On Some Fundamental Concepts in Relativistic Quantum Theory.

By

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Sur certaines notions fondamentales en théorie quantique relativiste.

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Translator

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INTRODUCTION

We know since the work of FOURIER and of CAUCHY that algebraic relations and differential equations with constant coefficients can, to some extent, be translated into each other through the FOURIER integral. Such correspondences are encountered frequently in quantum theory, and we will give some examples in the following.

SPACETIME

If in the four-dimensional Lorentz spacetime (spacetime or universe), the speed of light is equal to 1, the scalar square (x,x) of the radius vector x from the origin to the point $x = (x^0, x^1, x^2, x^3)$ is measured by the metric form

$$(x,x) = {x^0}^2 - {x^1}^2 - {x^2}^2 - {x^3}^2 = \sum_{jk=0}^{3} g_{jk} x^j x^k; \tag{1}$$

$$g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1;\ g_{jk} = 0,\ j \neq k.$$

By adopting the convention due to EINSTEIN, namely that, unless otherwise stated, a summation must be carried out for all pairs of identical indices (one upper, the other lower), we can also write

$$(x,x) = g_{jk} x^j x^k.$$

The coordinate x^0 corresponds to time, while x^1, x^2, x^3 are the space coordinates. By a *Lorentz transformation* we mean any linear transformation of the coordinates which preserves the metric form. If U is a vector with components $U^j\,(j = 0,1,2,3)$, we define the scalar square (U,U) by the formula, similar to (1), $(U,U) = g_{jk} U^j U^k$. If the coordinates are subjected to a LORENTZ transformation, the same transformation must be applied to the components U^j of a vector U, which implies that such a transformation conserves also the scalar square of an arbitrary vector. The components u^j are called the *contravariant* components of the vector U. We associate them with the *covariant* components, defined by the formulae $U_j = g_{jk} U^k$. To keep the symmetry, we put at the same time $U^j = g^{jk} U_k$, where, in this particular special case, $g^{jk} = g_{jk}$. In summary, the scalar square of a vector U can be written

$$(U,U) = g_{jk} U^j U^k = g^{jk} U_j U_k = U^j U_j. \tag{2}$$

A vector with real components is said to be 1) a *timelike* vector, 2) a *lightlike* vector of an *isotropic* vector, 3) a *spacelike* vector, according as its scalar square is 1) positive, 2) zero, 3) negative. A vector with scalar square ± 1 is called *unitary.* A timelike vector is said to be *positive* or *negative* according as its time component U^0 is positive or negative. The given definitions ate invariant with respect to LORENTZ transformations, the last with respect only to those which do not swap the two light semi-cones.

We define the scalar product (U,V) of two vectors U and V by the formula

$$(U,V) = (V,U) = g_{jk} U^j V^k = g^{jk} U_j V_k = U^j V_j = U_j V^j. \tag{3}$$

It is clear that this product is still invariant with respect to all LORENTZ transformations. The transformation of the covariant components has to be contragredient to that acting on the contravariant components. If, for one reason or another, an expression of the form $U^\mu V_\mu$ is invariant and we know that the U^μ are the contravariant components of a certain vector U, we can conclude that the V_μ are on their side the covariant components of a certain vector V. Applying this in particular to the total differential $df = \dfrac{\partial f}{\partial x^k}\, dx^k$ of a function f, we see that the quantities $\dfrac{\partial f}{\partial x^k}$ are the covariant components[2] $\mathrm{grad}_k f$ of the gradient of f, whose contravariant components are consequently,

$$\mathrm{grad}^j f = g^{jk} \frac{\partial f}{\partial x^k}. \tag{4}$$

Two vectors whose scalar product is zero are said to be *orthogonal.*

THE ENERGY-MOMENTUM SPACE

These preliminaries laid down, consider a material point $x = (x^0, x^1, x^2, x^3)$ with rest mass m which describes a world line. The infinitesimal displacement dx will be according to relativistic concepts (any mechanical speed is less than the speed of light) a timelike vector. The square of the element of arc ds (element of proper time of the particle) is given by the relation $ds^2 = (dx, dx) > 0$. Without specifying the positive direction of s, we form the four-velocity vector $v = dx/ds$ and the energy-momentum vector $p = m \cdot dx/ds$, which are both timelike vectors, with scalar squares $(v, v) = 1,\ (p, p) = m^2$ respectively.

That being so, we associate with the space x (spacetime) the space p (energy-momentum space). Only those points p will enter into our considerations, of a relativistic nature, which satisfy the condition $(p, p) > 0$ (or, at the limit, the condition $(p, p) = 0$). Geometrically speaking, we restrict ourselves to the interior (or to the boundary) of the light cone $(p, p) = 0$.

To the hypothesis, admitted in the following, namely that the rest mass has a given value m, there corresponds the algebraic relation

$$(p, p) = p_k p^k = m^2. \tag{5}$$

This relation means in geometric language that the point p is found on the two-sheeted hyperboloid H_m, whose equation is given by (5). Of course, this hyperboloid is a three dimensional manifold whose two sheets correspond respectively to positive and negative values of the energy p^0.

FOURIER'S INTEGRAL AND SCHRÖDINGER'S RELATIVISTIC EQUATION

That said, and equating $h/2\pi$ (h is PLANCK's constant) to 1, we form the integral

$$\psi(x) = \frac{1}{(\sqrt{2\pi})^3} \int_{H_m} \varphi(p) e^{i(p,x)} d\sigma, \tag{6}$$

where this integral is over the hyperboloid (5) and $d\sigma$ is an element of integration, attached to this surface, to which we shall return just now. As for the function $\varphi(p)$, it is sufficient to define it on H_m. Our integral provides the following translation of (5). By introducing the wave operator

$$\Box = \frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} - \frac{\partial^2}{\partial {x^2}^2} - \frac{\partial^2}{\partial {x^3}^2} = g^{jk} \frac{\partial^2}{\partial x^j \partial x^k} \tag{7}$$

it becomes under very general conditions

$$\Box\psi(x) = \frac{-1}{(\sqrt{2\pi})^3}\ g^{ik} p_j p_k \varphi(p) e^{i(p,x)} d\sigma.$$

Because the point p is restrained to the surface (5) we have $g^{jk} p_j p_k = (p, p) = m^2$, which gives

$$(\Box + m^2)\psi(x) = 0, \tag{8}$$

which is to say that the function $\psi(x)$ satisfies SCHRÖDINGER's relativistic equation, written for a free particle (no electromagnetic field) with rest mass m.[3]

Conversely, we can claim that any sufficiently regular solution of equation (8) can be represented directly or by passing to the limit (including FOURIER-STIELTJIES integrals) by FOURIER integrals. We do not dwell on this issue here, given that the FOURIER integrals that we shall consider belong to a very special class.

We will have to do in the sequel with integrals over certain surfaces S and we will need the Lorentzian surface element, an element invariant with respect to LORENTZ transformations.

To get such an element, consider first the most general case here. Consider in a space of dimension n an arbitrary quadratic metric, $(U, U) = g_{jk} U^j U^k, (g_{jk} = g_{kj})$, which provides us at the same time with the scalar product $(U, V) = g_{jk} U^j V^k$. Therefore, if $U, V, W, ...$ are a certain number, say ℓ, of vectors, finite or infinitesimal, we will define the "volume" v of the ℓ-dimensional parallelepiped constructed on these vectors by equating v^2 to the absolute value of the determinant

$$\begin{vmatrix} (U,U) & (U,V) & \cdots \\ (V,U) & (V,V) & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}.$$

This volume has all the essential properties of Euclidean volumes. Thus, for example, when the given vectors are mutually orthogonal, the "volume" v becomes equal to the product of the "lengths" of these vectors.

