vec{x}$ depends only on one variable, the depth. If we choose coordinates with the z-axis vertically up, the this law is expressed by writing

$$P = P(z) \tag{A.28}$$

The last law tells us that $P(z)$ is a monotonically decreasing function of z. In these coordinates, therefore, the internal stress of the fluid is given by

$$\sigma^i{}_j = P(z)\ \delta^i_j \tag{A.29}$$

A.7 Symmetry of the Stress Tensor

The stress tensor T^{ij} is symmetric. This follows from the dynamics that governs the rotational motion of the fluid. Consider a given fixed volume V in space with boundary surface S. Denote the total angular momentum about the origin of coordinates of the fluid occupying V by $\vec{L}$. We can calculate $\vec{L}$ by considering the angular momentum $\delta\vec{L}$ of a small quantity of fluid in V which occupies an element of volume δV. Then,

$$\delta\vec{L} = \vec{x} \times \delta\vec{p} = \vec{x} \times (\delta m\ \vec{v}) = \vec{x} \times (\rho\vec{v})\ \delta V \tag{A.30}$$

The total angular momentum of the fluid within V is therefore

$$\vec{L} = \int_V \vec{x} \times (\rho\vec{v})\ dV \tag{A.31}$$

or, in index notation,

$$L^i = \int_V \rho\varepsilon^{ijk} x_j v_k\ dV \tag{A.32}$$

The integrand represents the density of angular momentum of the fluid about the origin of coordinates. Note that we are treating the fluid as a collection of fluid particles. $\vec{L}$ is obtained by adding together the angular momenta possessed by these particles by virtue of their motion about the origin. Thus $\vec{L}$ is the total *orbital* angular momentum of the fluid. It does not include any form of intrinsic angular momentum possesses by the fluid particles.

In time, the total angular momentum of the fluid in V can be expected to change. These changes are produced by the action of torques on the fluid in V. These torques are produced by the action of two kinds of forces, the force field $\vec{f}$ acting on each element of fluid in V, and the surface stresses exerted by the surrounding fluid on the fluid in V at the surface S. We now calculate the total torque due to each of these two different kind of action.

The force $\delta\vec{F}$ exerted by the force field $\vec{f}$ on an element of fluid of mass δm is given by

$$\delta F^i = \delta m\ f^i = \rho f^i\ \delta V \tag{A.33}$$

The torque about the origin produced by this force is therefore

$$\delta\tau_V^i = \varepsilon^{ijk} x_j \ \delta F_k = \rho\varepsilon^{ijk} x_j f_k \ \delta V \tag{A.34}$$

Hence the total torque exerted by the force field $\vec{f}$ on the fluid in V is given by

$$\tau_V^i = \int_V \rho\varepsilon^{ijk} x_j f_k \ dV \tag{A.35}$$

The force $\delta\vec{F}$ exerted by the adjacent fluid of the fluid in volume V over a small element δS of the boundary surface S is given by

$$\delta F^i = \sigma^{ij} n_j \ \delta S \tag{A.36}$$

where $\vec{n}$ is the unit outward normal to the element of surface δS. The torque about the origin produced by this force is therefore

$$\delta\tau_S^i = \varepsilon^{ijk} x_j \ \delta F_k = \varepsilon^{ijk} x_j \sigma_{kr} n^r \ \delta S \tag{A.37}$$

Hence the total torque exerted on the fluid in V by its contact with the surrounding fluid is given by

$$\tau_S^i = \oint_S \varepsilon^{ijk} x_j {\sigma_k}^r n_r \ dS \tag{A.38}$$

This can be reexpressed as a volume integral using the divergence theorem,

$$\tau_S^i = \int_V \partial_r(\varepsilon^{ijk} x_j {\sigma_k}^r) \ dV \tag{A.39}$$

