Dirac's Equation in General Relativity

By

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L'equation de Dirac en Relativite Generale

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INTRODUCTION

The Dirac equation for a free particle of rest mass m, in suitable physical units, is

$$\left(\sum_{i=0}^{3}\gamma^k\frac{\partial}{\partial x^k}-im\right)\psi=0.$$

Here the x^k are the coordinates of a point x of the space-time of special relativity, the γ^k matrices of order 4 satisfying certain relations – those of Clifford – and ψ a column of four elements. Under a Lorentz transformation $x^k \to x^{k'}$ (with the γ^k remaining unchanged) the column ψ, called a spinor, undergoes a certain transformation of a type different from that undergone by tensors.

The purpose of the present work is to give an extension of the theory of Dirac in general relativity. Since the affine space-time is replaced by a Riemannian space of Lorentz signature, we construct at each point of this space a *local* Clifford algebra, based on the formulae $e_ie_j + e_je_i = 2g_{ij}$, where e_i is the basis vector that corresponds to the coordinate x^i and the g_{ij} are the coefficients of the metric form. At each point we introduce *all* the Lorentz systems of reference which can be obtained from the Riemannian system of reference formed by the e_i through continuous deformation.

Spinors are for us certain *collectives* of Clifford numbers which depend on the Lorentz systems of reference in a well-defined way. The principal difficulty in the problem of extension which occupies us is the differentiation of spinors. As far as we can judge, this difficulty is resolved here in a very natural and very simple way.

The geometric point of view dominates our presentation, where we believe that we have given a very satisfying synthesis between Riemannian tensors and spinors. Unfortunately, our presentation is not short. On the other hand, the covariant character of our definitions and of our procedures is clear from the beginning.

Besides the theoretical considerations, which dominate in our work, we give in n° 10 an overview of the actual calculation. The physicist will find there his usual tools, matrices and columns.

The problem addressed here has occupied several authors of great merit. It is appropriate to cite in particular E. Cartan who mentions it in the preface and treats it at the end of his *Leçons sur la théorie des spineurs*, I - II, Actualités Sci. Ind. 643, 701, Paris 1938. One finds there also a Bibliography to which it is appropriate to add a work by V. Bargmann, Sitzungsber. Akad. Berlin, 1932, and a work of W. Pauli, Ann. der Physik (5), t. 18 (1933). Cartan, who leaves aside the issues of covariance, judges his predecessors very harshly and upbraids them in particular for the absence of a geometric point of view. On the other side, the majority of the authors in question reprove each other because the covariance of the procedures which they use is difficult to see.

In spite of the differing points of view, the final results obtained by different methods seem to be identical. Finally, if we compare n° 10 of the present work and n° 174 of volume II of Cartan, we will find a perfect agreement.

I want to thank very warmly my former student Karl Greger who gave me valuable help in the preparation and editing of the present work.

1. RIEMANNIAN SPACE

Consider a Riemannian space (space-time of general relativity) with a metric $ds^2 = g_{ij}dx^idx^j$ $(i,j = 0,1,2,3)$ of *Lorentz signature*, which means that, reduced to a sum of squares, the metric form is made up of a positive square (time) and of three negative squares (space). We assume that $g_{00} > 0, g_{11} < 0, g_{22} < 0, g_{33} < 0$. The basis vectors[1] e_i corresponding to coordinates x^i $(i = 0,1,2,3)$ admit respective contravariant components $(1,0,0,0),(0,1,0,0,),\dots$.

The *scalar product* of two vectors A and B with contravariant components A^i and B^i $A = A^ie_i, B = B^ie_i$ is defined by the formula

$$(A,B) = g_{ij}A^iB^j \tag{1}$$

and, in particular,

$$(e_i,e_j) = g_{ij}. \tag{2}$$

By putting, to abbreviate, $\partial/\partial x^k = \partial_k$, we have

$$\partial_ie_j = \partial_je_i = \Gamma_{ij}{}^re_r, \qquad \Gamma_{ij}{}^r = \Gamma_{ji}{}^r,\ ^2 \tag{3}$$

from which we obtain, for any vector $A = A^je_j$, the *covariant differential*

$$dA = dA^j\,e_j + A^j\Gamma_{ij}{}^re_r\,dx^i. \tag{3^{again}}$$

Infinitesimal *parallel transport* is given by $dA = 0$, that is, by

$$dA^j + \Gamma_{ij}{}^r A^j \, dx^i = 0. \tag{4}$$

We introduce also the *dual basis* by means of the formulae

$$(e^i, e_j) = \delta^i_j \text{ where } e^i = g^{ij} e_j. \tag{5}$$

We deduce from (2) and from (5)

$$\partial_i e^j = -\Gamma_{ir}{}^j e^r. \tag{6}$$

During a transformation $x^i \to x^{i'}$ of Riemannian coordinates, the e_i and the e^i transform according to the usual laws

$$e_{i'} = \partial_{i'} x^j \cdot e_j, \qquad e^{i'} = \partial_j x^{i'} \cdot e^j. \tag{7}$$

We allow only such transformations of coordinates as 1) preserve the respective signs of the g_{ii}, stated at the beginning, 2) preserve the orientation (that is, such that the functional determinant $|\partial_i x^{j'}| > 0$, 3) do not invert the future and the past (that is, such that the partial derivative $\partial_0 x^{0'} > 0$). The stated conditions are symmetric and transitive.

According as the scalar square (A, A) of a vector A, with real components A^i, has a positive, zero or negative value, A will be classified as a *timelike, lightlike*, or *spacelike vector*. Because of our convention regarding the signs of the g_{ii}, e_0 is a timelike vector, while e_1, e_2, e_3 are spacelike vectors. A timelike or lightlike vector A will be said to be *positive* or *negative* according as the components A^0 is positive or negative. In particular, e_0 will be a *positive timelike vector*. Thanks to condition 3) which we have just put down, the positive or negative nature of a vector is independent of the system of coordinates.