In the present case of a Lorentzian surface element, we will have $n = 4, \ell = 3$ and the vectors will be the infinitesimal tangents of three families of curves traced on the surface S.

This point cleared up, let us return to our FOURIER integral. Therefore, denoting by dH_m the Lorentzian surface element of H_m, we will define the integration element $d\sigma$ appearing in (6), by the relation $d\sigma = dH_m/m$. I say that

$$d\sigma = \frac{dH_m}{m} = \frac{dp_1 dp_2 dp_3}{|p_0|}, \tag{9}$$

by attributing to all the differentials positive values.

To avoid explicit calculations, we can allow ourselves to be guided by the corresponding Euclidean formulae. Consider in the Euclidean space of n-dimensions the sphere $\sum_1^n y_k^2 = r^2$ and denote by dS its element of surface. We will obviously have

$$|dy_1 ... dy_{k-1} dy_{k+1} ... dy_n| = |dS \, \cos(y_k, r)| = \frac{|y_k|}{r} \, dS$$

and hence

$$\frac{dS}{r} = \frac{|dy_1 ... dy_{k-1} dy_{k+1} ... dy_n|}{|y_k|}.$$

This point made, we see that the last member of (9) is an invariant since it is so for the second member. On the other hand, this one has no meaning for $m = 0$, the value corresponding to the light cone, because the numerator and the denominator are both zero, while the last member reduces to the well known expression $dp_1 dp_2 dp_3 : (p_1^2 + p_2^2 + p_3^2)^{\frac{1}{2}}$. These preliminaries laid down, we confine ourselves in the following to the case where integral (6) is only over the sheet of H_m for which the energy p_0 is positive, which sheet will be denoted by H_m^+. Our FOURIER integral finally becomes

$$\psi(x) = \frac{1}{(\sqrt{2\pi})^3} \int_{H_m^+} \varphi(p) e^{i(p,x)} d\sigma, \tag{10}$$

where $d\sigma$ is defined by (9). As regards $\varphi(p)$, we will assume now that the integral

$$A = \int_{H_m^+} |\varphi(p)|^2 d\sigma \tag{11}$$

has a finite value. Moreover, nothing prevents us from making additional hypotheses about $\varphi(p)$ as p goes to infinity (remaining always on H_m^+) to ensure, for example, that the operation $\Box\psi$ can be carried out without difficulty.

THE CORRESPONDING PROBABILITY DISTRIBUTIONS IN THE TWO SPACES. AN IDENTITY.

It is now very natural to interpret $A^{-1}|\varphi(p)|^2 d\sigma$ as the probability because the point representing the energy-momentum vector of a free particle, with given rest mass m, belongs to the element $d\sigma$ around the point p. (To be rigorous, we should speak of the element of surface dH_m^+ corresponding to $d\sigma$.)

The question arises as to what concept of probability corresponds in the space x to the element of probability with respect to the space p that we have just defined. It is easy to show that if the quantity A, given by formula (11), is defined, it will be the same as the triple integral

$$\iiint_{-\infty}^{+\infty} |\psi|^2 dx_1 dx_2 dx_3$$

for any fixed value of the time x^0. Now this integral converges even too well, it converges as soon as it is so of the integral that we get from (11) by replacing $d\sigma$ by $d\sigma/p_0$. Furthermore, by varying x^0, the value of the integral will not be conserved. This state of things, well known in principle from the beginnings of relativistic quantum theory, suggests that by looking for an integral formed in the space x which has a value identical to (11), we will have to resort to an expression for which there exists a conservation law. One such expression is provided by the time component of the four-vector considered by GORDON (1926) and by KLEIN (1927).

In his well known *Bakerian Lecture*[4], Mr Dirac has established (in rather vague terms) that the probabilities provided on the one hand by the expression $|\varphi|^2 d\sigma$ and on the other by the said four-vector have to correspond. We will deduce now an identity by which this correspondence will be precisely defined.

Since the function $\psi(x)$ is a solution of equation (8), it will be so also for the conjugate imaginary function $\overline{\psi}(x)$. Form the gradients of these functions with respect to the four variables x^μ. The gradient of $\psi(x)$ will have for example the covariant components $\dfrac{\partial\psi}{\partial x^\mu}$. Put therefore the real four-vector

$$G = \frac{1}{2i}(\overline{\psi} \text{ grad } \psi - \psi \text{ grad } \overline{\psi}). \tag{12}$$

Due to equation (8) being satisfied by the functions ψ and $\overline{\psi}$, the divergence (that is, the four-divergence) of this vector will be zero. The contravariant components G^k of G are in effect

$$G^k = \frac{1}{2i}(\overline{\psi} \text{ grad}^k \psi - \psi \text{ grad}^k \overline{\psi}),$$

thus

$$\text{div } G = \frac{\partial G^k}{\partial x^k} = \frac{1}{2i}\left(\frac{\partial\overline{\psi}}{\partial x^k} \text{ grad}^k \psi + \overline{\psi}\,\frac{\partial \text{ grad}^k \psi}{\partial x^k} - \frac{\partial\psi}{\partial x^k} \text{ grad}^k \overline{\psi} - \psi\,\frac{\partial \text{ grad}^k \overline{\psi}}{\partial x^k}\right),$$

The four terms destroy themselves two by two. In fact, given that $\partial\psi/\partial x^k = \text{grad}_k\psi$, $\partial\overline{\psi}/\partial x^k = \text{grad}_k\overline{\psi}$, we will have according to (3) two terms $\pm(\text{grad } \psi, \text{ grad } \overline{\psi})$. On the other hand, because of formelae (7) and (8) we will obtain two terms $\pm m^2\psi\,\overline{\psi}$.

Consider then an oriented surface S in space[5] which extends to infinity. Take at each point the unit normal n directed towards increasing x^0 and form the scalar product $(G, n) = G_k n^k$, which could be interpreted as the projection of G onto n or the component of G along n. This product is obviously invariant with respect to any Lorentz transformation. We put furthermore to abbreviate $(G, n) = G_{<n>}$. Under the usual conditions on the form of ψ at infinity, the relation div $G = 0$ translates by virtue of Gauss' theorem into the fact that the integral $\int_\Sigma G_{<n>} d\Sigma$ is independent of the surface Σ. It is the general form of the conservation law for the vector G. In particular, if the surface Σ is reduced to a plane $x^0 = \text{constant}$, $G_{<n>}$ becomes G^0, and we find

$$\int_\Sigma G_{<n>} d\Sigma = \iiint_{-\infty}^{+\infty} G^0(x^0, x^1, x^2, x^3) dx^1 dx^2 dx^3. \tag{13}$$

The fact that this last integral is independent of x^0 represents the conservation law in its usual form.

We shall now establish the identity, valid for $m \geq 0$,

$$\int_\Sigma G_{<n>} d\Sigma = \int_{H_m^+} |\varphi|^2 d\sigma, \tag{14}$$

which is an immediate corollary of the theorem of Parseval-Plancherel. Here is the theorem in question.

