Elie Cartan

Lectures on
INTEGRAL INVARIANTS

An English Translation of Elie Cartan's Book

Leçons sur les Invariant Intégraux

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Introduction.

This book is a reproduction of a course given during the summer term 1920-1921 at the Paris Faculty of Sciences.

The theory of integral invariants was founded by H. Poincaré and explained by him in Volume III of his *Méthodes nouvelles de la Mécanique céleste*.

In two notes to the Comptes Rendus de l'Académie des Sciences (16 and 30 June 1902), the author was led, in the study of differential equations that admit given transformations, to consider certain differential forms which he called *integral forms*: they were characterised by the property of being able to be expressed solely by means of first integrals of the given differential equations and their differentials. By deepening his research in the same sequence of ideas, the author, on the one hand, came to base his method for integrating systems of partial differential equations that have characteristics that depend only on arbitrary constants (Cauchy characteristics) and, on the other hand, his theory of the structure of finite and infinite continuous groups of transformations.

Now it is found that the concept of an integral form is not essentially different from that of an integral invariant. The comparison of these two concepts forms the basis of this book.

Consider, for example, a system of three first-order differential equations with three unknown functions x, y, z, of the independent variable t; we can regard them as defining an infinite number of trajectories of a moving point. A differential form, such as $P\,dx+Q\,dy+R\,dz+H\,dt$ for example, can be thought of as a quantity attached to a *state* (x,y,z,t) of the moving point and to an infinitely close *state* $(x+dx, y+dy, z+dz, t+dt)$. To say that this form is *integral* (or *invariant*, according to the terminology adopted in these lectures), obviously means that this quantity depends only on the trajectory that contains the first state and on the infinitely close trajectory that contains the second state. In other words, an invariant form does not change its value if we move the two states (x,y,z,t), $(x+dx, y+dy, z+dz, t+dt)$ along their trajectories. If we then consider a linear continuous sequence of trajectories and extend the integral $\int P\,dx+Q\,dy+R\,dz$ to the arc of the locus curve of the positions taken by the moving body on these trajectories at the same time t, this integral is independent of t: it is an integral invariant in the sense of H. Poincaré. Conversely, there is a very simple way to go from an integral invariant $\int P\,dx+Q\,dy+R\,dz$ of H. Poincar to the corresponding invariant form $P\,dx+Q\,dy+R\,dz+H\,dt$.

These considerations are not limited to linear differential forms. Any invariant differential form that can be put under an integration sign, single or multiple, gives rise to an integral invariant in the sense of H. Poincar, if we remove the terms which contain the differential(s) of the independent variable.[1]

[1] Mr R. Hargrea.tes, in a paper in the *Transactions of the Cambridge Philosophical Society* (Vol. XXI, 1912), had already considered integrals containing the differential of the independent variable; but his point of view is quite different from that of the text, and he always gives the independent variable a separate role.

Ultimately, *the quantity under the integral sign in an invariant integral of H. Poincar is nothing other than a truncated invariant differential form*. The invariant character of the *completed* integral is preserved if it is over any set of states, *simultaneous or not*.

The consequences of this reconciliation between the two concepts of an invariant integral and an invariant differential form are many. First, all the properties relating to the formation of integral invariants, their invariants, and their derivation from one another, become self-evident. The same is true of the applications to the integration of differential equations.

Another consequence, relating to the principles of Mechanics, must be pointed out. H. Poincar proved that the general equations of Dynamics possess the property that they admit an linear (relative) integral invariant, namely

$$\int p_1\,dq_1 + p_2\,dq_2 + \cdots + p_n\,dq_n, \tag{1}$$

where the q_i and the p_i denote Hamilton's canonical variables. If we complete the differential form under the sign $\int$, the integral invariant takes the form

$$\int p_1\,dq_1 + p_2\,dq_2 + \cdots + p_n\,dq_n - H\,dt, \tag{2}$$

where H is Hamilton's function. We thus see the appearance, alongside the *momenta* $(p_1, ..., p_n)$ of the material system considered, its energy H. The form under the sign $\int$ thus acquires an extremely important mechanical significance; we can call it the *energy-momentum tensor*.[2] Hamilton's elementary *action* is none other than this tensor considered along a trajectory: the concept of *action* is thus connected with those of momentum and energy.