2. LOCAL CLIFFORD ALGEBRAS

Form at each point x of the Riemannian space a local Clifford algebra C_x defined over the field of complex numbers. In such an algebra, we have an associative and distributive multiplication, whose rules are deduced from the formulae

$$e_i e_j + e_j e_i = 2g_{ij}. \tag{8}$$

From these formulae it follows that, for two arbitrary vectors $A = A^i e_i$ and $B = B^i e_i$,

$$AB + BA = 2(A, B). \tag{9}$$

A general element c of the algebra C_x can be written in one and only one way as the sum of a scalar, a vector, a bivector, a trivector, and a pseudoscalar

$$c = c^S + c^i e_i + \frac{c^{ij}}{2} e_i e_j + \frac{c^{ijk}}{3!} e_i e_j e_k + \frac{c^{ijk\ell}}{3!} e_i e_j e_k e_\ell, \tag{10}$$

where the coefficients are complex numbers and the $c^{ij}, c^{ijk}, c^{ijk\ell}$ are *antisymmetric.*

The coefficient c^S is called the *scalar part* of c. This scalar part is *invariant* with respect to any coordinate transformation. Moreover, the scalar part of the product of an arbitrary number of Clifford numbers is invariant with respect to cyclic permutations of the factors:

$$(c_1 c_2 ... c_n)^S = (c_2 ... c_n c_1)^S. \tag{11}$$

The Clifford number c given by formula (10) will be considered as *real* if all its coefficients are real. This convention, which makes all questions of reality of the final result, extremely simple, is entirely different from the usual conventions.

By introducing, in a general way, the commutator (or the exterior product) of order p of p arbitrary quantities $U_1, U_2, ..., U_p$

$$[U_1, U_2, ..., U_p] = \varepsilon^{i_1 i_2 \dots i_p} U_{i_1} U_{i_2} ... U_{i_p},$$

where $\epsilon^{i_1 i_2 \dots i_p}$ is the totally antisymmetric tensor of order p, we obtain 16 linearly independent elements which provide a basis for the algebra, namely 1 scalar = 1, 4 vectors = e_i, 6 bivectors = $[e_i, e_j]/2!$, $i < j$, 4 trivectors = $[e_i, e_j, e_k]/3!$, $i < j < k$, 1 pseudoscalar = $[e_0, e_1, e_2, e_3]/4!$.

The calculation is greatly simplified if we replace the e_i by the linear combinations f_i which form an *orthogonal* system (not necessarily normalised), $(f_i, f_j) = 0 \; i \neq j$. We have in this case, according to (9),

$$f_i f_j + f_j f_i = 0, \qquad i \neq j, \tag{12}$$

which states that the f_i are *anti-commuting.* The 16 basis elements can, in this case, be written as

$$1;\; e_i;\; f_i f_j,\; i < j;\; f_i f_j f_k,\; i < j < k;\; f_0 f_1 f_2 f_3. \tag{13}$$

In the proof of the statements above, we can with advantage make use of such orthogonal systems.

We emphasise further that – if the space is not flat[4] – *only* Clifford quantities defined at the *same point* or at *infinitely close points* can be composed with one another. In the latter case, *we agree to identify the quantities which are deduced from one another by parallel transport.*

The algebras C_x admit two anti automorphisms, *reversal* and *conjugate reversal* (and one automorphism, conjugation), which we will now consider.[5]

We begin by writing expression (10) in the condensed form $c = \sum c^A e_A$.

1) Reversal. This operation consists in the reversal in each product e_A of the order of the factors e_i. For example $1 \mapsto 1,\; e_0 \mapsto e_0,\; e_2e_0e_3 \mapsto e_3e_0e_2,\; e_2e_1e_0e_3 \mapsto e_3e_0e_1e_2$. The operation in question will be denoted by $e_A \mapsto \tilde{e}_A, c = \sum c^A e_A \mapsto \tilde{c} = \sum c^A \tilde{e}_A$.

2) Conjugation. Each coefficient c^A is replaced by its complex conjugate $\bar{c}^{\,A}$, that is to say that $c = \sum c^A e_A \mapsto \sum \bar{c}^{\,A} e_A = \bar{c}$.

3) Conjugate reversal. The two operations above are commutative. Their product consists in $c \mapsto \hat{c} = \sum \bar{c}^{\,A} \tilde{e}_A$.

Note that all the operations defined above are independent of the choice of vector basis. It is sufficient to show this for the last one, which can be characterised in the following way, where the e_i are not involved.

The operation $c \mapsto \hat{c}$ is an *anti-automorphism* ($c' \mapsto \hat{c}\,',\; c'' \mapsto \hat{c}\,''$ implies $c'c'' \mapsto \hat{c}\,''\hat{c}\,'$) of the Clifford algebra which conserves the real scalars and vectors, and changes the scalar $i = \sqrt{-1}$ into $-i$.

We will need the following theorem whose proof is found in my "Conference" (p. 136).

If U and V are two positive timelike vectors (in the Clifford algebra) and c is a Clifford number $\neq 0$, we have

$$(U\hat{c}Vc)^S > 0. \tag{14}$$

In the following, *the differential operator, invariant with respect to any coordinate transformation*,

$$\nabla = e^i \partial_i \tag{15}$$

will play an important role. By means of formula (6). we verify easily the relation

$$\nabla^2 = \Delta = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|}\, g^{ij} \partial_j), \qquad \text{with} \qquad g = |g_{ij}|. \tag{15$^{\text{again}}$}$$

That is to say that the square of the operator ∇ is equal to Beltrami's second order differential operator.

3. LOCAL LORENTZ SYSTEMS OF REFERENCE

It is clear from the signature of the metric that we can introduce at each point, in an infinity of ways, Lorentz systems of reference which give rise to the metric form

$$(A, A) = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2.$$

That said, we form at each point all the Lorentz frames $\Sigma_x, \dot{\Sigma}_x, \ddot{\Sigma}_x, \ldots$ or, more briefly $\Sigma, \dot{\Sigma}, \ddot{\Sigma}, \ldots$ with basis vectors $\gamma_{k\,\Sigma}, \gamma_{k\,\dot{\Sigma}}, \gamma_{k\,\ddot{\Sigma}}, \ldots$, which have the following two properties. The vectors $\gamma_{0\,\Sigma}, \gamma_{0\,\dot{\Sigma}}, \gamma_{0\,\ddot{\Sigma}}, \ldots$ are positive timelike vectors, the frames themselves have the same orientation as the Riemannian basis e_0, e_1, e_2, e_3.

The Lorentz basis vectors $\gamma_{0\,\Sigma}, \gamma_{0\,\dot{\Sigma}}, \ldots$ satisfy, according to (9), the relations

$$(\gamma_{k\,\Sigma}, \gamma_{\ell\,\Sigma}) = (\gamma_{k\,\dot{\Sigma}}, \gamma_{\ell\,\dot{\Sigma}}) = \ldots = \delta_{k\ell} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{16}$$

These basis vectors are linear combinations with real coefficients of the vectors e_i and conversely; considered as Clifford numbers, the satisfy, by virtue of (9) and (16), the relations

$$\gamma_{k\,\Sigma}\gamma_{\ell\,\Sigma} + \gamma_{\ell\,\Sigma}\gamma_{k\,\Sigma} = ... = 2\delta_{k\ell}. \tag{17}$$

Clearly, any Clifford number c attached to the point x can also be expressed with the help of the $\gamma_{k\,\Sigma}, \gamma_{k\,\dot{\Sigma}}, ...$ attached to the same point. (Thanks to formulae (17) and (13), expression (10) will be greatly simplified.)