In an n-dimensional space put $\displaystyle\sum_0^n x_k y_k = (x, y)$, $dx_1 dx_2 ... dx_n = dx$, $dy_1 dy_2 ... dy_n = dy$. Furthermore, let the functions $f(x)$ and $g(x)$ be measurable in the sense of Lebesgue and such that the integrals $\int |f|^2 dx$ and $\int |g|^2 dx$, over the entire space, are finite. Then the Fourier transforms

$$T(f(x)) = \frac{1}{(\sqrt{2\pi})^n}\int f(y)e^{i(x,y)}dy \quad \text{and} \quad T(g(x)) = \frac{1}{(\sqrt{2\pi})^n}\int g(y)e^{i(x,y)}dy$$

exist almost everywhere (in the sense of convergence in the mean) and we have

$$\int T(f(x))\,\overline{T(g(x))}\,dx = \int f(y)\overline{g(y)}\,dy,$$

the last integrals being absolutely convergent. Of course, all the integrals are over the entire space.

Apply this theorem to the present case. We have

$$\left.\begin{aligned} \psi(x) &= \frac{1}{(\sqrt{2\pi})^3}\iiint_{-\infty}^{+\infty} \varphi(p)e^{i(p,x)}\frac{dp_1 dp_2 dp_3}{p_0}, \qquad p_0^2 = m^2 + p_1^2 + p_2^2 + p_3^2 > 0. \\ \frac{\partial}{\partial x^0}\psi(x) &= \frac{1}{(\sqrt{2\pi})^3}\iiint_{-\infty}^{+\infty} \varphi(p)e^{i(p,x)}\,dp_1 dp_2 dp_3. \end{aligned}\right\} \tag{15}$$

We deduce

$$\iiint_{-\infty}^{+\infty} \overline{\psi(x)} \frac{\partial}{\partial x^0}\psi(x)\ dx^1dx^2dx^3 = \iiint_{-\infty}^{+\infty} \frac{\overline{\varphi(p)}e^{-ip_0x^0}}{p_0}\ i\varphi(p)e^{ip_0x^0}\ dp_1dp_2dp_3 = i\int_{H_m^+} |\varphi(p)|^2 d\sigma.$$

By relation (13), identity (14) follows from the identity that has just been obtained and from its complex conjugate.

Of course, since the factor p_0 does not appear in the denominator in the second formula (15), to ensure the legitimacy of this deduction, we will have to assume that the integral (11) remains convergent if we replace the element of the integration $d\sigma$ by $p_0\,d\sigma = dp_1dp_2dp_3$.

As element of probability, we have always found fault with the expression $G^0dx^1dx^2dx^3$ because G^0 is not necessarily positive. By our result, we see at least that the total probability provided by G^0 can be expressed by an integral taken in the space p where the element of integration $|\varphi|^2d\sigma$ is always ≥ 0.

In summary, to the probability scalar $|\varphi|^2$ relating to the space p there corresponds in the space x a probability provided by a component of a vector, to the positive probability a probability which can admit negative values. The state of things will be much more satisfactory if we pass from the Schrödinger equation to that of Dirac.

CLIFFORD NUMBERS

Recall first the definition and some properties of Clifford numbers. Each vector U can clearly be put into the form $U = U^ke_k$, where e_k is the (basis) vector whose k^{th} contravariant component is 1, while the others are zero. We look now for the conditions such that we can write

$$(U,U) = g_{jk}U^jU^k = (U^ke_k)^2 = U^2. \tag{16}$$

These conditions are evidently

$$e_je_k + e_ke_j = 2g_{jk}. \tag{17}$$

We deduce from either one of relations (16) and (17) that for any two vectors

$$UV + VU = 2(U,V), \tag{18}$$

which shows that two arbitrary orthogonal vectors anticommute ($(U,V) = 0$). The preceding considerations apply to an arbitrary metric and dimension. We are interested here only in the case where the dimension is four and the g_{jk} have the values given in (1), that is to say that the rules of calculation (17) are precisely adapted to the Lorentz space. In detail, we then have

$$e_0^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1, \quad e_je_k + e_ke_j = 0 \quad j \neq k.^{?} \tag{19}$$

By setting moreover $e^j = g^{jk}e_k$, we consequently see that $e^je^k + e^ke^j = 2g^{jk}$. Note further that, in the present case, we have $e^0 = e_0, e^k = -e_k$ for $k = 1,2,3$.

Before going further, we note here the following fact which will be of major importance in the following. We have

$$e_j^{-1} = e_0e_je_0 = e_0e_je_0^{-1}, \quad j = 0,1,2,3. \tag{20}$$

In fact, this is equivalent to $e_0e_j \cdot e_0e_j$, which is only an immediate corollary of relations (19). The above relation is characteristic of the Lorentz signature: only one of the squares e_j^2 is positive.

The quantities e_j generate an algebra, or an hypercomplex system, or a ring, of order 16, with associative multiplication. A basis is provided by the 16 monomials that appear in the expansion of the product $(1+e_0)(1+e_1)(1+e_2)(1+e_3)$. The monomials in question are anti symmetric in the sense that the interchange of any two factors in $e_je_ke_\ell$, for example, changes the sign: $e_ke_je_\ell = -e_je_ke_\ell = e_je_\ell e_k$ (j,k,ℓ are assumed integral). We denote these 16 monomials by $e_A, e_B, \ldots$, the order of the factor in each being indifferent, but determined once for all. A Clifford number c will then have the form $c = \sum c_Ae_A$, where the c_A are ordinary complex numbers, which commute with the elements e_A: $c_Ae_A = e_Ac_A$. We put $c = 0$ if all the coefficients c_A are zero, and in this case only.

To avoid confusion, we emphasise here that we consider the primary symbols e_j and their products e_A as *real* quantities. A Clifford number c will be considered as a real quantity if all its coefficients c_A are ordinary real numbers. We will then not be hindered here by the inconveniences that one encounters when determining the real nature of certain Clifford quantities of the Lorentz space and represented by inconvenient matrices, which are remedied by passing from the matrices γ_k to the matrices α_k (*Cf.* Pauli, Handb. d. Phys. 24_1 (1934) p. 221 and Rev. Modern Phys. **13** (1941) p. 221).

The scalar basis element 1, which commutes with any CLIFFORD number will be alternatively denoted e_S. Its coefficient c_S gives the *scalar part* of the number c. We note the following properties of this scalar part which plays a large role in what follows.

1) c_S is invariant with respect to any LORENTZ transformation.

2) If c' and c'' are two CLIFFORD numbers, we have $(c'c'')_S = (c''c')_S$. (It follows that in products containing more factors cyclic permutations also do not change the scalar part.)

The first property depends on the fact that each homogeneous part forms an antisymmetric tensor (by extension also including scalars and vectors) and it transforms, according the laws of transformation of these tensors, into an homogeneous form of the same degree.

To prove the second property, note that $(c'c'')_S = \sum_A c'_A c''_A e_A^2$ where $e_A^2 = \pm 1$.

NORM OF A CLIFFORD NUMBER. A TENSOR OF SECOND ORDER.

We will define as norm $\|c\|$ of a CLIFFORD number the quantity $\|c\| = \sum_A |c_A|^2$ which is clearly *positive* if $c \neq 0$. We will see that this value, which is an (invariant) scalar in the Euclidean space, is, in the LORENTZ space, the components T_{00} of a tensor of second order. To arrive at this result, we must consider certain automorphisms of the CLIFFORD ring.

1) *Reversal.* This automorphism consists of reversing in each product e_A of the order of the factors e_j. For example, $e_1e_3 \to e_3e_1$, $e_2e_0e_3 \to e_3e_0e_2$. This operation will be denoted by $e_A \to \tilde{e}_A$, $c = \sum c_A e_A \to \sum c_A \tilde{e}_A = \tilde{c}$.