The total torque acting on the fluid in V is therefore

$$\tau^i = \tau_V^i + \tau_S^i = \int_V \left[\rho\varepsilon^{ijk} x_j f_k + \partial_r(\varepsilon^{ijk} x_j {\sigma_k}^r)\right] \ dV \tag{A.40}$$

The effect of the action of a torque on the fluid in V is to increase its angular momentum at a rate given by

$$\tau^i = \frac{dL^i}{dt} \tag{A.41}$$

We thus have

$$\int_V \left[\rho\varepsilon^{ijk} x_j f_k + \partial_r(\varepsilon^{ijk} x_j {\sigma_k}^r)\right] \ dV = \frac{d}{dt}\int_V \rho\varepsilon^{ijk} x_j v_k \ dV \tag{A.42}$$

We now use the equations of motion to evaluate the time derivative on right hand side of this equation,

$$\begin{aligned} \frac{d}{dt}\int_V \rho\varepsilon^{ijk} x_j v_k \ dV &= \int_V \frac{\partial}{\partial t}\left(\rho\varepsilon^{ijk} x_j v_k\right) dV \\ &= \int_V \varepsilon^{ijk} x_j \frac{\partial}{\partial t}\left(\rho v_k\right) dV \end{aligned}$$

We have already shown from the equations of motion for the fluid that

$$\frac{D}{Dt}\left(\rho v^k\right) = \partial_r\sigma^{kr} + \rho f^k \tag{A.43}$$

so that

$$\frac{d}{dt}\int_V \rho\varepsilon^{ijk}x_j v_k \; dV = \int_V \varepsilon^{ijk}x_j\left(\partial_r \sigma_k{}^r + \rho f_k\right) dV$$
$$= \int_V \left(\varepsilon^{ijk}x_j\partial_r\sigma_k{}^r + \rho\varepsilon^{ijk}x_j f_k\right) dV$$
$$= \int_V \left[\partial_r\left(\varepsilon^{ijk}x_j\sigma_k{}^r\right) - \varepsilon^{ijk}(\partial_r x_j)\sigma_k{}^r + \rho\varepsilon^{ijk}x_j f_k\right] dV$$
$$= \int_V \left[\partial_r\left(\varepsilon^{ijk}x_j\sigma_k{}^r\right) - \varepsilon^{ijk}\delta_{rj}\sigma_k{}^r + \rho\varepsilon^{ijk}x_j f_k\right] dV$$
$$= \int_V \left[\partial_r\left(\varepsilon^{ijk}x_j\sigma_k{}^r\right) - \varepsilon^{ijk}\sigma_{kj} + \rho\varepsilon^{ijk}x_j f_k\right] dV$$

where we have used the fact that, in Cartesian coordinates, $g_{ij} = \delta_{ij}$, and hence $\delta_{rj}\sigma_k{}^r = \sigma_{kj}$. So, finally,

$$\frac{d}{dt}\int_V \rho\varepsilon^{ijk}x_j v_k \; dV = \int_V \left[\partial_r\left(\varepsilon^{ijk}x_j\sigma_k{}^r\right) - \varepsilon^{ijk}\sigma_{kj} + \rho\varepsilon^{ijk}x_j f_k\right] dV$$

Inserting this into (A.42) gives

$$\int_V \left[\rho\varepsilon^{ijk}x_j f_k + \partial_r(\varepsilon^{ijk}x_j\sigma_k{}^r)\right] \; dV$$
$$= \int_V \left[\partial_r\left(\varepsilon^{ijk}x_j\sigma_k{}^r\right) - \varepsilon^{ijk}\sigma_{kj} + \rho\varepsilon^{ijk}x_j f_k\right] dV$$

or

$$\int_V \varepsilon^{ijk}\sigma_{kj} \; dV = 0 \tag{A.44}$$

Since this result is true for arbitrarily selected volumes V,

$$0 = \varepsilon^{ijk}\sigma_{kj} \tag{A.45}$$

This gives

$$0 = \varepsilon_{irs}\varepsilon^{ijk}\sigma_{kj} = \delta^{jk}_{rs}\sigma_{kj} = \sigma_{rs} - \sigma_{sr}$$

or

$$\sigma_{rs} = \sigma_{sr} \tag{A.46}$$

showing that σ_{ij} is symmetric.