Note that in the sequence of ideas of Riemannian geometry there does not exist any (integrable) relation between the Lorentz frames attached to two non-adjacent points x and y. Thus, if one of the frames attached to point x is denoted by Σ_x, it does not matter which of the frames attached to the point y is denoted by Σ_y. It is only in the case of frames attached to infinitely close points that we can establish a well defined correspondence.

Let Σ be an arbitrary reference system attached to the point x. (We suppress here the respective indices x and $x + dx$ so as not to encumber our formulae.) The reference system attached to the point $x + dx$ which is obtained from Σ by parallel transport will be denoted by $\Sigma^{\|}$. If the basis vectors of Σ are $\gamma_{k\,\Sigma}$, those of $\Sigma^{\|}$ will be denoted by $\gamma_{k\,\Sigma^{\|}}$. Considered as Clifford numbers, $\gamma_{k\,\Sigma}$ and $\gamma_{k\,\Sigma^{\|}}$ $(k = 0, 1, 2, 3)$ are to be regarded as *identical*; see a convention set down in n$^{\text{o}}$ 2.

4. STABLE ELEMENTS AND ROTORS

Any element of the algebra C, formed at a point x, which is independent of the Lorentz reference system attached to this point will be called a *stable element* of the algebra. Such elements are, for example, the scalar 1, the vectors of the Riemannian basis e_i, their commutators, etc. For reasons that we will see a little later (n$^{\text{o}}$ 7), we say that a stable elements is of *type* $\{\ \}$. Alongside stable elements, we consider also certain *collectives* of Clifford numbers (rotors, spinors) that depend on the frames in question. As an example we can cite the basis vectors $\gamma_{k\,\Sigma}, \gamma_{k\,\dot{\Sigma}}, \gamma_{k\,\ddot{\Sigma}}, ...$

We consider first a polynomial type

$$a(u_0, u_1, u_2, u_3) = a^S + \sum_k a^k u_k + \sum_{k<\ell} a^{k\ell} u_k u_\ell + \sum_{k<\ell<m} a^{k\ell m} u_k u_\ell u_m + a^{0123} u_0 u_1 u_2 u_3, \tag{18}$$

where the u_k are indeterminates that satisfy the relations $u_k u_\ell + u_\ell u_k = 2\delta_{k\ell}$ (similar to (17)) and the coefficients $a^S, a^k, ...$ are arbitrary complex numbers.

If we substitute for the u_k the basis vectors of the Lorentz frames $\Sigma, \dot{\Sigma}, \ddot{\Sigma}, ...$ attached to a *definite point* x, we obtain the elements of the local algebra C_x, namely

$$a_\Sigma = a(\gamma_{0\,\Sigma}, \gamma_{1\,\Sigma}, \gamma_{2\,\Sigma}, \gamma_{3\,\Sigma}), \qquad a_{\dot{\Sigma}} = ..., \qquad a_{\ddot{\Sigma}} = ... \qquad ..., \tag{19}$$

We say that *these elements are of identical structure* with respect to different frames. The *collective* formed by these elements will be called a *rotor* and will be denoted by a. We write

$$a = \{\vdots\, a_\Sigma, a_{\dot{\Sigma}} a_{\ddot{\Sigma}} ... \vdots\} = \{\vdots\, a_\Sigma \vdots\} \tag{20}$$

and say that a rotor is a collective *of the type* $\{\vdots\ \vdots\}$. The above notations will be explained later (n$^{\text{o}}$ 7). The numbers $a_\Sigma, a_{\dot{\Sigma}}, a_{\ddot{\Sigma}} ...$ are called *the components* of the rotor with respect to the frames $\Sigma, \dot{\Sigma}, \ddot{\Sigma} ...$.

All the components of a rotor clearly have the same scalar part. The scalars, and only the scalars, can be considered either as stable elements or as rotors.

Here is another important *extension* of one of the definitions that we have set down. If we substitute for u_k the basis vectors of two arbitrary frames attached to two *different* points x and y, we obtain two elements of the respective algebras C_x and C_y that we always consider as being of *identical structure*.

5. LEFT AND RIGHT (MINIMAL) IDEALS

By a left ideal I in an algebra C we mean a set such that $\eta_1 \in I, \eta_2 \in I$ implies $\eta_1 - \eta_2 \in I$ and such that $\eta \in I, c \in C$ implies that $c\eta \in I$. A similar definition for a right ideal. A left (right) ideal is said to be *minimal* if it does not contain any left (right) ideal except itself and the null ideal, which consists of the single element 0. Until further notice, we will deal only with left ideals; we will return later to right ideals and to certain correspondences between left and right ideals (n$^{\text{o}}$ 11).

To frames $\Sigma, \dot{\Sigma}, \ddot{\Sigma}$... we associate the ideals $I_\Sigma, I_{\dot{\Sigma}}, I_{\ddot{\Sigma}}$... of the algebra C_x. These ideals must be of identical structure, by which we mean that they are formed by elements of identical structure. By putting

$$I_\Sigma = \{\eta_\Sigma, \zeta_\Sigma, \upsilon_\Sigma, ...\} \qquad I_{\dot{\Sigma}} = \{\eta_{\dot{\Sigma}}, ...\} \qquad I_{\ddot{\Sigma}} = \{\eta_{\ddot{\Sigma}}, ...\}, \qquad ... \tag{21}$$

where, to simplify notation, we have suppressed the index x, the numbers $\eta_\Sigma, \eta_{\dot{\Sigma}}, \eta_{\ddot{\Sigma}}, ...$ for example are just the components of the rotor η, that is, the elements of identical structure of C_x.

Ideals belonging to different points x and y should also be of identical structure with respect to the respective frames; *cf.* the extension of the definition of the identity of structure of the elements belonging to the different algebras C_x and C_y (end of n° 4).

We construct all these ideals with the help of the indeterminates u_k, introduced in n° 4. In the algebra type, generated by these indeterminates, we form a (minimal) left ideal. The elements of this ideal will be given by certain polynomials $\eta(u_k), \zeta(u_k), \tau(u_k), ...$ (*cf.* formula (18)). Finally, we substitute for u_k, just as we did above, the basis vectors of the respective frames. All these ideals are minimal or not at the same time.

Trivial examples of ideals: the entire algebra and the null ideal; for non trivial examples, see my "Conference" and Note 1 of the present work.

While the complete algebra has 16 basis elements, a (non null) minimal ideal only has 4. The unknown ψ in Dirac's equation will be with us, in a certain sense, an element of a minimal ideal. It can then be given, as with Dirac, by four numerical (complex valued) functions; *cf.* n° 10 and Note 1.