2) *Conjugation.* Each coefficient c_A is replaced by its complex conjugate $\bar{c}_A$, that is to say that $c = \sum c_A e_A \to \sum \bar{c}_A e_A = \bar{c}$.

3) *Conjugate reversal.* The two operations above are commutative. Their product is $c \to c^\dagger = \widetilde{(\bar{c})} = \overline{(\tilde{c})} = \sum \bar{c}_A \tilde{e}_A$.

Note that all the operations defined above are *independent of the system of coordinates* (invariant under the transformations which preserve the metric). It is sufficient to show this for the last one. It can be characterised in the following way, which does not involve the primary symbols e_j.

The operation $c \to c^\dagger$ is an anti-automorphism ($c' \to c'^\dagger$, $c'' \to c''^\dagger$ implies $c'c'' \to c''^\dagger c'^\dagger$) of the textscClifford ring which conserves the scalars and the real vectors[7] and changes the scalar $i = \sqrt{-1}$ into $-i$.

It follows immediately from this fact that the norm $\|c\|$ is a scalar in a CLIFFORD ring for an Euclidean space. In fact, in such a space we have, as an example, $e_A = e_je_k : e_A\tilde{e}_A = e_je_ke_ke_j = e_je_j = 1$, that is to say that $\|c\| = \sum |c_A|^2 = (cc^\dagger)_S$. It is not so in a LORENTZ space where $e_A\tilde{e}_A = \pm 1$. To identify the tensorial character of the norm in such a space, it is necessary to define further a fourth automorphism which however will depend on the choice of the system of coordinates or, more precisely, on the choice of the direction of the x^0-axis, without depending on the positive sense of this axis.

4) *Hermitian adjoint.* $c = \sum c_A e_A \to c^* = \sum \bar{c}_A {e_A}^{-1}$.[8] This time round we have indeed

$$\|c\| = (cc^*)_S = (c^*c)_S. \tag{21}$$

It is easy to show that $c^* = e_0 c^\dagger e_0 = e_0 c^\dagger {e_0}^{-1}$ and hence

$$\|c\| = (e_0 c^* e_0 c)_S. \tag{22}$$

In fact, the relation reduces to ${e_A}^{-1} = e_0 \tilde{e}_A {e_0}^{-1}$. Now by virtue of (20) we have for example for $e_A = e_je_k$, ${e_A}^{-1} = {e_k}^{-1}{e_j}^{-1} = e_0e_k{e_0}^{-1} \cdot e_0e_j{e_0}^{-1} = e_0e_ke_je_0 = e_0\tilde{e}_A{e_0}^{-1}$.

By forming the tensor of second order

$$T_{jk} = (e_j c^\dagger e_k c)_S, \tag{23}$$

we clearly have $\|c\| = T_{00}$.

The tensor T_{jk} is *real*, but it is not symmetric in general. The first point is established as follows. We have

$$T_{jk} = \sum_{A,B} \bar{c}_A c_B \tilde{e}_A e_B,$$

the summation is over pairs of indices A and B for which the product $e_j\tilde{e}_Ae_ke_B$ is a scalar. Now such being the case, it will be the same for the product $e_j\tilde{e}_Be_ke_A$ which appears with the coefficient $\bar{c}_Bc_A$. If we can show that these two products are equal, their common value will appear with coefficient $\bar{c}_Ac_B + \bar{c}_Bc_A$, which is real. Now a product which is scalar is changed neither by reversal nor by a cyclic permutation of the factors. We thus have $e_j\tilde{e}_Ae_ke_B = \tilde{e}_Be_ke_Ae_j = e_j\tilde{e}_Be_ke_B$. Q.E.D.

As regards the lack of symmetry in the general case, we will see examples later. Here however, to show what interest the tensor T_{jk} deserves, we will show that the MAXWELL-MINKOWSKI energy-momentum tensor which appears in the theory of electromagnetism is a particular case — always symmetric — of the above tensor.

The antisymmetric tensor with twice contravariant components F^{jk}, which corresponds to an electromagnetic field, can always be represented by the CLIFFORD number $F = \sum_{p<q} F^{pq} e_p e_q$. We know that the energy density, expressed in the notation of MAXWELL as $\frac{1}{2}(|E|^2 + |H|^2)$ and in that of MINKOWSKI as $\frac{1}{2}\sum_{p<q} F^{pq\,2}$, is not an invariant but that it is the component S_{00} of said tensor of energy-momentum S_{jk}. On the other hand, by observing that the field and, hence, the F^{jk} are always assumed real, we have $F^{pq\,2} = |F^{pq}|^2$, and thus by virtue of (21),

$$2S_{00} = \sum F^{pq\,2} = \|F\| = (e_0 F^\dagger e_0 F)_S = T_{00},$$

where we have put more generally $T_{jk} = (e_j F^\dagger e_k F)_S$. We show that the tensors T_{jk} and S_{jk} are identical up to a factor of $\frac{1}{2}$. Now, we know that the latter tensor is symmetric. Let us show that it is the same for T_{jk}. In fact, we have because $e_q e_p = -e_p e_q$, $F^\dagger = -F$ and, by a cyclic permutation of factors,

$$T_{jk} = (e_j F^\dagger e_k F)_S = -(e_j F e_k F)_S = -(e_k F e_j F)_S = (e_k F^\dagger e_j F)_S = T_{kj}.$$

This being so, we see that the tensor $W_{jk} = S_{jk} - \frac{1}{2}T_{jk}$ is symmetric and furthermore that $W_{00} = 0$ in every system of Lorentz coordinates. It follows by an easy calculation that $W_{jk} = 0$ for every pair of indices j, k.

Return to the general case. We have seen that $(e_0 c^\dagger e_0 c)_S > 0$ for all CLIFFORD numbers $c \neq 0$, since the quantity above is $= \|c\|$. This fact admits the following generalisation which we will need in the following.

If U and V are two positive (or negative) timelike vectors and c a CLIFFORD *number $\neq 0$, we have*

$$(U c^\dagger V c)_S > 0. \tag{24}$$

With no loss of generality, we can assume that U and V are unit vectors. That said, inequality (24) is exact for $U = V$, as we see by introducing a system of Lorentz coordinates x'^k where the basis vector e'_0 coincides with U.[9] In the general case, we will rely on the following lemma which we will prove just now.

If U and V are two positive unit timelike vectors, there exists a positive unit timelike vector such that the two equivalent relations

$$V = KUK \quad \text{and} \quad U = KVK \tag{25}$$

are valid.

Accepting this lemma, we will have

$$(U c^\dagger V c)_S = (U c^\dagger K U K c)_S = (U (Kc)^\dagger (Kc))_S,$$

by which the general case reduces to the particular case.