But there is more. Not only do the differential equations of motion admit the integral invariant (2), but they are also the only differential equations that have this property. We can then put the following principle at the foundation of Mechanics which we could call the PRINCIPLE OF CONSERVATION OF MOMENTUM AND ENERGY:

The motions of a material system (with perfect holonomic constraints, subject to forces that derive from a force function) are governed by first-order differential equations between time, the position parameters and the velocity parameters, and these differential equations are characterised by the property that the integral of the ENERGY-MOMENTUM *tensor, over any closed linear continuous sequence of states of the system, does not change its value when we move these states in any way whatever along their respective trajectories*.

In this statement, the expression *state* means the set of quantities that define the position of the system in space, the time at which it is considered, and the velocities at that instant.

[2] The form pointed out appears quite naturally when we calculate the variation of Hamilton's action integral. of Hamilton's action integral; it has already been pointed out from this point of view. It is moreover introduced in this way in these lectures.

The previous statement is more abstract and less intuitive than Hamilton's least action, for example. It does nevertheless have one advantage over Hamilton's equation that is important to point out. Lagrange's equations allow us to give the laws of mechanics a form that is *independent of the system of reference adopted for space*, and it is this that gives them their importance.

But time still has a privileged position. On the contrary, the principle of conservation of energy and momentum gives the laws of mechanics a form that is independent of the system of reference adopted for the Universe (space-time): if we perform a change of variables that acts on both the position parameters of the system and on the time, it is sufficient to know the form taken by the ENERGY-MOMENTUM tensor in space-time in the new system of coordinates to be able to deduce the deduce the equations of motion. We thus obtain a scheme to which all mechanical theories must be subordinated, and to which relativistic mechanics itself is in fact subordinated.

It is important to note that this scheme only applies to material systems that depend on a finite number of parameters.

The present work leaves aside a large number of applications of the theory of Integral Invariants; in particular those topics, extremely important in Celestial Mechanics, which are related to the theory of periodic solutions of the three-body problem, to the theory of Poisson stability, are systematically left aside. We have confined ourselves mainly to applications relating to the integration of differential equations; but, even in this sequence of ideas, the problem is only just initiated.

However, we have endeavoured to show that this problem cannot be considered in isolation; we only narrow it down if we do not look at it as a particular aspect of a more general problem in which must consider not only integral invariants, but even invariant Pfaffian equations for the given differential equations, as well as the infinitesimal transformations which preserve these differential equations. A complete exposition of the problem would have gone far beyond the scope of these lectures and would, moreover, have required some knowledge of the theory of continuous groups. We have confined ourselves to showing on some occasions the fundamental role played in the final analysis by the group G of transformations which, applied to the integrals of the given differential equations, leave invariant all the information known a priori about these integrals.[3] Any system of differential equations can be reduced to typical systems, each of which corresponds to a *simple* group G. If this simple group is *finite*, we obtain systems of differential equations which have been studied especially by S. Lie and M. E. Vessiot, who called them *Lie systems*. They are related to the theory of integral invariants in the sense that, by the addition of auxiliary unknown functions if necessary, they admit as many linear integral invariants as there are unknown functions. We will find some general guidelines in Chapter XV of these lectures by looking at them from this latter point of view.

If the simple group G is infinite, and if we exclude the case where it is the most general group with n variables, in which case we know nothing about the corresponding system of differential equations, it admits either an integral invariant of maximum degree (theory of the Jacobi multiplier), or a linear relative integral invariant (theory of equations reducible to the canonical form),

[3] Cf. E. Cartan. *Les sous-groupes des groupes continus de transformations*; Ann. Ec. Norm. (3), Vol. XXV (1908), p. 57-194 (Chap. I).

or an invariant Pfaffian equation (equations reducing to a partial differential equation of first order). Chapters XI-XIV are devoted to these classic theories.

The concept of an integral invariant can be considered from a point of view that is slightly different from the usual point of view, which is that of H. Poincaré, and which is, in short, the point of view we have adopted in these lectures. Instead of considering a multiple integral as attached to a system of differential equations with respect to which it has a property of invariance, we can consider it as attached to a group of transformations with respect to which it is invariant. In fact, the two points of view are related. The latter is the point of view adopted by S. Lie and which seemed to him for some time to be the only true one. Here again the concept of integral invariant plays an important role since, as the author has shown,[4] any group of transformations can be defined, if necessary by the addition of auxiliary variables, as the set of transformations which admit a certain number of linear integral invariants. This aspect of the concept of integral invariant is completely omitted in these lectures.