6. THE SUBGROUP OF PROPER LORENTZ ROTATIONS

We will study more closely the Lorentz transformations which take the Lorentz frames attached to a given point x into one another. These transformations obviously form a subgroup, denoted by L, of the group of Lorentz transformations that operate on the local vector space attached to the point x. The subgroup L is made up of those elements of the entire group which are connected to the identity. We can further characterise this subgroup as consisting of the proper Lorentz rotations, that are only the Lorentz transformations which preserve the orientation of space as well as the past and the future.

Consider an arbitrary element ℓ of the subgroup of proper Lorentz rotations L. The rotation ℓ, having a clear sense for any vector of C_x, we can extend ℓ such that it operates on the entire algebra C_x by the conventions $\ell(c'c'') = \ell c' \cdot \ell c''$ and $\ell(\lambda' c' + \lambda'' c'') = \lambda' \, \ell c' + \lambda'' \, \ell c''$, set down for two arbitrary elements c' and c'' of the algebra C_x and two equally arbitrary complex numbers λ' and λ''. The above conventions show that ℓ generates an automorphism of the algebra C_x.

We can show[7] that to any transformation ℓ there corresponds a real element R_ℓ of C_x, *determined up to sign*, such that $R_\ell^{-1} = \tilde{R}_\ell$ (see n° 2 for this notation and for the notation $\hat{R}_\ell$ which follows) and that we have for each vector $U \in C_x$, $\ell U = R_\ell^{-1} U R_\ell$. It follows, for an arbitrary element c of C_x, that

$$\ell c = R_\ell^{-1} c R_\ell. \tag{22}$$

Note that since R_ℓ is real, we have $\tilde{R}_\ell = \hat{R}_\ell$, that is

$$R_\ell^{-1} = \tilde{R}_\ell = \hat{R}_\ell. \tag{23}$$

The R_ℓ obviously form a group; it is $R_{\ell'} R_{\ell''}$ that corresponds to the transformation $\ell'' \ell'$.

Formula (22) emphasises again the fact that all transformations ℓ provide an automorphism of the algebra C_x.

We add that for any number R_ℓ there exists a real bivector[8] F_ℓ (actually, there exist in general an infinity) such that

$$R_\ell = e^{\frac{1}{2} F_\ell}. \tag{24}$$

Conversely, if F is an arbitrary real bivector, $R = e^{\frac{1}{2} F}$ will have the properties 1) $R^{-1} = \tilde{R}$ (which follows from $-F = \tilde{F}$), 2) the transformation $U \to R^{-1} U R$ is a proper Lorentz rotation.

We note furthermore that in the case where ℓ is an *infinitesimal rotation* — the most important case in what follows — we can rid ourselves of the bivalence of R_ℓ and of the multivalence of F_ℓ by limiting ourselves to that of the two values of R_ℓ that is closer to the identity and to that of F_ℓ which is infinitesimal. That said, R_ℓ and F_ℓ will be completely determined by ℓ.[9] See also Note III.

7. RETURN TO LOCAL LORENTZ FRAMES

Consider two arbitrary frames attached to a definite point x and denoted by $\dot{\Sigma}$ and $\ddot{\Sigma}$. There exists a well defined rotation ℓ that takes $\dot{\Sigma}$ to $\ddot{\Sigma}$ and which will be denoted by $\ell_{\ddot{\Sigma}/\dot{\Sigma}}$. (We will not be alarmed by the obvious fact that this rotation corresponds to an infinity of pairs of frames, where one can be chosen arbitrarily.) To abbreviate, we shall write in the following ℓ in place of $\ell_{\ddot{\Sigma}/\dot{\Sigma}}$.

The formulae

$$\gamma_{k\ddot{\Sigma}} = \ell\gamma_{k\dot{\Sigma}}, \qquad \ddot{\Sigma} = \ell\dot{\Sigma} \tag{25}$$

need not be explained. The first of these formulae implies

$$a_{\ddot{\Sigma}} = \ell a_{\dot{\Sigma}}, \tag{26}$$

where $a_{\dot{\Sigma}}$ and $a_{\ddot{\Sigma}}$ are the respective components with respect to the frames $\dot{\Sigma}$ and $\ddot{\Sigma}$ of the rotor a.

The bivalent number R_ℓ which corresponds to ℓ will be denoted by $R_{\ddot{\Sigma}/\dot{\Sigma}}$ or more briefly by R. By virtue of formulae (25) and (22), we have

$$\gamma_{k\ddot{\Sigma}} = R^{-1}\gamma_{k\dot{\Sigma}}R \tag{27}$$

or in condensed form

$$\ddot{\Sigma} = R^{-1}\dot{\Sigma}R.$$

Finally, by virtue of formulae (25) and (22),

$$a_{\ddot{\Sigma}} = R^{-1}a_{\dot{\Sigma}}R. \tag{28}$$

This point established, it is easy to explain the notation $\{\vdots a_\Sigma \vdots\}$ (formula(20)) and the phrase "of type $\{\vdots a_\Sigma \vdots\}$". The two columns of points indicate that the passage $a_{\dot{\Sigma}} \to a_{\ddot{\Sigma}}$, that is the passage from one component of a rotor to another, can be obtained by two multiplications, the one by R^{-1} from the left and the other by R from the right. We write symbolically

$$\{\vdots ... \vdots\} = \{R^{-1}...R\}. \tag{29}$$

That said, the statement that any stable elements (that is, any element independent of the Lorentz reference system (n° 4)) is of type { } explains itself.

Formula (28) follows from here, but it follows also from the definition of a rotor which, if we define the product of two rotors $\alpha = \{\vdots \alpha_\Sigma \vdots\}$ and $\beta = \{\vdots \beta_\Sigma \vdots\}$ by $\alpha\beta = \{\vdots \alpha_\Sigma\beta_\Sigma \vdots\}$, we will have

$$\text{rotor} \times \text{rotor} = \text{rotor} \quad \text{or} \quad \{\vdots ... \vdots\} \times \{\vdots ... \vdots\} = \{\vdots ... \vdots\}. \tag{30}$$

Note in passing that if we complete the above definition by the definition $\alpha - \beta = \{\vdots(\alpha_\Sigma - \beta_\Sigma)\vdots\}$, we see that the rotors attached to a point x form an algebra Γ_x. Any mapping $\alpha \to \alpha_\Sigma$ (where Σ is given) establishes an isomorphism between then algebras Γ_x and C_x.