It remains to prove the Lemma. Note first that $(U, U) = (V, V)$ can also be written as $(U + V, U - V) = 0$, which leads by formula (3) to these last vectors anti-commuting. By putting now

$$U = \frac{U+V}{2} + \frac{U-V}{2}, \qquad V = \frac{U+V}{2} - \frac{U-V}{2}, \qquad K = \mu(U+V),$$

where μ is determined such that $K^2 = (K, K) = 1$, we get

$$KUK = K \cdot \frac{U+V}{2} K + K \cdot \frac{U-V}{2} K = K^2 \cdot \frac{U+V}{2} - K^2 \cdot \frac{U-V}{2} = \frac{U+V}{2} - \frac{U-V}{2} = V.$$

Our Lemma is intimately related to the following known theorem.[10]

If L is a non-isotropic vector and U is any vector, $V = LUL^{-1}$ is a vector, namely the reflection of U with respect to the line which contains L.[11]

Decompose U into two vectors U' and U'', the first parallel and the second orthogonal to L. Therefore $LUL^{-1} = L(U' + U'')L^{-1} = U'LL^{-1} - U''LL^{-1} = U' - U''$. — Moreover, we also have (formula (3))

$$LUL^{-1} = (LU + UL)L^{-1} - UL \cdot L^{-1} = 2(L, U)L^{-1} - U. \tag*{(25)$^{\text{2nd version}}$}$$

ZERO DIVISORS

An essential feature of the CLIFFORD ring is of containing zero divisors, that is quantities f and g such that $fg = 0$ while $f \neq 0$ and $g \neq 0$. If f is scalar, $fg = 0$, $g \neq 0$, we have necessarily $f = 0$. In fact, in the opposite case, f^{-1} would exist and we would have $g = f^{-1}.fg = f^{-1}.0 = 0$. This can also be expressed by saying that *there do not exist scalar divisors of zero.* The divisors of zero that interest us in the first place are of the form $U - u$, where U is a vector $\neq 0$ and u is a scalar. On this subject, we have the following theorem.

If U ($\neq 0$) is a timelike or a spacelike vector, there exist two scalar quantities $u = \pm\sqrt{(U,U)}$ such that $U - u$ is a zero divisor. Any isotropic vector is itself a zero divisor.

The latter statement is obvious, since if U is isotropic, we have $U^2 = (U,U) = 0$. To prove the first statement, denote any one of the square roots $\sqrt{(U,U)}$ by $|U|$. We then have $(U - |U|)(U + |U|) = U^2 - |U|^2 = U^2 - (U,U) = 0$, which means that $U - |U|$ and $U + |U|$ are zero divisors, a fact which will play a decisive role in the following. Conversely, $U - u$ can be a zero divisor only if $u = \pm|U|$. In fact, suppose that $(U - u)c = 0$, while $c \neq 0$. Multiplication from the left by $U + u$ gives us $(U^2 - u^2)c = \big((U,U) - u^2\big)\, c = 0$. Since the first factor is a scalar, it must be zero. What we have said about $U - u$ as a left divisor clearly applies to $U - u$ as a right divisor. We will now prove the following theorem.

For the relation

$$(U - u)c = 0, \qquad u = \pm|U|, \tag{26}$$

to be valid for a given CLIFFORD *number c, it is necessary and sufficient that there exist a* CLIFFORD *number g such that*

$$c = (U + u)g. \tag{27}$$

In fact, it is clear that condition (27) is sufficient. On the other hand, any number c can be written as

$$c = \frac{1}{2}\left[\left(1 - \frac{U}{u}\right) + \left(1 + \frac{U}{u}\right)\right] c.$$

When (26) is satisfied, $c = \frac{1}{2}(1 + U/u)c$, that is to say that (27) is also satisfied, if we put $g = c/2u$.

The above theorem remains valid for isotropic vectors $U^2 = (U,U) = 0$ or $u = 0$, even though the above proof fails. Suppose then that $Uc = 0$ and choose a vector V such that $(U,V) = \frac{1}{2}$ or (see formula (18)) $UV + VU = 1$. Hence $c = (UV + VU)c = U \cdot Vc = U \cdot g$.

Let us conclude these considerations with the remark, evident from that which precedes, that the necessary and sufficient condition for the relation $c(U \mp |U|) = 0$ to be valid is that $c = g(U \pm |U|)$.

In our applications, U will be a timelike vector. We shall mean by $|U|$ the positive square root of (U,U).

DIRAC'S EQUATION. FOURIER'S INTEGRAL.

With the help of the symbols introduced, we can now write

$$\Box + m^2 = g^{jk}\frac{\partial^2}{\partial x^j \partial x^k} + m^2 = \left(\frac{1}{i}\, e^j \frac{\partial}{\partial x^j} - m\right)\left(-\frac{1}{i}\, e^k \frac{\partial}{\partial x^k} - m\right). \tag{28}$$

By putting

$$\nabla = e^k \frac{\partial}{\partial x^k}, \tag{29}$$

the equation

$$\left(\frac{1}{i}\,\nabla - m\right)\psi(x) = 0, \tag{30}$$

will be called DIRAC's equation for a free particle — absence of the field — whose rest mass is m. In DIRAC, the quantities e^k are represented by matrices γ^k with four rows and columns, while $\psi(x)$ is a column with four elements or, if you wish, a *spinor.* $\psi(x)$ contains only four complex components, which certainly must be considered as a considerable advantage. In our work, $\psi(x)$ denotes, until further notice, a general CLIFFORD number (which clearly depends on x) which means that it is a quantity with sixteen complex components. After having developed the theory in the general case, we will specialise the $\psi(x)$, so that the number of components will reduce to four. By decomposition (28), it is clear that $\psi(x)$ satisfies the SCHRÖDINGER equation $(\Box + m^2)\psi(x) = 0$. Since the operator $\Box + m^2$ is a scalar, any component ψ_A of $\psi = \sum_A \psi_A e_A$ must satisfy the same equation.

Put formally

$$\psi(x) = \frac{1}{(\sqrt{2\pi})^3} \int \varphi(p) e^{i(p,x)} d\tau, \tag{31}$$

where neither the domain nor the element of integration will be specified for the moment. By the analysis which will follow, $d\tau$ will become identical to the element $d\sigma$, but *a priori* we do not yet exclude the case where $d\tau$ is a four-dimensional element. As regards $\varphi(p)$, it is a CLIFFORD number that depends on (the components) of the vector $p = p_k e^k$ and defined in the domain of integration which is still to be determined. In the following calculations, p is itself considered as a CLIFFORD number (vector).

Assuming that it is possible to differentiate under the sign $\int$, it becomes

$$\left(\frac{1}{i}\nabla - m\right)\psi(x) = \left(\frac{1}{i}\, e^k \frac{\partial}{\partial x^k} - m\right)\psi(x) = \frac{1}{(\sqrt{2\pi})^3} \int (e^k p_k - m)\varphi(p) e^{i(p,x)} d\tau$$
$$= \frac{1}{(\sqrt{2\pi})^3} \int (p - m)\varphi(p) e^{i(p,x)} d\tau.$$

For the left hand side to be identically zero, we must have

$$(p - m)\varphi(p)\, d\tau = 0. \tag{32}$$

Here p is a vector, m is a fixed scalar, $\varphi(p)$ is a general CLIFFORD number (which depends on p), $d\tau$ is an element of integration, which has not yet been specified. On the other hand according to a theorem demonstrated above $(p - m)$ is a divisor of zero only in the case where $m = \pm|p| = \pm\sqrt{(p,p)}$ or in other words that $(p,p) = m^2$, that is to say that the point p which represents the energy-momentum vector is situated on the hyperboloid (5) which we have denoted by H_m. The quantity $\varphi(p)\, d\tau$ must thus be zero at all points of the space p which are not situated on H_m. This implies that the integration is only over H_m and that $d\tau$ will have to be replaced by an element of integration relative to this surface, that is the invariant element $d\sigma$ used above (formulae (9) and (6)). The integral then has exactly the same form as (6), except that this time φ is an hypercomplex quantity. That is sufficient for $\psi = \sum \psi_A e_A$ and all the components ψ_A to satisfy the equation $(\Box + m^2)\psi = 0$ or $(\Box + m^2)\psi_A = 0$. For $\psi(x)$ to satisfy the DIRAC equation, the necessary and sufficient condition is that we also have $(p - m)\varphi(p) = 0$ on H_m. By analysing above the zero divisors, we showed that this equality will hold when $\varphi(p) = (p + m)\chi(p)$ and in this case only, where $\chi(p)$ is an arbitrary function — defined on H_m — which, of course, satisfies suitable regularity conditions. In summary:

The general solution of the DIRAC equation (30) is provided by the FOURIER integral

$$\psi(x) = \frac{1}{(\sqrt{2\pi})^3} \int_{H_m} \varphi(p) e^{i(p,x)} d\sigma \tag{33}$$

where

$$(p - m)\varphi(p) = 0. \tag{34}$$

THE TENSOR T_{jk} FORMED FROM A SOLUTION OF DIRAC'S EQUATION

Leaving aside the Fourier integral for a moment, let ψ be a CLIFFORD quantity (sufficiently regular as a function of x) which satisfies the DIRAC equation

$$\frac{1}{i}\, e^k \frac{\partial \psi}{\partial x^k} - m\psi = 0. \tag{35}$$

Form, according to the prescriptions given above, the quantity $\psi^\dagger(x)$ which we obtain from ψ by a conjugate transpose. The same operation applied to the left hand side, which is zero, gives us

$$-\frac{1}{i} \frac{\partial \psi^\dagger}{\partial x^k}\, e^k - m\psi = 0 \qquad \text{or} \qquad \frac{1}{i} \frac{\partial \psi^\dagger}{\partial x^k}\, e^k + m\psi = 0. \tag{36}$$

This being so, form with ψ and $\psi^\dagger$ the tensor T_{jk}, or rather the corresponding mixed tensor,

$$T^k{}_j = (e_j \psi^\dagger e^k \psi)_S. \tag{37}$$

We find by a known method that the tensor divergence $\partial T^k{}_j/\partial x^k = 0$. We have, in fact, concerning equations (35) and (36),

$$\frac{\partial T^k{}_j}{\partial x^k} = \left(e_j \frac{\partial \psi^\dagger}{\partial x^k} e^k \psi + e_j \psi^\dagger e^k \frac{\partial \psi}{\partial x^k} \right)_S = i(-m\psi^\dagger\psi + m\psi^\dagger\psi) = 0. \tag{38}$$

Hence if Σ is an oriented surface in space, form the normal components

$$T_{j<n>} = T_{jk}n^k = (e_j\psi^\dagger n^k e_k \psi)_S = (e_j \psi^\dagger n \psi)_S, \qquad (j = 0, 1, 2, 3). \tag{39}$$

A conservation law follows from (32) by the Gauss theorem, namely that the integral

$$\int_\Sigma T_{j<n>} d\Sigma \tag{40}$$

is independent of Σ. In particular, if the surface Σ reduces to a plane $x^0 =$ constant, we obtain the special conservation law which says that the integral

$$\iiint_{-\infty}^{+\infty} T_{j0}(x^0, x^1, x^2, x^3)\, dx^1 dx^2 dx^3 \tag{41}$$

is independent of x^0.

Note again that application of relation (24) to the right hand side of (39), with $j = 0$, shows the important fact that *the normal component $T_{0<n>}$ is always positive.*

Comment. The law of conservation remains valid (and its proof remains roughly the same) if we substitute for equation (35) the more general equation which corresponds to the presence of an electromagnetic field. This is also the case for the positive character of $T_{0<n>}$ which is entirely of algebraic origin.

AN IDENTITY

Return now to the Fourier integral. By supposing as above that the integral is only over a single sheet of H_m, over H_m^+ for example, we will establish some identities between integrals (40) and structural integrals over H_m^+. Put then in analogy with formula (10)

$$\psi(x) = \frac{1}{(\sqrt{2\pi})^3} \int_{H_m^+} \varphi(p) e^{i(p,x)} d\sigma, \tag{42}$$

where the Clifford quantity $\varphi(p)$ satisfies relation (34), which entails that $\psi(x)$ satisfies the Dirac equation. Form with $\varphi(p)$ the tensor analogous to (32),

$$M_{jk} = (e_j \varphi^\dagger e_k \varphi)_S, \tag{43}$$

denote by n the unit normal $n^k e_k$ of H_m^+, directed to the interior of H_m^+, and put $M_{j<n>} = M_{jk}n^k$. I claim that we have the identities

$$\int_\Sigma T_{j<n>} d\Sigma = \int_{H_m^+} M_{j<n>} \frac{d\sigma}{m} \qquad (j = 0, 1, 2, 3).^{12} \tag{44}$$

As regards the left hand side, it is sufficient to consider them in the particular form (41). Note on the other hand that relation (44) written for $\varphi = \sum \varphi_A e_A$ and $\psi = \sum \psi_A e_A$ implies an analogous relationship between each pair of components φ_A and ψ_A. Finally, by observing that $M_{jk} = \sum \pm \varphi_A \overline{\varphi}_B$ and $T_{jk} = \sum \pm \psi_A \overline{\psi}_B$ contain the same pair of indices and the same distribution of signs, it follows from the theorem of Parseval-Plancherel, and recalling expression (9) for $d\sigma$, that

$$\iiint_{-\infty}^{+\infty} T_{j0} dx^1 dx^2 dx^3 = \iiint_{-\infty}^{+\infty} M_{j0} \frac{dp_1 dp_2 dp_3}{p_0^2}. \tag{46}$$

There is nothing to object to in the left hand side of this relation. We know already its invariant significance. It is not the same for the right hand side; neither M_{j0}, nor $p_0^{-2} dp_1 dp_2 dp_3$ nor their product look invariant. However, this is due to the fact that we have so far not used the fundamental relation (34). This relation give us

$$\varphi = \frac{p}{m}\varphi \quad \text{and} \quad \varphi^\dagger = \varphi^\dagger \frac{p}{m}.$$

By this

$$\varphi^\dagger e_0 \varphi = \varphi^\dagger \frac{p}{m}\, e_0\, \frac{p}{m}\, \varphi,$$

or by forming the arithmetic mean of the two sides

$$\varphi^\dagger e_0 \varphi = \varphi^\dagger \cdot \frac{1}{2}\left(e_0 + \frac{p}{m}\, e_0\, \frac{p}{m}\right)\varphi. \tag{47}$$

Because $\left(\frac{p}{m}\right)^2 = \frac{p\cdot p}{m^2} = 1$, we find, by using (18),

$$\frac{1}{2}\left(e_0 + \frac{p}{m}\, e_0\, \frac{p}{m}\right) = \frac{p}{m}\cdot\frac{1}{2}\left(e_0 + \frac{p}{m}\, e_0\, \frac{p}{m}\right) = \frac{p}{m}\cdot\left(\frac{p}{m}, e_0\right) = \frac{p}{m}\,\frac{p_0}{m}\,.$$ [13]

The unit Lorentz normal $n = n^k e_k$ of H_m^+ directed toward its interior is clearly identical to the radius vector p, normalised, that is to say p/m. This being so, the right hand side of the above relation can also be written as $n \cdot p_0/m$. Using this in (47), it becomes

$$\varphi^\dagger e_0 \varphi = \frac{p_0}{m}\, \varphi^\dagger n \varphi.$$

This point made, it follows that

$$\begin{aligned} M_{j0} &= (e_j \varphi^\dagger e_0 \varphi)_S = \frac{p_0}{m}\,(e_j \varphi^\dagger n \varphi)_S = \frac{p_0}{m}\,(e_j \varphi^\dagger n^k e_k \varphi)_S \\ &= \frac{p_0}{m}\,(e_j \varphi^\dagger e_k \varphi)_S\, n^k = \frac{p_0}{m}\, M_{jk} n^k = \frac{p_0}{m}\, M_{j<n>}. \end{aligned}$$

If we substitute for M_{j0} in (46) the value $\frac{p_0}{m}\, M_{j<n>}$ that we have just obtained and take into account (9), the invariant identity (44) will be proved.