The basic assumption that has produced this result is contained in equation (A.42), namely, that $\vec{L}$ *is the total angular momentum of the fluid*. Since $\vec{L}$ is the total *orbital* angular momentum of the fluid elements about the origin of coordinates, we have assumed implicitly that the fluid elements each have zero intrinsic angular momentum. In a fluid in which this is not the case, we would need to add into the total angular momentum a term representing the total intrinsic angular momentum of the fluid. This would then change this result by giving the stress tensor σ_{ij} a non-zero antisymmetric part.

Appendix B Strain

B.1 Infinitesimal Displacement of a Fluid

Consider a fluid in motion. Each of its points will change its position with time. We are interested in the motion of its points in an infinitesimal time interval δt. Denote the position of a of a given fluid point at time t by $\vec{x}$. Its velocity at that instant is therefore $\vec{v}(\vec{x}, t)$. In time δt, it will move to a new position $\vec{x} + \vec{v}(\vec{x}, t)\ \delta t$. Consider now an adjacent fluid point which at time t is at position $\vec{x} + d\vec{x}$. Its velocity is therefore $\vec{v}(\vec{x} + d\vec{x}, t)$. In time δt, it will move to new position $\vec{x} + d\vec{x} + \vec{v}(\vec{x} + d\vec{x}, t)\ \delta t$. This is displayed in Figure B.1. The displacement of the

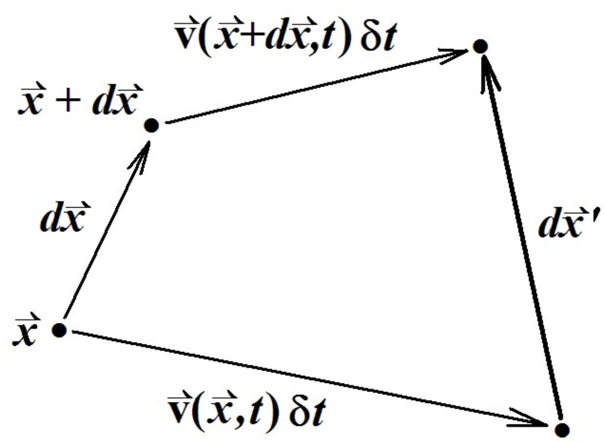

Figure B.1 Displacement of fluid points in time δt.

second fluid point from the first at time t is given by $\vec{x} + d\vec{x} - \vec{x} = d\vec{x}$. At time $t + \delta t$, however, its displacement relative to the first has changed to

$$d\vec{x}' = [\vec{x} + d\vec{x} + \vec{v}(\vec{x} + \delta\vec{x}, t)\ \delta t] - [\vec{x} + \vec{v}(\vec{x}, t)\ \delta t] = d\vec{x} + \frac{\partial \vec{v}}{\partial x^k}(\vec{x}, t)\ dx^k\ \delta t$$

Appendix C Continuity equations and Conservation Laws

In this chapter, we review the classical theory of currents and densities. We then use this theory to establish the continuity equation for conserved and for non-conserved quantities, and interpret the results physically. Lastly, we apply the theory to the description of conservation of mass in the fluid continuum. In later chapters, we also apply the theory developed here to the description of the conservation of energy, momentum, and angular momentum in fluids.

C.1 Fluid Element

An essential concept in the theory of fluids is that of a *fluid element*. A *fluid element* is a given fixed set of points in the fluid continuum which we isolate mentally from the surrounding fluid, and whose motion we track as the fluid moves. The fluid element is the theoretician's equivalent of injecting dye into a fluid in order to track the motion of the fluid.