In summary, if $\ell \in L$, we have for two *stable elements* c' and c'', $\ell(c'c'') = \ell c' \cdot \ell c''$, for two *rotors* α' and α'', $\ell(\alpha'\alpha'') = \ell\alpha' \cdot \ell\alpha''$; in contrast, for a stable element c and a rotor α, $\ell(c\alpha) \neq \ell c \cdot \ell\alpha (= c \cdot \ell\alpha)$, except the case where c and α are scalars. This means that the product stable elements × rotor does not have a natural transformation law. But in the following we will need collectives ψ such that $\ell(c\psi) = c \cdot \ell\psi$; That is, it is an isomorphism with operators, according to the usual algebraic terminology. The reason is that the invariant differential operator $\nabla = e^j\partial_j$ does not depend on the Lorentz frames.

8. (LEFT) SPINORS

As above, we attach left ideals $I_\Sigma, I_{\dot{\Sigma}}, I_{\ddot{\Sigma}}, ...$ to Lorentz frames $\Sigma, \dot{\Sigma}, \ddot{\Sigma}, ...$ (In the applications we shall restrict ourselves to minimal ideals, for the reasons discussed above (n° 5).) Formula (21) establishes by its very appearance a correspondence by elements between the ideals I, such that the corresponding collectives of elements are the rotors. With an easy interpretation of formula (28) we can write

$$I_{\ddot{\Sigma}} = R^{-1}I_{\dot{\Sigma}}R \quad \text{where} \quad R = R_{\ddot{\Sigma}/\dot{\Sigma}}, \tag{31}$$

Since all the I's are left ideals, we also have

$$I_{\ddot{\Sigma}} = I_{\dot{\Sigma}} R \quad \text{since} \quad R^{-1} I_{\dot{\Sigma}} = I_{\dot{\Sigma}}. \tag{32}$$

This last formula defines a new correspondence by elements between the ideals $I_\Sigma, I_{\dot{\Sigma}}, I_{\ddot{\Sigma}}, \ldots$. If we take an element from each ideal, these elements corresponding with each other in the new way (formula (32)), we obtain a collective of elements called a *(left) spinor*.

In more detail, formula (32) can be written as

$$I_\Sigma = \{\psi_\Sigma, \varphi_\Sigma, \ldots\}, \quad I_{\dot{\Sigma}} = \{\psi_{\dot{\Sigma}}, \varphi_{\dot{\Sigma}}, \ldots\}, \quad I_{\ddot{\Sigma}} = \{\psi_{\ddot{\Sigma}}, \varphi_{\ddot{\Sigma}}, \ldots\}, \quad \ldots,$$

where, for example,

$$\psi_{\ddot{\Sigma}} = \psi_{\dot{\Sigma}} R, \quad \varphi_{\ddot{\Sigma}} = \varphi_{\dot{\Sigma}} R, \quad \ldots, R = R_{\ddot{\Sigma}/\dot{\Sigma}}, \tag{33}$$

for an arbitrary pair $\dot{\Sigma}$ and $\ddot{\Sigma}$. The elements $\psi_\Sigma, \psi_{\dot{\Sigma}}, \psi_{\ddot{\Sigma}}, \ldots$ are called *components* of the spinor ψ with respect to the frames $\Sigma, \dot{\Sigma}, \ddot{\Sigma}, \ldots$. We write

$$\psi = \{\psi_\Sigma, \psi_{\dot{\Sigma}} \psi_{\ddot{\Sigma}} \vdots\} = \{\psi \vdots\}, \tag{34}$$

and we say that the collective ψ is *of type* $\{\ \vdots\}$, a notation in good agreement with the conventions set down above. We define the *addition* and the *subtraction* of spinors by

$$\psi \pm \varphi = \{\psi_\Sigma \pm \varphi_\Sigma \vdots\} = \text{ spinor}. \tag{35}$$

If the element c is *stable* and ψ is a *spinor* we put $c\psi = \{(c\psi_\Sigma)\vdots\}$, which is to say that stable element $\times$ spinor $=$ spinor, or symbolically $\{\ \} \times \{\ \vdots\} = \{\ \vdots\}$.

If ψ is a *spinor* and α a rotor we put $\psi\alpha = \{(\psi_\Sigma \alpha_\Sigma)\vdots\}$, which is to say that spinor $\times$ rotor $=$ spinor, symbolically $\{\ \vdots\} \times \{\vdots\ \vdots\} = \{\ \vdots\}$.

The products rotor $\times$ spinor and spinor $\times$ spinor are of no interest.

Transformation of spinors. The result of this paragraph will not be applied anywhere. Let $\psi = \{\psi_\Sigma, \psi_{\dot{\Sigma}}, \psi_{\ddot{\Sigma}}, \ldots\vdots\}$ be a spinor and $\ell \in L$ an automorphism of the Clifford algebra C_x. By $\ell\psi$ we mean the set $(\ell\psi_\Sigma, \ell\psi_{\dot{\Sigma}}, \ell\psi_{\ddot{\Sigma}}, \ldots)$ interpreted in a suitable way. The transformation $\psi_\Sigma \mapsto \ell\psi_\Sigma$ can be written as $(\psi_\Sigma \to R_\ell^{-1}\psi_\Sigma R_\ell)$ and can be implemented in two steps: $(\psi_\Sigma \to \psi_\Sigma R_\ell)$ and $(\psi_\Sigma R_\ell \to R_\ell^{-1}\psi_\Sigma R_\ell)$. The first step $(\psi_\Sigma \to \psi_\Sigma R_\ell)$ transforms the set (ψ_Σ) into itself however by permuting the ψ_Σ. The second step multiplies by R_ℓ^{-1} from the left all of the elements of the set; all this can be written $(\ell\psi) = R_\ell^{-1}(\psi_\Sigma)$. We conclude that the elements of the set $(\ell\psi)$ are identical to the components of the spinor $R_\ell^{-1}\psi$. We write symbolically $\ell\psi = R_\ell^{-1}\psi$, which is in agreement with the law of transformation of spinor columns.

Differentiation in a spinor field

At each point x of the Riemannian space we choose among all the Lorentz systems of reference attached to this point an arbitrary system of reference, denoted by Σ_x. The basis vectors of Σ_x are expressed through the Riemannian vectors $e_j(x)$ with the help of a matrix a_{ij}. The set of these systems of reference is called a *field of systems of reference*, if the matrix a_{ij} is sufficiently differentiable.

If a spinor $\psi(x)$ is attached to each point x, we will say that these spinors form a field of spinors, if the component ψ_{Σ_x}, described in the form (10), admits (sufficiently) differentiable coefficients, and this for any field of systems of reference Σ_x. As regards the existence and the construction of such fields *cf.* n^o 10.