Several chapters are devoted to the rules for calculating the differential forms which occur under multiple integration signs. M. Goursat gives these forms the name of symbolic expressions; I propose to call them differential forms with exterior multiplication or, more briefly, *exterior differential forms*, because they obey the rules of H. Grassmann's exterior multiplication. Similarly, I propose the name of *exterior derivative* for the operation that allows us to go from a multiple integral of degree $p-1$ over a $(p-1)$-dimensional closed manifold to an equal multiple integral of degree p over the p-dimensional manifold bounded by the former.[5] This operation, which reduces to classic derivation operations when the coefficients of the differential form under the sign $\int$ admit partial derivatives of the first order, may still make sense when this is no longer the case. Interesting problems arise in this respect which have not yet been studied systematically and which deserve to be. The book ends with two chapters, very brief indeed, on the relationship of the theory of integral invariants to the calculus of variations and to the principles of optics.

At the end of the volume you will find a list, which does not claim to be complete, of the main works on the theory of integral invariants. Papers on the classic theories of the Jacobi multiplier, the canonical equations, and first-order partial differential equations are cited only when they relate directly to the theory of integral invariants.

Le Chesnay, 4 November 1921.

[4] E. Cartan. *Sur la structure des groupes infinis de transformations*; Ann. Ec. Norm. (3), Vol. XXI (1904), p. 153-206; Vol. XXII (1905), p. 219-308.

[5] This is M. Goursat's operation *D*.

Chapter I

Hamilton's Principle of Least Action and the Energy-Momentum Tensor

I. — *The Case of an Unconstrained Particle.*

1. We can build all of analytical mechanics on a principle which reduces the determination of the motion of a material system to the solution of a problem in the calculus of variations: it is Hamilton's principle of least action. We will first present it for the case of an unconstrained point particle[1,2] subject to a force derived from a force function U, given as a function of rectangular coordinates x, y, z of the point and of the time t.[3]

In this simple case, Hamilton's principle can be stated in this way:

Among all possible motions that take the point particle from a given position (x_0, y_0, z_0) at time t_0 to another given position (x_1, y_1, z_1) at time t_1, the actual motion is the one which minimises the definite integral

$$W = \int_{t_0}^{t_1} \left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) + U \right] \, dt$$

In this expression m denotes the mass of the point particle; x', y', z' the components of its velocity; the quantity under the integral sign is called *the elementary action*, and the integral W is the *action* in the time interval (t_0, t_1).

To prove this principle, regard x, y, z as functions of t and of an arbitrary parameter α and calculate the *variation* of W when we increment α by $\delta\alpha$, assuming that (x, y, z) reduce to (x_0, y_0, z_0) for $t = t_0$ and to (x_1, y_1, z_1) for $t = t_1$, *and that whatever the value of* α. We have

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[1] Fr. *un point matériel libre*.

[2] TRANSLATOR'S NOTE. The word *libre* means *free*. In English, a *free material point* is a point particle on which no forces act. However, Cartan immediately adds that the particle is subject to a force. I have therefore translated *un point matériel libre* here as *an unconstrained point particle*.

[3] TRANSLATOR'S NOTE. Cartan's *force function* is the *negative* of what English readers call a *force potential* or, more simply, *the potential*.

$$\delta W = \int_{t_0}^{t_1} \left[m(x'\delta x' + y'\delta y' + z'\delta z') + \frac{\partial U}{\partial x}\, \delta x + \frac{\partial U}{\partial y}\, \delta y + \frac{\partial U}{\partial z}\, \delta z \right] dt;$$

now, we have

$$\delta x' = \frac{\partial}{\partial \alpha} \left(\frac{\partial x}{\partial t} \right) \delta \alpha = \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial \alpha} \delta \alpha \right) = \frac{\partial \,(\delta x)}{\partial t};$$

an integration by parts then gives, noting that $\delta x, \delta y, \delta z$ are zero at the limits,

$$\delta W = \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\, \frac{d^2 x}{dt^2} \right) \delta x + \left(\frac{\partial U}{\partial y} - m\, \frac{d^2 y}{dt^2} \right) \delta y + \left(\frac{\partial U}{\partial z} - m\, \frac{d^2 z}{dt^2} \right) \delta z \right] dt.$$