For ease of visualisation, imagine that at time $t = 0$ we paint red all of the fluid points within a selected volume in order to distinguish them visibly from the remaining points of the fluid continuum. The selected volume may be finite or infinitesimal. If finite, we speak of a *finite fluid element*; and if infinitesimal, of an *infinitesimal fluid element*. The volume occupied by the marked fluid points is of no consequence: in general, it will change continuously as the motion unfolds. Only the marked fluid points are of consequence. Together, these constitute the entity whose progress we wish to track.

It is not difficult to 'paint the fluid points red'. In a Lagrangian description of the motion, each fluid point is assigned a set of coordinates a^α which identify it thereafter for all time. The position at time t of the fluid point whose Lagrangian coordinates are a^α is given by equations of the form

$$x^i = x^i(\vec{a}, t)$$

To distinguish, or 'paint red', a fluid element, we identify the domain $\mathcal{D} \subset \mathbb{R}^3$ in the Lagrangian coordinate space occupied by these fluid points. We therefore distinguish the fluid element by specifying a simply-connected set $\mathcal{D}$ of points. If our fluid element has a boundary, we choose $\mathcal{D}$ to be closed set; if we want to consider an element without its boundary, we choose $\mathcal{D}$ to be an open set.

It is important to understand that, in this model, the fluid points are not real physical particles, nor is the fluid element a collection of real physical particles. Rather, the fluid element is a collection of an infinite number of mathematical points that we use to model the motion of a finite collection of real physical particles. The success of the fluid model depends on how closely we succeed in making the fluid element imitate the motion of the real physical particles.

C.2 Mass in the Fluid Model

The fluid points are mathematical points. They have no mass associated with them. Nor can any mass be associated with them. Suppose we use a fluid element to model the motion of a given set of physical fluid particles. The number of physical particles contained in a finite volume is finite. Therefore, their total mass is finite. The number of mathematical points in a fluid element is infinite. If we attempt to divide out a finite mass between them, each fluid point must necessarily be assigned a zero mass. Conversely, if we attempt to attribute a finite mass to each fluid point, the total mass of all the fluid points that make up a fluid element will be infinite. Mass therefore cannot be attributed to the fluid points.

To obtain a description of mass in a continuum theory, we need to introduce the concept of a mass density. The density field is defined in such a way that, integrated at time t over the volume occupied by any a fluid element, we obtain the total mass of all the physical particles contained at time t in volume of the element. Put differently, mass is a property of a fluid element, not of the fluid points that make up that element, and the total mass of the element is obtained by integration of the density over its volume.

In principle, we may think of the mass-density function as follows. At time t, construct a small volume δV around the space point $\vec{x}$. The volume δV will contain real physical particles with total mass δm. Form the quotient

$$\frac{\delta m}{\delta V}$$

This is the *average density* at position $\vec{x}$ at time t for the given volume element. As the volume δV is made smaller, the value of this quotient will change and will begin to approach a limiting value. If the volume is made smaller still, the value of quotient will eventually begin to change discontinuously as the number of physical particles that it contains becomes smaller, and eventually either fall to zero value if δV contains no physical particles, or diverge to infinity in the rare event that a physical particle is actually found at position $\vec{x}$. We thus assign to the density ρ at $\vec{x}$ at time t the limiting value that the quotient at first appears to approach.

An alternative way to think about this is as follows. At position $\vec{x}$ at time t, select an element of volume, say of spherical shape, that is small compared to the dimensions of the domain occupied by the fluid, but large compared to the typical distance between the real physical particles. Determine the total mass of the particles contained in this volume, and divide this mass by δV. Take this as the value of the density $\rho(\vec{x}, t)$ at position $\vec{x}$ at time t. Repeat this procedure for all other points $\vec{x}$ at time t using a volume element of identical size and shape at each point in space. This procedure is called *coarse-graining*.