According to the definition given in n^o 3 we mean by $\Sigma^{\|}_{x+dx}$, or more briefly $\Sigma^{\|}$, the system of reference arising from Σ_x, or more briefly from Σ, by parallel transport $(x \to x + dx)$. We put by definition

$$d\psi_\Sigma(x) = \psi_{\Sigma^{\|}}(x + dx) - \psi_\Sigma(x) \text{ and } d\psi(x) = \{d\psi_\Sigma(x)\vdots\}. \tag{36}$$

The notations are legitimate and $d\psi_\Sigma(x)$ is a spinor because $d\psi_\Sigma(x) \in I_\Sigma$ and $d\psi_{\ddot{\Sigma}}(x) = d\psi_{\dot{\Sigma}}(x) \cdot R_{\ddot{\Sigma}/\dot{\Sigma}}$.

9. THE DIRAC EQUATION

By formula (36), the partial derivatives $\partial_i\psi(x)$ are defined at the same time. The Dirac equation for a particle of rest mass m, of charge e and in the presence of an electromagnetic field given by the vector potential $A = A^j e_j = A_j e^j$ has the same appearance as in Dirac, namely

$$(\nabla - ieA - im)\psi = 0, \tag{37}$$

of, more explicitly, $(e^j(\partial_j - ieA_j) - im)\psi$.

Of course, it is necessary to interpret the entire equation and in particular the differentiation of the spinor ψ in the way explained above.

10. ACTUAL CALCULATION

At each point x of the Riemannian space we consider a *particular* Lorentz system of reference, that is Σ_x, where these systems of reference form a *field of reference systems* according to the definition we have just given. We write, to abbreviate, $\Sigma_x = \Sigma$ and $\Sigma_{x+dx} = \Sigma^*$. During a parallel transport corresponding to $x \to x + dx$ we will pass from $\Sigma \to \Sigma^{\|}$. If the systems of reference Σ^* and $\Sigma^{\|}$ is attached to the same point $x + dx$, we can form $R_{\Sigma^{\|}/\Sigma^*}$ and we have $\Sigma^{\|} = R^{-1}\Sigma^* R$.

We specify a spinor $\psi(x)$ by its components ψ_{Σ_x} or more briefly ψ_Σ. We thus arrive at a field of spinors. We have $\psi_{\Sigma^{\|}}(x + dx) = \psi_{\Sigma^*}(x + dx)R$.

Bases of the ideals

We introduce in each ideal I_{Σ_x} basis elements $\sigma_1(x), \sigma_2(x), ...$, where the basis elements belonging to different ideals are of identical structure. We can write $\psi_{\Sigma_x}(x) = \sum a_p(x)\sigma_p(x)$, where the $a_p(x)$ are complex numbers. With the conventions laid down at the end of n° 3, we identify the basis elements σ_p and $\sigma_p^{\|}$ relating to the ideals attached to parallel systems of reference Σ and $\Sigma^{\|}$.

Let $\psi_\Sigma = \sum a_p\sigma_p$ and $\psi_{\Sigma^*} = \sum b_p\sigma_p^*, a_p = a_p(x)$ and $b_p = a_p(x + dx)$. We want to calculate $\psi_{\Sigma^{\|}}$. We have $\psi_{\Sigma^{\|}} = \psi_{\Sigma^*}R = \sum b_p\sigma_p^* R = \sum b_p(R\sigma_p^{\|}R^{-1})R = \sum b_p(R\sigma_p R^{-1})R = \sum b_p R\sigma_p = R\sum b_p\sigma_p$.

Representation of spinors and Clifford numbers

Let $\psi_\Sigma \in I_\Sigma$ be a component of the spinor ψ, where $\sigma_1, \sigma_2, ...$ is a basis of I_Σ, we can write $\psi_\Sigma = \sum a_p\sigma_p$, which gives a representation of $\psi_\Sigma = \sum a_p\sigma_p$, which gives a representation of ψ_Σ by the column

$$\begin{pmatrix} \vdots \\ a_p \\ \vdots \end{pmatrix}_\Sigma .$$

Let $c \in C_x$ be any Clifford number. From $c\sigma_k = \sum_j c_{jk}\sigma_j$ we obtain a representation of c by the matrix $(c_{jk})_\Sigma$.

The product $c\psi_\Sigma$ is represented by

$$\begin{pmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \\ \vdots & & \end{pmatrix}_\Sigma \begin{pmatrix} \vdots \\ a_p \\ \vdots \end{pmatrix}_\Sigma .$$

If the ideal I_Σ is minimal, the number of elements σ_p as well as the order of the matrices (c_{jk}) is equal to 4. That said, the matrices with arbitrary complex elements provide a faithful representation of the algebra C_x.

If

$$\psi_\Sigma(x) = \begin{pmatrix} \vdots \\ a_p(x) \\ \vdots \end{pmatrix}_\Sigma \quad \text{and} \quad \psi_{\Sigma^*}(x + dx) = \begin{pmatrix} \vdots \\ a_p(x + dx) \\ \vdots \end{pmatrix}_{\Sigma^*} ,$$

we have, by means of the result obtained earlier,

$$\psi_{\Sigma\|} = R \begin{pmatrix} \vdots \\ a_p(x+dx) \\ \vdots \end{pmatrix}_{\Sigma^*} .$$

According to the previous formulae,

$$d\psi_\Sigma(x) = \psi_{\Sigma\|}(x+dx) - \psi_\Sigma(x) = R \begin{pmatrix} \vdots \\ a_p(x+dx) \\ \vdots \end{pmatrix}_{\Sigma} - \begin{pmatrix} \vdots \\ a_p(x) \\ \vdots \end{pmatrix}_{\Sigma} .$$

We write according to formula (24) $R = e^{\frac{1}{2}F} = 1 + \frac{1}{2}F + \cdots$ where F is a real infinitesimal bivector of the form $F = \sum_{i=0}^{3} F_i dx^i$ where the F_i real bivectors that depend on x.

Therefore

$$d\psi_{\Sigma_x}(x) = \begin{pmatrix} \vdots \\ da_p(x) \\ \vdots \end{pmatrix}_{\Sigma_x} + \frac{1}{2}F \begin{pmatrix} \vdots \\ da_p(x) \\ \vdots \end{pmatrix}_{\Sigma_x} .$$

In the above formulae, R and F are the matrices that represent the Clifford numbers to which they correspond.

11. RIGHT SPINORS

A right spinor φ is a collective of the form

$$\varphi = \{\vdots\varphi_\Sigma, \varphi_{\dot\Sigma}, \varphi_{\ddot\Sigma}, ...\} = \{\vdots\varphi_\Sigma\}, \tag{38}$$

where $\varphi_{\ddot\Sigma} = R^{-1}_{\ddot\Sigma/\dot\Sigma}\varphi_{\dot\Sigma}$. Such a spinor is a collective of the type $\{\vdots\ \}$. To a left spinor $\psi = \{\psi_\Sigma\vdots\}$ there corresponds a right spinor $\widehat{\psi} = \{\vdots\widehat{\psi}_\Sigma\}$ (*see* n° 2) and *vice versa*.