If we want δW to be zero for $\alpha = 0$ whatever functions δx, δy, δz may be of t which are zero at the limits, it is necessary and sufficient, by applying a classic argument, that we have, for $\alpha = 0$,

$$\left.\begin{aligned} m\, \frac{d^2 x}{dt^2} &= \frac{\partial U}{\partial x}, \\ m\, \frac{d^2 y}{dt^2} &= \frac{\partial U}{\partial y}, \\ m\, \frac{d^2 z}{dt^2} &= \frac{\partial U}{\partial z}. \end{aligned}\right\} \tag{1}$$

It follows from this that *the motions that the point particle takes under the action of the given force realise the extremum of the integral W with respect to all the possible infinitely close motions that correspond to the same initial and final positions of the point, and furthermore that these motions are the only ones that have this property.*

Strictly, we can only speak of the *extremum* of the action and not of the *minimum*, because the condition that the first variation δW be zero is a necessary but not a sufficient condition for a minimum.

2. The elementary action

$$\left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) + U \right]\, dt$$

seems to have been introduced here as a mere calculational device for stating the laws of motion in a condensed form. We shall see that we can substitute another equivalent principle for Hamilton's principle which reveals also a linear expression in dx, dy, dz, dt, but of which all the coefficients have a simple mechanical meaning.

In fact, return to the action W, but now assuming that t_0 and t_1 are themselves functions of the parameter α, where the corresponding values $x_0, y_0, z_0, x_1, y_1, z_1$ are also functions of α. By applying the methods for deriving a definite integral, the calculation of δW gives,

$$\delta W = \left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) + U\right]_{t=t_1} \delta t_1 - \left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) + U\right]_{t=t_0} \delta t_0$$
$$+ \left[mx'\delta x + my'\delta y + mz'\delta z\right]_{t=t_1} - \left[mx'\delta x + my'\delta y + mz'\delta z\right]_{t=t_0}$$
$$+ \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\,\frac{d^2x}{dt^2}\right)\delta x + \left(\frac{\partial U}{\partial y} - m\,\frac{d^2y}{dt^2}\right)\delta y + \left(\frac{\partial U}{\partial z} - m\,\frac{d^2z}{dt^2}\right)\delta z\right] dt.$$

Note now that we have

$$[\delta x]_{t=t_1} = \left[\frac{\partial x}{\partial \alpha}\right]_{t=t_1} \delta\alpha, \quad \delta x_1 = \left[\frac{\partial x}{\partial t}\right]_{t=t_1} \delta t_1 + \left[\frac{\partial x}{\partial \alpha}\right]_{t=t_1} \delta\alpha$$

and consequently

$$[\delta x]_{t=t_1} = \delta x_1 - x_1'\ \delta t_1.$$

The formula for δW is thus

$$\begin{aligned}
\delta W = {} & mx_1'(\delta x_1 - x_1'\ \delta t_1) + my_1'(\delta y_1 - y_1'\ \delta t_1) + mz_1'(\delta z_1 - z_1'\ \delta t_1) \\
& + \left[\frac{1}{2}\, m({x_1'}^2 + {y_1'}^2 + {z_1'}^2) + U_1\right] \delta t_1 \\
& - \Big\{ mx_0'(\delta x_0 - x_0'\ \delta t_0) + my_0'(\delta y_0 - y_0'\ \delta t_0) + mz_0'(\delta z_0 - z_0'\ \delta t_0) \\
& + \left[\frac{1}{2}\, m({x_0'}^2 + {y_0'}^2 + {z_0'}^2) + U_0\right] \delta t_0 \Big\} \\
& + \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\,\frac{d^2x}{dt^2}\right)\delta x + \left(\frac{\partial U}{\partial y} - m\,\frac{d^2y}{dt^2}\right)\delta y + \left(\frac{\partial U}{\partial z} - m\,\frac{d^2z}{dt^2}\right)\delta z\right] dt.
\end{aligned} \tag{2}$$

Put

$$\left.\begin{aligned}
\omega_\delta &= mx'(\delta x - x'\,\delta t) + my'(\delta y - y'\,\delta t) + mz'(\delta z - z'\,\delta t) \\
&\quad + \left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) + U\right] \delta t \\
&= mx'\ \delta x + my'\ \delta y + mz'\ \delta z - \left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) - U\right] \delta t.
\end{aligned}\right\} \tag{3}$$

The differential expression thus introduced has as coefficients, first

$$mx', \quad my', \quad mz',$$

that is, the components of the *momentum* of the moving point, and then

$$\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) - U$$

that is, the *energy* E.