Of course, both of the procedures described above are impractical. We cannot repeat a procedure for an infinite number of space positions. At best, we can repeat it for a finite number of positions at one time, and then interpolate. In the end, therefore, any choice of density function to describe the mass distribution of the fluid is an assumption of the theory, and must be regarded as part of the modelling procedure.

In summary, mass is described in the fluid model by a density function $\rho(\vec{x}, t)$. An important part of the modelling procedure establishing differential equations that will determine how ρ evolves in time, and also choosing a function that correctly models the distribution of mass in the fluid at the initial time $t = 0$.

C.3 Densities and Current-densities

All that has been said in the previous section about the description of mass can be repeated for every other extensive property of the fluid, like momentum, energy, angular momentum, internal energy, entropy, etc. No extensive parameter can be attributed individually to the fluid points, but must be regarded as properties of fluid elements of finite size. All extensive parameters must therefore be described by density functions that evolve in time from some appropriately chosen initial density function according to some suitable partial differential equation. The density of any extensive parameter can be conceptualised as indicated above as some 'coarse-graining' procedure, and the value of the extensive for any selected finite element of fluid is calculated by integration of the density over the volume of the fluid element.

Suppose now that the density of some extensive parameter at a given point in space changes with time. A change in its density can occur only if the total value of the extensive for a given volume of space around that point has changed. Such a change can take place by only one of two mechanisms:

1. By fluxing, that is, by the flow of that extensive into, or out of, the given element of volume, or
2. By creation (or destruction) of the extensive within the given element of volume.

The rate at which an extensive property passes through a surface, real or imagined, in the fluid is called the *current* of that extensive. The rate per unit area at which the extensive property passes through the surface is called the *current-density* of that extensive. Furthermore, if the extensive is generated or depleted at any point in space, the sites at which the creation or depletion occur are called *sources* and *sinks* respectively. The presence of currents and of sources and sinks in the fluid determine how the density of the given extensive property evolves in time. To model the fluid, we need to determine equations for how each of the extensive densities of the fluid evolve with time.

C.4 Current Density

Flux is described in the fluid model by the concept of current density. Consider an arbitrary element of volume in space within the continuously distributed fluxing substance. Denote the volume by V, and the boundary surface that encloses this volume by S. Consider the flow of some extensive property across a small element of this closed surface. Denote the surface area of the element by ΔS and its normal vector in the outward direction by $\vec{n}$ (Figure C.1). The element

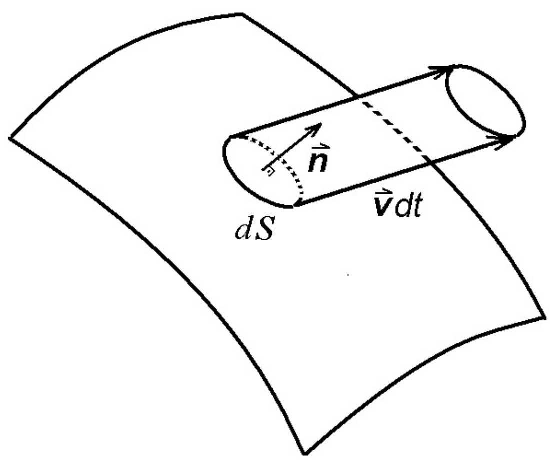

Figure C.1 Flux through an element of surface $d\vec{S}$.

of area must be chosen sufficiently small for us to be able to approximate the velocity of the extensive over that surface element by a single value $\vec{v}$. Then in a brief time Δt, the extensive initially at the surface will move in the direction of $\vec{v}$ by amount $\vec{v}\Delta t$, sweeping out a small skew cylinder in space. The total value of the extensive in this small cylinder is the extensive that crosses the element ΔS in time Δt. If the density of the extensive is ρ, then the total amount of the extensive that has flowed across ΔS in time Δt is

$$
\begin{aligned}
\rho \times \text{volume of cylinder} &= \rho \times \text{base area} \times \text{perpendicular height} \\
&= \rho \times \Delta S \times \vec{n} \cdot \vec{v}\ \Delta t
\end{aligned}
$$