We have clearly

$$\begin{aligned} \{\vdots\ \} \times \{\ \ \} &= \{\vdots\ \} \\ \{\vdots\ \vdots\} \times \{\vdots\ \} &= \{\vdots\ \} \\ \{\ \vdots\} \times \{\vdots\ \} &= \{\ \ \} \\ \{\vdots\ \} \times \{\ \vdots\} &= \{\vdots\ \vdots\} \qquad (\textit{cf.}\ \text{n}^\circ\ 8). \end{aligned} \tag{39}$$

Besides $\widehat{\psi}$, deduced from ψ, we also form the right spinor (cf. (39b))

$$\psi^\dagger = \gamma_0\hat{\psi}, \tag{40}$$

where $\gamma_0 = \{\vdots\gamma_{0\Sigma}\vdots\}$ is a rotor. From equation (37) it follows that, for a conjugate reversal (n° 2), $\widehat{\psi}((\partial_j + ieA_j)e^j + im) = 0$ whence

$$\psi^\dagger((\overleftarrow{\partial}_j + ieA_j)e^j + im) = 0. \tag{41}$$

This last move is legitimate, because $d\psi^\dagger = d(\gamma_0\widehat{\psi}) = \gamma_0 d\widehat{\psi}$, seeing that $d\psi^\dagger_\Sigma = d(\gamma_{0\Sigma}\widehat{\psi}_\Sigma) = \gamma_{0\Sigma\|}\widehat{\psi}_{\Sigma\|} - \gamma_{0\Sigma}\widehat{\psi}_\Sigma = \gamma_{0\Sigma}(\widehat{\psi}_{\Sigma\|} - \widehat{\psi}_\Sigma) = \gamma_{0\Sigma}d\widehat{\psi}_\Sigma$. We call equation (41) the *adjoint* to (37).

12. THE DIRAC CURRENT VECTOR

We form the product

$$\psi\psi^\dagger = \cdots + j^i e_i + \cdots \tag{42}$$

where only the vector part is written explicitly. According to (39c), this product is a stable element, which implies that the contravariant components j^i of the current vector are independent of the Lorentz reference systems while transforming in the usual way during Riemannian transformations.

We get easily[10]

$$j^i = (\psi\psi^\dagger e^i)^S = (\psi^\dagger e^i \psi)^S. \tag{43}$$

We deduce from inequality (14) that j^0 is real and > 0. It will be the same in all admissible Riemannian systems, which leads to j being a positive timelike vector.

Finally we see easily that

$$\text{div } j = \frac{1}{\sqrt{|g|}}\,\partial_j(\sqrt{|g|}\, j^i) = 0, \qquad \text{where } g = |g_{ij}|. \tag{44}$$

This follows from equations (37) and (41) completed by the formula $\partial_i(\sqrt{|g|}\, e^i) = 0$. This is a consequence of (6) since ${\Gamma_{ir}}^i = \partial_r(\ln\sqrt{|g|})$.

NOTE I. EXAMPLES OF MINIMAL IDEALS

The first example corresponds to the representation given by v. d. Waerden and Élie Cartan.

Let u_0, u_1, u_2, u_3 be indeterminates that satisfy the relations $u_j u_k + u_k u_j = 2\delta_{jk}$ (*Cf.* (18)). They generate a Clifford algebra C. Form

$$\mu_1 = \frac{iu_1 + u_2}{2}, \qquad \mu_1' = \frac{iu_1 - u_2}{2}, \qquad \mu_2 = \frac{u_0 + u_3}{2}, \qquad \mu_2' = \frac{u_0 - u_3}{2}.$$

These new quantities satisfy, for their part, the relations

$$\mu_j\mu_k + \mu_k\mu_j = \mu_j'\mu_k' + \mu_k'\mu_j' = 0,$$
$$\mu_j\mu_k' + \mu_k'\mu_j = \varepsilon_{jk}, \qquad \text{where } (\varepsilon_{jk}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We write furthermore the following relations, which are immediate consequences of those which precede,

$$\mu_j\mu_k'\mu_j = \varepsilon_{jk}\mu_j, \qquad \mu_j'\mu_k\mu_j' = \varepsilon_{jk}\mu_j'.$$

Let us form $N_1 = \mu_1\mu_1', N_2 = \mu_2\mu_2'$ and $\sigma_0 = N_1N_2 = N_2N_1$. We have $N_1^2 = N_1, \sigma_0^2 = \sigma_0$. From $(\sigma_0)^S = \frac{1}{4}$ follows that $\sigma_0 \neq 0$. Consider the left ideal generated by σ_0, that is, the ideal $C\sigma_0$. We verify that this ideal is minimal. Note first that, if c is any Clifford number, $\sigma_0 c\sigma_0 = \lambda\sigma_0$, where λ is a scalar. This is an immediate consequence of the fact that $\sigma_0^2 = \sigma_0$ and that $\sigma_0\pi\sigma_0 = 0$ when π is a product of the μ_j and the μ_j' except in the case where π is equal to $\pm N_1$ or $\pm N_2$ or $\pm N_1N_2$, in which case $\sigma_0\pi\sigma_0 = \pm\sigma_0$.

We have need also of

Lemma 1. *Let $a\sigma_0 \neq 0$ be any element of the left ideal generated by σ_0. There exists then a Clifford number b such that $ba\sigma_0 = \sigma_0$.*

Observe ((*cf.* inequality (14)) that $(u_0\widehat{\sigma}_0\widehat{a}u_0a\sigma_0)^S > 0$, which gives, on noting that $u_0\widehat{\sigma}_0u_0 = \sigma_0, 0 \neq u_0\widehat{\sigma}_0\widehat{a}u_0a\sigma_0 = (u_0\widehat{\sigma}_0u_0)(u_0\widehat{a}u_0)a\sigma_0 = \sigma_0(u_0\widehat{a}u_0)a\sigma_0 = \lambda\sigma_0$, the scalar $\lambda \neq 0$. We can put $b = \frac{1}{\lambda}\sigma_0u_0\widehat{a}u_0$. This shows that the ideal can be generated by any one of its non-zero elements and it is, consequently, a minimal ideal.

The numbers $\sigma_0, \sigma_1 = \mu_1'\sigma_0, \sigma_2 = \mu_2'\sigma_0, \sigma_3 = \mu_1'\mu_2'\sigma_0$ and 0 transform into each other, up to a sign, by multiplication from the left with $1, \mu_i$ or μ_i'. The elements $\sigma_0, ..., \sigma_3$ are linearly independent and we conclude that they form a basis of the ideal $C\sigma_0$.