AN IDENTITY FOR DIRAC'S CURRENT VECTOR

Before comparing our tensor T_{jk} with the Dirac current vector s_k, which it resembles in several respects, we will present for the latter an identity similar to (44). In doing so, we will use ordinary matrix language. Consider then, without writing them out explicitly, the four Hermitian matrices of order four of Dirac, usually denoted b y γ_k and which satisfy the relations

$$\gamma_\ell \gamma_m + \gamma_m \gamma_\ell = 2\delta_{\ell m}, \qquad \delta_{\ell m} = \begin{cases} 1, & \ell = m, \\ 0 & \ell \neq m. \end{cases} \qquad \ell, m = 1,2,3,4. \tag{48}$$

By setting $\check\gamma_0, \check\gamma_j = i\gamma_j,\ j = 1,2,3$, the four matrices $\check\gamma_k$ $(k = 0,1,2,3)$, provide a representation of our symbols e_k, characterised by formulae (19). Denote by ψ a column of four elements ψ_r, by ψ^* a row whose elements are the complex conjugates ψ_r^* of the ψ_r. Also form the new row $\psi^\dagger = \psi^* \gamma_0$ and the vector s, called the current vector, with covariant components

$$s_k = \psi^\dagger \check\gamma_0 \check\gamma_k \psi, \qquad k = 0,1,2,3. \tag{49}$$

For $k = 0$, we get from the last formula $s_0 = \psi^*\psi = \sum \psi_r^* \psi_r = \sum |\psi|^2 > 0$. If ψ satisfies the Dirac equation,

$$\left(\frac{1}{i}\,\check\gamma^k\,\frac{\partial}{\partial x^k} - m\right)\psi = 0, \tag{50}$$

the four-divergence $\partial s_k/\partial x^k$ of the vector s will be zero.

All this being accepted, take up again the surfaces Σ with orientation of the space considered above, form the components $s_{<n>} = s_k n^k$; we see as above (for $T_{0<n>}$) that $s_{<n>} > 0$. By Gauss's theorem, for any sufficiently regular solution of (35), there again follows the conservation law, consisting in that the integral

$$\int_\Sigma s_{<n>} d\Sigma \tag{51}$$

is independent of Σ.

As regards the energy-momentum space, the vector p will be represented by the matrix $p = p_k\check{\gamma}^k$. We will always restrict ourselves to points p situated on the sheet H_m^+, $(p,p) = m^2$, $p_0 > 0$. If we form a column $\varphi = \varphi(p)$, which satisfies the relation $(p-m)\varphi(p) = 0$, the column $\psi(x)$ provided by the FOURIER integral

$$\psi(x) = \frac{1}{(\sqrt{2\pi})^2}\int_{H+m}\varphi(p)e^{i(p,x)}d\sigma \tag{52}$$

will satisfy the DIRAC equation (35).

This being the case, form as above the rows corresponding to φ^* and $\varphi^\dagger$ and, finally, the vector v with components

$$v_k = \varphi^\dagger\check{\gamma}_k\varphi = \varphi^*\check{\gamma}_0\check{\gamma}_k\varphi. \tag{53}$$

Denoting always by $n = p/m$ the unit normal of H_m^+ (directed towards the interior of H_m^+) and by $v_{<n>} = v_k n^k$ the normal component of v, we obtain, as above, the identity

$$\int_\Sigma s_{<n>}d\Sigma = \int_{H_m^+} v_{<n>}\frac{d\sigma}{m}. \tag{54}$$

PRIMITIVE NUMBERS AND SIMPLE IDEALS. COLUMNS, ROWS, SPINORS.

A CLIFFORD number $\pi \neq 0$ is said to be *primitive* if, for an *arbitrary* CLIFFORD number c, we have a relation f the form $\pi c\pi = t\pi$, where t is a scalar (an ordinary complex number) which obviously depends on c. One such primitive number is for example the idempotent

$$\varepsilon = \frac{1+e_0}{2}\frac{1+ie_1e_2}{2}, \qquad \varepsilon^2 = \varepsilon.$$

In fact, for the numbers $e_A = 1, e_0, ie_1e_2, ie_0e_1e_2$ which up to a factor i are elements of the basis of the CLIFFORD ring, we have $\varepsilon e_A\varepsilon = \varepsilon$, whereas for all other elements e_B of the basis we have $\varepsilon e_B\varepsilon = 0$. This is due to the following facts: 1) The e_A and the idempotents $\varepsilon_1 = \dfrac{1+e_0}{2}, \varepsilon_1' = \dfrac{1-e_0}{2}, \varepsilon_2 = \dfrac{1+ie_1e_2}{2}, \varepsilon_2' = \dfrac{1-ie_1e_2}{2}$ all commute with one another. 2) $e_0\varepsilon_1 = \varepsilon_1$, $ie_1e_2\varepsilon_2 = \varepsilon_2$. 3) $\varepsilon_1\varepsilon_1' = \varepsilon_1'\varepsilon_1 = \varepsilon_2\varepsilon_2' = \varepsilon_2'\varepsilon_2 = 0$. 4) Any e_B satisfies, for at least one of the indices $j = 1, 2$, a relation of the form $e_B\varepsilon_j = \varepsilon_j' e_B$.

If the CLIFFORD number a is arbitrary and π is primitive, the numbers $a\pi$ and πa are primitive. In fact, $a\pi\cdot c\cdot a\pi = a\cdot\pi(ca)\pi = a\cdot t\pi = t\cdot a\pi$.

We mean by a *left ideal* I a set of CLIFFORD numbers such that $z_1 \in I$, $z_2 \in I$ leads to $(z_1 \pm z_2) \in I$ and that $z \in I$ leads to $az \in I$ where a is an arbitrary CLIFFORD number. A similar definition for a *right ideal*. A left (right) ideal is said to be *simple* if it does not contain any left (right) ideal except itself and the null ideal, which consists of the only element 0. On the subject of such ideals, we have the following theorems:

I. *Any primitive number π generates a simple left ideal $I = \{a\pi\}$, where a ranges over the entire* CLIFFORD *ring.*

II. Any elements of a simple left ideal is primitive and the ideal is generated by any one of its elements $\neq 0$. (Similar theorems for right ideals.)

The fact that the numbers $I = \{a\pi\}$ form form a left ideal is clear for any fixed number π, primitive or not. The remainder of our statement requires more delicate considerations. I will return to these issues on another occasion, by going deeper into them in many ways. Here, to get quickly to the point, I content myself by pointing out that our theorems follow almost immediately from the known fact that, in the complex field, the CLIFFORD ring is isomorphic to a total matrix algebra of order 4, that is, to a set of matrices of said order that admit arbitrary complex elements.

To primitive elements π of the ring there correspond matrices P of rank 1, which we will call primitive matrices. If A is an arbitrary matrix, the matrices AP and PA will be primitive. Any primitive matrix can be written as a product: column times row, from which it follows that $PAP = tP$, where t is a scalar.