Thanks to this notation, we can write

$$\delta W = [\omega_\delta]_0^1 + \int_{t_0}^{t_1} \left[\left(\frac{\partial U}{\partial x} - m\,\frac{d^2x}{dt^2} \right) \delta x + \left(\frac{\partial U}{\partial y} - m\,\frac{d^2y}{dt^2} \right) \delta y + \left(\frac{\partial U}{\partial z} - m\,\frac{d^2z}{dt^2} \right) \delta z \right] dt.$$

Suppose now that we consider a sequence of actual trajectories that depend on a parameter α, and that we limit each trajectory to a time interval (t_0, t_1) which varies with α. The formula which gives the variation of the action along these variable trajectories reduces to

$$\delta W = (\omega_\delta)_1 - (\omega_\delta)_0.$$

Finally, suppose that we consider a *tube of trajectories*, that is, a *closed* linear continuous sequence of trajectories each of which is limited to a time interval (t_0, t_1); the total variation of the action when we have returned to the initial trajectory is obviously zero, so that, integrating with respect to α, we have

$$\int (\omega_\delta)_1 = \int (\omega_\delta)_0.$$

3. To interpret the result just obtained, let us agree to call the set of seven quantities

$$x, y, z, x', y', z', t$$

the *state* of the material point, where the first three define the position of the point, the next three its velocity, and the last the time at which the point is considered. We can regard a *state* as a point in a seven dimensional space, the *state space*. A *trajectory* can be defined as the sequence of states that correspond to the same actual motion of the point, that is, in summary, as a solution of the system of the differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= x', & m\,\frac{dx'}{dt} &= \frac{\partial U}{\partial x}, \\ \frac{dy}{dt} &= y', & m\,\frac{dy'}{dt} &= \frac{\partial U}{\partial y}, \\ \frac{dz}{dt} &= z', & m\,\frac{dz'}{dt} &= \frac{\partial U}{\partial z}. \end{aligned}\right\} \tag{4}$$

According to this, *the curvilinear integral*

$$\int \omega_\delta = \int mx'\ \delta x + my'\ \delta y + mz'\ \delta z - E\ \delta t$$

over an any closed curve in the state space does not change if we move each of the states of which it is composed in any way whatsoever along the trajectory that corresponds to that state.

We can also say that *given any tube of trajectories, the integral* $\int \omega_\delta$ *over a closed curve that goes around this tube, is independent of this curve and depends only on the tube.*

Note that the expression ω_δ can be viewed as the elementary work of a vector in the four-dimensional universe (x, y, z, t): this vector would have the three ordinary components of momentum as its spatial components and energy as its time component. We can call it the *energy-momentum tensor*; each of its components thus has a simple mechanical meaning.

4. If we consider a sequence of *simultaneous* states, that is, if we suppose that $\delta t = 0$, the integral $\int \omega_\delta$ reduces to

$$\int m(x'\,\delta x + y'\,\delta y + z'\,\delta z);$$

adopting this last point of view, we obtain the following theorem:

If we consider a closed sequence of trajectories, and if we take on these trajectories the state corresponding to any given time t, the integral $\int m(x'\,\delta x + y'\,\delta y + z'\,\delta z)$ *over the closed sequence of states thus obtained is independent of* t.

This theorem is due to H. Poincaré, who characterised the property thus obtained by giving the name of *integral invariant* to the integral

$$\int m(x'\,\delta x + y'\,\delta y + z'\,\delta z)$$

over a closed contour.

The concept of energy is not involved in Poincaré's approach; it necessarily appears if, instead of considering a closed sequence of *simultaneous* states, we consider any closed sequence of states.

We will say that the integral $\int \omega_\delta$ of the energy-momentum tensor is a *complete integral invariant*, — or more simply an *integral invariant*, when no confusion is to be feared — for the differential equation of motion. The Poincaré integral invariant is thus the complete integral invariant of the energy-momentum tensor considered from a particular point of view.

It is remarkable that if, instead of considering a sequence of simultaneous states, we consider a sequence of states that satisfy the relations

$$\delta x = x'\,\delta t, \quad \delta y = y'\,\delta t, \quad \delta z = z'\,\delta z$$

the tensor ω_δ reduces to Hamilton's elementary action

$$\left[\frac{1}{2}\, m(x'^2+y'^2+z'^2)+U\right]\delta t.$$

Consequently, *the integral invariant of H. Poincaré and Hamilton's action are two different aspects of the energy-momentum integral*, even though at first sight there is no connection between these two notions.