We can confine ourselves to showing the linear independence of the σ_j. Let $\Lambda = \lambda_0\sigma_0 + \lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3 = 0$. Then $\sigma_0\Lambda = \lambda_0\sigma_0 = 0, \mu_1\mu_2\Lambda = -\lambda_3\sigma_0 = 0$, which gives $\lambda_0 = \lambda_3 = 0$. This being so, $\mu_1\Lambda = \lambda_1\sigma_0 = 0, \mu_2\Lambda = \lambda_2\sigma_0 = 0$, which gives $\lambda_1 = \lambda_2 = 0$.

Here is a second example of a minimal ideal which corresponds to the representation given by Dirac. As basis elements we choose

$$\sigma_0 = \frac{(1+u_0)(1+iu_1u_2)}{2}, \qquad \sigma_1 = u_1\sigma_0, \qquad \sigma_2 = u_3\sigma_0, \qquad \sigma_3 = u_1u_3\sigma_0.$$

For some details *see* our "Conference", p. 144-147 and A. Sommerfeld, Atombau und Spektrallinien, t. II, Braunschweig 1939. p. 217-263, in particular, p. 249.

NOTE II. SOME COMMENTS CONCERNING THE DIRAC EQUATION

In a flat space, the spacetime of special relativity, for example, our method reduces immediately to that of Dirac. At each point, choose a single local Lorentz reference system (*cf.* n^o 10), parallel to the Lorentz reference system to which the space is referred. This done, we can neglect the local frames. Afterwards, we identify all the local Clifford algebras.

Here is a note concerning the Maxwell equations and a comparison of these equations with the Dirac equation. At each point x of the spacetime of general relativity we form the local Clifford algebra C_x. From then on, the Maxwell equation can be condensed in a single equation

$$\nabla F = -s,$$

where $\nabla = e^j\partial_j$, $F \in C_x$ is a bivector representing the electromagnetic field and $s \in C_x$ is a vector, the vector current given in advance, which must satisfy the condition div $s = 0$.

To simplify, consider for an instant the case of special relativity. During a Lorentz rotation $x \to x'$, $\nabla F = -s$ transforms into $\nabla' F' = -s'$, where the transformation follows the tensor laws. The two equalities are correct at the same time. The transformation in question can thus be obtained by multiplying the first equation by R^{-1} from the left and by R from the right (with a suitable choice of R). If, in place of the bilateral multiplication, we implement only an unilateral multiplication (for example by multiplying from the left by R^{-1}), we again arrive at a correct equality, but thios one is of no interest, since the tensor character of our quantities is destroyed.

In Dirac, the operator $\nabla - ... \to R^{-1}(\nabla - ...)R$ and $\psi \to R^{-1}\psi$, where the transformation is correct. It would be the same if we applied to ψ a bilateral multiplication. However, the spinor ψ is an element of a (minimal) left ideal and not a tensor and it is all about staying in the same ideal. For this reason, it is the first transformation and not the second that we must choose.

We move on now to a comparison between our scheme and that of Dirac as regards the transformation laws. Dirac transforms the coordinates $x^k \to x^{k'}$, whereas the γ^k remain unaltered. This shows that for him $\nabla = \gamma^j\partial_j$ is a rotor. For us, where $\nabla = \gamma^j\partial_j$, the whole being Riemannian, it is natural not to rotate the e^j for example (which nevertheless would be possible), but to consider ∇ as a stable element. All the same, the problem of the specific transformation of spinors arises for us in a relatively natural way: *cf.* n^o 8, *Transformation of spinors.*

NOTE III. ON THE BIVALENCE OF R_ℓ

We can, to some extent, eliminate the bivalence of the R_ℓ by restricting them, at each point x, to a connected bundle of reference systems that vary continuously with x. We assume that the mutual deviation of the reference systems belonging to the same bundle is very small so that the scalar part of $R_{\ddot\Sigma/\dot\Sigma}$ is not zero for any pair $\dot\Sigma, \ddot\Sigma$. Under these conditions, we will denote by $R_{\ddot\Sigma/\dot\Sigma}$ that of the two possible values whose scalar part is positive. The R's determined in this way vary continuously with $\dot\Sigma$ and $\ddot\Sigma$.

We can arrive at a complete elimination of the bivalence by constructing a covering group of the group of Lorentz rotations. This covering group is isomorphic to the group of the R_ℓ. I will return to this issue on another occasion.

NOTE IV. TENSORS, ROTORS, SPINORS

The tensors attached to different points of a Riemannian space display certain properties of the structure of the space. Thus the possibility of passing from a given system of coordinates to an arbitrary system, or of contravariant components to covariant components and conversely, or also the possibility of infinitesimal parallel transport. The collectives introduced above, namely those of rotors and of spinors, do more, since they display moreover the following property of Riemannian spaces. Such a space can, at each of its points, be equipped with a family of local frames, where the frames attached to the same point or to different points are equal to each other. The family of frames attached to the same point is invariant with respect to the group of Lorentz rotations. The families attached to nearby points can be brought into coincidence by parallel transport.

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[1] *Cf.* Leçons sur la géométrie des espaces de Riemann, 2[e] éd., Paris 1946, p. 31 ff.

[2] , *Cf.* Cartan *l.c.* p. 35.

[3] For terminology and for different facts concerning Clifford numbers, *see* our conference made at the 10[th] Congress of Scandinavian Mathematicians held at Copenhagen 1946 (København 1947); cited in the following as Conference. Note however that, in this conference, spinors are defined as elements of the Clifford algebra, whereas here they are defined as collectives of such elements.

[4] *Cf.* the note in n[o] 3.

[5] The exposition given here is taken from my Conference. One of the notations is slightly changed.

[6] We assume here that it is a general Riemannian space and not a flat space, such as the space-time of special relativity.

[7] See Cartan, Spinors I, n[os] 10, 58 and II, n[os] 96-98, and Conference, p. 136-137.

[8] An element F of C_x is called a real bivector if $F = \frac{1}{2}F^{ij}e_ie_j$, $F^{ij} - F^{ji}$ and F^{ij} real.

[9] To prove (24) in the infinitesimal case, see Cartan, Spinors, I, n[o] 19. In the general case, the proof is very delicate; I think that I shall return to it in another work.

[10] It is of some interest to note that $\psi^\dagger c^i \psi$ is a rotor (by virtue of formulae (39)). According to a remark made in n[o] 4, the scalar part of a rotor, in the actual case j^i, is independent of the Lorentz frame.