Any primitive matrix P generates a simple left ideal $I = \{AP\}$, where A ranges over all the matrices of order 4, and conversely, a simple left ideal contains only primitive matrices and it is generated by any one of its elements $\neq= 0$. We can go a little further. If A is an arbitrary matrix and P is a primitive matrix, that is, the product of a *fixed* column with a *fixed* row, the matrices AP which form the ideal I *will be the product of an arbitrary column with a fixed row*. By this, we see that a simple left[14] ideal admits four basis elements, provided for example by the particular columns

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{etc.}$$

The ideals that we have just considered are part of the CLIFFORD ring. The function ψ that appears in DIRAC's equation, which at first sight has 16 components, will have only 4, if we restrict it to belong to a simple left ideal. On the other hand we have just seen that since the CLIFFORD ring is represented by a matrix algebra, a left ideal will be represented by the product of an arbitrary column and a fixed row. Now this last manifold is clearly isomorphic to the manifold of arbitrary columns (spinors) that we encounter in the usual presentation of the DIRAC theory, a presentation which is to be preferred from a technical point of view. However, by this very natural simplification, the ideal exits from the CLIFFORD domain, from which arise the known complications related to the transformation of spinors. Another thing which, precisely in the case of a matrix representation, favours the left and right ideals in place of the columns and rows is that a CLIFFORD number, represented by a matrix, cannot appear as a left factor is a column and as a right factor when the other factor is a row. Of course, these are not too serious points.

Let us return to the Fourier integral (42). When the function $\varphi(p)$ belongs to a given simple ideal, it will be the same for the function $\psi(x)$, the solution to the DIRAC equation. It is very easy to see that in this case the tensors M_{jk} and T_{jk} will be of rank *one*, which means that we will have $M_{jk} = a_j b_k$ and $T_{jk} = c_j d_k$. In fact, if ε and ε' are two primitive numbers, we see by matrix representation that, for an arbitrary CLIFFORD number c, we have $\varepsilon' c \varepsilon = t\varepsilon''$ where t is a scalar and ε'' is a primitive number that depends only on ε and ε'. On the other hand, if e_j and e_ℓ are two basis vectors, we will always be able to choose the scalars v_j and v_ℓ, where at least one is $\neq 0$, such that the scalar part of $(v_j e_j + v_\ell e_\ell)\varepsilon''$ is zero.

On the other hand, if the primitive number ψ is written in a matrix representation in the form μ times ν, where μ is a column and ν is a row, we will form, as above, by means of μ and the row μ^* whose elements are the complex conjugates of those of μ the current vector s_k of DIRAC. Hence we will have for $j = 0, 1, 2, 3$ $T_{jk} = t_j s_k$

Dirac's matrices γ_k are closely related to the idempotents

$$\varepsilon = \frac{1 \pm e_0}{2} \, \frac{1 \pm i e_1 e_2}{2}, \tag{55}$$

which satisfy the relations $\varepsilon = \varepsilon^\dagger = \varepsilon^*$. We thus have, if ψ belongs to the left ideal $\{a\varepsilon\}$,

$$T_{jk} = \left(e_j (a\varepsilon)^\dagger e_k \, a\varepsilon\right)_S = \left(e_j \varepsilon^\dagger a^\dagger e_k \, a\varepsilon\right)_S = \left(e_j \varepsilon a^\dagger e_k \, a\varepsilon\right)_S,$$

or, since ε is a primitive number,

$$T_{jk} = (e_j s_k \varepsilon)_S\,,$$

where s_k is a scalar. Now, for $j = 1, 2, 3$, the scalar part of $e_j\varepsilon$ is zero. So we have

$$T_{jk} = 0, \qquad j = 1, 2, 3.$$

For $j = 0$ we have, according to the sign of e_0 in (55), $e_0\varepsilon = \pm\varepsilon$ and

$$T_{0k} = \pm s_k \varepsilon_S = \pm \frac{1}{4}\, s_k.$$

We must still add some comments concerning the law of transformation of the quantities T_{jk} (or M_{jk}). By passing to a new system of coordinates, the quantities ψ and $\psi^\dagger$ are invariant. In fact, ψ_A and e_A transform by contragredient transformations into ψ'_A and e'_A such that $\psi = \sum \psi_A e_A = \sum \psi'_A e'_A$ remains invariant. However, e_j and e_k transform into e'_j and e'_k such that $T_{jk} = (e_j \psi^\dagger e_k \psi)_S$ transforms into $T_{jk} = (e'_j \psi^\dagger e'_k \psi)_S$, which is in agreement with the tensorial notation.

As regards the physical significance of the tensor T_{jk}, it would be premature to express any opinion.

In the particular case considered above, each row T_{jk} will be proportional to DIRAC's current vector s_k, where the factors of proportionality are constants in every system of coordinates and they transform like the components of a vector.

I shall indicate here briefly how we can arrive by the mechanism developed above at the quantities which transform like the components of the vector of DIRAC. We need only put $s_k = (b\psi^\dagger e_k \psi)_S$, where b is a fixed positive timelike vector. By a Lorentz transformation, $e_k \to o_k^r e_r$ it becomes $s_k \to o_k^r s_r$. We can show by means of a theorem given above (p. 137) that, in the case where $o_0^0 > 0$ there exists a CLIFFORD number R, determined up to sign, such that $r^{-1} = R^\dagger$ and that $o_k^r e_r = R^{-1} e_k R$.

Hence the Lorentz transformation relative to s_k can also be expressed by the formula $s_k \to (b\psi^\dagger R^\dagger e_k R\psi)_S = (b\psi'^\dagger e_k \psi')_S$ [15] where $\psi' = R\psi$. In the last transformation we have left e_k invariant and transformed ψ in accordance with the theory of Dirac.

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[1] Comp. Rend. X[me] Cong. Math. Scan. 1946 (Kopenhagen).

[2] The index k which appears in the operation $\partial/\partial x^k$ must be considered as a lower index (covariant).

[3] The following remark may not be devoid of interest, even though it is only a geometric transcription of a fact well known elsewhere. Consider in the space p of four dimensions the paraboloid

$$p_0 = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2).$$

The function $\varphi(x)$ (x is a spacetime point) provided by an integral analogous to (6), but this time over the surface of the aforesaid paraboloid, satisfies the Schrödinger equation.

[4] Proc. Roy. Soc. A. **180**; 1-40; see in particular p. 6-7.

[5] A surface is said to be oriented in space if an arbitrary light cone which has its vertex on this surface has only this point in common with it.

[6] In Euclidean space we have $e_j^2 = 1$ and $e^j = 1$ for all values of j.

[7] Of course, the products of such vectors are not conserved in general.

[8] We have just seen that in a Clifford ring for an Euclidean space we have $c^* = c^\dagger$.

[9] We will have, by using the new system of coordinates

$$c = \sum_A c_A e_A = \sum_A c'_A e'_A \qquad \text{and} \qquad (e'_0 c^\dagger e'_0 c)_S = \sum_A |c'_A|^2.$$

[10] This theorem encompasses relations (20).

[11] $L^{-1} = L/(L, L)$.

[12] We will return soon to the normal of H_m^+. For the moment, we will content ourselves by noting that since H_m^+ is oriented in space, the normal in question will be a positive timelike vector. The normal component $M_{0\langle m\rangle}$ is thus positive for the same reasons as $T_{0\langle n\rangle}$.

[13] We also could have applied formula ($25^{2^{\text{nd}}\,\text{version}}$).

[14] The obvious facts that we stated here for left ideals translate immediately to right ideals.

[15] Translator's note. In Riesz's paper this formula appears with the undefined symbol c_k in place of e_k. This appears to be a typographical error, since in the sentence that follows this formula he refers to e_k rather than c_k. I have therefore taken the liberty of correcting Riesz's formula.