5. Above, we simply deduced from Hamilton's principle a *property* of the energy-momentum tensor, namely, that the integral of this tensor along a closed line of states does not change when we deform this closed line without changing the trajectories on which it rests. Now we will show that *this property can replace Hamilton's principle*, that is, that *the differential equations of motion are the only ones which admit as integral invariant the integral* $\int \omega_\delta$ *over any closed contour.*

In fact, let

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{dx'}{X'}=\frac{dy'}{Y'}=\frac{dz'}{Z'}=\frac{dt}{T}, \tag{5}$$

be any system of differential equations, where the denominators are specified functions of the seven variables x,y,z,x',y',z',t. Imagine a tube of integral curves of this system that depends on one parameter α; this parameter will vary, for example, from 0 to ℓ, where the integral curve that corresponds to $\alpha=\ell$ coincides with that corresponding to $\alpha=0$. To express that the integral $\int \omega_\delta$ over a closed curve that goes around this tube does not depend on the chosen closed curve, we will imagine that the coordinates x,y,z,x',y',z',t of any state on the tube are functions of the parameter α and of another parameter u. By giving u a fixed value, we will have a closed curve that that goes around the tube. By moving along an integral curve of the tube we will have

$$\rho\, du=\frac{dx}{X}=\frac{dy}{Y}=\cdots=\frac{dt}{T},$$

where ρ denotes an arbitrary factor *which we can always choose such as to obtain for* $u=C^{\text{st}}$ *any sequence of closed contours given in advance that go around the tube*.

That said, the integral $I=\int_{(C)}\omega_\delta$, in which we give u a particular value value, is a function of u and, if we reserve the symbol d for a displacement in which only u varies, we have

$$dI=\int_{(C)} m\, dx'\ \delta x+m\, dy'\ \delta y+m\, dz'\ \delta z-dE\ \delta t+mx'\ d(\delta x)+my'\ d(\delta y)$$
$$+mz'\ d(\delta z)-E\ d(\delta t),$$

or, by exchanging the order of the differentiations d and δ and integrating by parts,

$$
\begin{aligned}
dI = {} & \left[mx'\,dx + my'\,dy + mz'\,dz - E\,dt\right]_C \\
& + \int_C \big(m\,dx'\;\delta x + m\,dy'\;\delta y + m\,dz'\;\delta z - dE\;\delta t \\
& \qquad - m\,dx\;\delta x' - m\,dy'\;\delta y' - m\,dz\;\delta z' + dt\;\delta E\big)\,.
\end{aligned}
$$

The fully integrated part is clearly zero *since the contour of integration is closed*. As for integral that remains on the right hand side, for $\int \omega_\delta$ to be an integral invariant for the differential system considered, it is necessary and sufficient that this integral vanish when we replace

$$dx,\; dy,\; dz,\; dx',\, dy',\, dz',\, dt$$

respectively by

$$\rho X,\; \rho Y,\; \rho Z,\; \rho X',\; \rho Y',\; \rho Z',\; \rho T,$$

and this *whatever the closed contour* (C) *and whatever the function* ρ. We deduce easily that the coefficients of

$$dx,\; dy,\; dz,\; dx',\, dy',\, dz',\, dt$$

must be identically zero. Consequently, *for a system of differential equations to admit the integral invariant* $\int \omega_\delta$, *it is necessary and sufficient that the equations*

$$
\left.
\begin{aligned}
m\,dx' + \frac{\partial E}{\partial x}\,dt &= 0,\\
m\,dy' + \frac{\partial E}{\partial y}\,dt &= 0,\\
m\,dz' + \frac{\partial E}{\partial z}\,dt &= 0,\\
-m\,dx + \frac{\partial E}{\partial x'}\,dt &= 0,\\
-m\,dy + \frac{\partial E}{\partial y'}\,dt &= 0,\\
-m\,dz + \frac{\partial E}{\partial z'}\,dt &= 0,\\
-dE + \frac{\partial E}{\partial t}\,dt &= 0,
\end{aligned}
\right\}
\quad \text{or} \quad
\left.
\begin{aligned}
& m\,dx' - \frac{\partial U}{\partial x}\,dt = 0,\\
& m\,dy' - \frac{\partial U}{\partial y}\,dt = 0,\\
& m\,dz' - \frac{\partial U}{\partial z}\,dt = 0,\\
& -m\,dx + mx'\,dt = 0,\\
& -m\,dy + my'\,dt = 0,\\
& -m\,dz + mz'\,dt = 0,\\
& -m\left(x'\,dx' + y'\,dy' + z'\,dz'\right) + dU - \frac{\partial U}{\partial t}\,dt = 0
\end{aligned}
\right\}
\qquad (6)
$$

be consequences of the differential equations of the system.