Thus the total amount of the extensive flowing out of the volume V per unit time through the element of surface ΔS is

$$\rho\ \vec{n} \cdot \vec{v}\ \Delta S$$

and the total amount of the extensive flowing out of the volume V per unit time is

$$\oint_S \rho\ \vec{n} \cdot \vec{v}\ dS$$

The quantities $\vec{n}$ and dS are properties of the infinitesimal surface element considered. We combine them into a single vector

$$d\vec{S} = \vec{n}\ dS$$

called the vector area of the element. Thus $d\vec{S}$ is a vector in the outward normal direction of the bounding surface of V with magnitude equal to the area dS of the surface element. The quantities ρ and $\vec{v}$ are properties of the flowing extensive. We combine them into a single quantity

$$\vec{J} = \rho\vec{v}$$

called the *current density* of the extensive. Thus the total amount of extensive flowing out of the volume V per unit time is

$$\oint_S \vec{J} \cdot d\vec{S}$$

We can express this as a volume integral via the Gauss divergence theorem,

$$\oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J}\ dV$$

C.5 The Conservation Equation

Consider now a case in which the extensive property of interest is conserved. By *conserved* we mean that the substance can neither be created nor destroyed. According to CM, mass and charge can neither be created nor destroyed, and so are conserved quantities. In this case, there are neither sources nor sinks for the substance, so the only way that its density can change is by fluxing through the boundary of a given volume element. Now, the total amount of substance contained in V at any time t is given by

$$Q(t) = \int_V \rho\ dV$$

The rate at which this amount increases with time is therefore

$$\begin{aligned} \frac{dQ}{dt}(t) &= \frac{d}{dt} \int_V \rho\ dV \\ &= \int_V \frac{\partial \rho}{\partial t}\ dV \end{aligned}$$

The rate at which substance is *exiting* the volume V is given by

$$\int_V \nabla \cdot \vec{J}\, dV$$

Since the substance is conserved, we must have

rate of increase of substance in V = −rate of fluxing of substance out of V

or

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla \cdot \vec{J}\, dV$$

Since this result is true for arbitrary volume elements V, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \tag{C.1}$$

Equation (C.1) is called the *continuity equation* for the conserved substance whose density is ρ. It expresses the conservation of the quantity whose density is measured by ρ. But that is not all. A quantity can be conserved by a spontaneous decrease of its concentration on earth accompanied by a simultaneous spontaneous increase of its concentration on the Andromeda galaxy in such a way that its total amount in the universe is constant. This would be conservation by mystic transference, or paranormal channelling, and would require transport of substance from one location to another without passage of the substance through the intervening space. Equation (C.1) forbids this kind of conservation. It requires any substance that exits the given volume to do so by passing through its bounding surface. It thus requires a mediating flow that carries the transferred quantity from its point of origin to its final destination, continuously and contiguously through the intervening space. Thus equation (C.1) describes continuous transference across adjacent portions of space. Hence its name, *the equation of continuity.*

Because (C.1) expresses not only the process of continuous transfer but also the conservation of the substance concerned, it is also sometimes called the *conservation equation.*

C.6 General Continuity Equation

Some processes permit creation (or destruction) of substance. Thus heat may be generated at sources within the given volume or withdrawn at sinks. Similarly for entropy. The analysis of the previous section may be modified to take such processes into account. Denote by σ the nett rate of generation of substance per unit volume. Then

rate of increase of substance in V =
− rate of fluxing of substance out of V + rate of creation of material in V

or

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla \cdot \vec{J}\, dV + \int_V \sigma\, dV$$

Since this result is true for arbitrary volume elements V, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = \sigma \tag{C.2}$$

Equation (C.2) is the *continuity equation* for non-conservative flows. It expresses the process of continuous and contiguous transfer of substance through space together with the process of creation (or destruction) at continuously distributed sources (or sinks).