The first six of these equations are none other than the classical differential equations of motion; as for the seventh, *it gives the* ***vis-viva*** *theorem*, [4,5] which is a consequence.

6. From the above, we see the fundamental role played by the energy-momentum tensor. *If we accept that a trajectory is defined as a succession of states forming a solution of a system of ordinary differential equations, among all conceivable systems of differential equations this system is characterised by the property that it admits as an integral invariant the curvilinear integral of the energy-momentum tensor over any closed curve of states.*

We thus obtain a new principle that could be called the *principle of the conservation of energy-momentum*. As we saw in the previous number, the *vis viva* theorem is a particular consequence of this principle.

II. — *The General Case.*

7. All of the preceding can be extended to material systems such as we usually consider in analytical mechanics. We will assume that these systems satisfy three conditions.

1° *The constraints*[6] *to which they are subject are perfect*, that is, that at each time t the sum of the elementary work of the constraint forces[7] is zero for any virtual displacement consistent with the constraints that exist at time t. Under these conditions, d'Alembert's principle is valid and it is stated as follows:

D'ALEMBERT'S PRINCIPLE. — *If we consider the motion under the action of given forces of a material system subject to perfect constraints, the sum of the elementary work of the given forces and of the inertial forces at each instant is zero for any virtual displacement of the system consistent with the constraints that exist at that instant t.*

d'Alembert's principle is expressed by the formula

$$\sum\left[\left(X - m\,\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\,\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\,\frac{d^2z}{dt^2}\right)\delta z\right] = 0, \tag{7}$$

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[4] Fr. *forces vives*.

[5] TRANSLATOR'S NOTE. — See Footnote 2 of n° **183** for more detail. Here, Cartan appears to be referring to a generalisation of the theorem known to English readers as the *work-enegy theorem* in which the potential is time dependent.

[6] Fr. *liaisons*.

[7] Fr. *forces de liaisons*.

where X, Y, Z denote the components of the given force applied to the point (x, y, z) of mass m and where $\delta x, \delta y, \delta z$ denote the components of the most general elementary displacement consistent with the constraints.

Of all the systems with perfect constraints, we will now consider only those whose constrains are *holonomic*, that is:

2° *We will assume that the constraints can be expressed by finite equations between the coordinates of the points of the system and the time t*. Again, this amounts to saying that it is possible to express the coordinates of the various points of the system by formulae such as

$$\begin{aligned} x_i &= f_i(q_1, \ldots, q_n, t), \\ y_i &= g_i(q_1, \ldots, q_n, t), \qquad\qquad (i = 1, 2, \ldots) \\ z_i &= h_i(q_1, \ldots, q_n, t), \end{aligned}$$

with n arbitrary parameters q. To each system of values of the q and t corresponds one and one only position of the system consistent with the constraints that exist at time t. Any virtual displacement consistent with the constraints that exist at time t is obtained by giving arbitrary increments $dq_1, \ldots, dq_n$ to $q_1, \ldots, q_n$.

Finally, we make one last assumption:

3° *The sum of the elementary work of the given forces, for any virtual displacement consistent with the constraints that exist at time t, is the total differential of a certain function U of the q and of t*, that is,

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = \frac{\partial U}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial U}{\partial q_n}\,\delta q_n;$$

we have not included the term $\dfrac{\partial U}{\partial t}\,\delta t$ on the right hand side because the virtual displacements mentioned in d'Alembert's principle assume that t remains constant.

8. Hamilton's principle of least action generalise without difficulty to the preceding systems. Put

$$W = \int_{t_0}^{t_1} \left[\frac{1}{2}\sum m(x'^2 + y'^2 + z'^2) + U\right] dt.$$

Regard the parameters $q_1, \ldots, q_n$ as functions of t and of a parameter α, where the lower and upper limits of the integral could depend on α. A calculation identical to the one performed above (n° 2) gives us the variation δW of the *action*, when we give α a variation $\delta\alpha$. We get

$$\delta W = [\omega_\delta]_1 - [\omega_\delta]_0 + \int_{t_0}^{t_1} \left[\delta U - \sum m\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z\right)\right] dt, \tag{8}$$

where we have put

$$\left.\begin{aligned}
\omega_\delta \;&= \sum m(x'\,\delta x + y'\,\delta y + z'\,\delta z) - \left[\frac{1}{2}\sum m(x'^2+y'^2+z'^2) - U\right]\delta t,\\
[\omega_\delta]_1 &= \sum m(x'_1\,\delta x_1 + y'_1\,\delta y_1 + z'_1\,\delta z_1) - \left[\frac{1}{2}\sum m(x'^2_1+y'^2_1+z'^2_1) - U_1\right]\delta t_1,\\
[\omega_\delta]_0 &= \sum m(x'_0\,\delta x_0 + y'_0\,\delta y_0 + z'_0\,\delta z_0) - \left[\frac{1}{2}\sum m(x'^2_0+y'^2_0+z'^2_0) - U_0\right]\delta t_0.
\end{aligned}\right\} \tag{9}$$

That said, the principle of d'Alembert immediately shows that, given an actual motion of the system, if we consider this motion in any time interval (t_0, t_1), it attains an extremum of the action W with respect to all infinitely close conceivable motions that correspond to the same initial *position* and to the same final *position* of the system. Conversely, the only motions which have this property are the actual motions of the system: *this is Hamilton's principle of least action.*

Moreover, formula (8) shows that the integral $\int \omega_\delta$ over a closed contour of *states* of the system (consistent with the constraints) does not change if we deform this closed contour by moving each of its constituent states in an arbitrary way along the corresponding trajectory of the system. In other words, *the integral* $\int \omega_\delta$ *is an invariant integral for the differential equations of motion*.

The differential form ω_δ, where we assume that we consider only states of the system that are consistent with the constraints, can again be called the energy-momentum tensor of the system.

9. In general, the differentials $\delta x, \delta y, \delta z, \delta t$ that enter the expression ω_δ are not arbitrary, because they must satisfy the equations obtained by finding the total differential of the constraint equations of the system. We can also express them in terms of the

$$\delta q_1,\ \delta q_2,\ \ldots,\ \delta q_n,\ \delta t,$$

if we have introduced the n position parameters of the system. We will now adopt this point of view and, on the one hand, determine the differential equations of motion and, on the other, the energy-momentum tensor. For this, it is sufficient to calculate δW, assuming that the elementary action has been expressed in terms of the parameters q and the time t. Put

$$T = \sum \frac{1}{2}\, m(x'^2+y'^2+z'^2);$$

the kinetic energy T is a function of second degree with respect to the derivatives $\dfrac{dq_i}{dt}$, which we will write as q'_i, and which we shall regard as arguments that are independent of the q_i and of t. Provisionally put

$$F = T + U, \qquad W = \int_{t_0}^{t_1} F\, dt.$$

A simple calculation gives,

$$\delta W = F_1\,\delta t_1 - F_0\,\delta t_0 + \int_{t_0}^{t_1} \sum \left[\frac{\partial F}{\partial q_i}\,\delta q_i + \frac{\partial F}{\partial q'_i}\,\delta q'_i\right] dt;$$

now,

$$\delta q'_i\,dt = \delta \frac{\partial q_i}{\partial t}\,dt = \frac{\partial}{\partial t}(\delta q_i)\,dt = d(\delta q_i);$$

we thus have, integrating by parts,

$$\delta W = F_1\,\delta t_1 - F_0\,\delta t_0 + \left[\sum \frac{\partial F}{\partial q'_i}\,\delta q_i\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum \left[\frac{\partial F}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial q'_i}\right)\right] \delta q_i\,dt.$$

Note finally that we have

$$[\delta q_i]_{t_0} = \frac{\partial}{\partial \alpha} q_i(t_0,\alpha)\,\delta\alpha \quad \text{and} \quad \delta(q_i^0) = \frac{\partial q_i(t_0,\alpha)}{\partial t}\,\delta t_0 + \frac{\partial q_i(t_0,\alpha)}{\partial \alpha}\,\delta\alpha,$$

from which,

$$[\delta q_i]_{t_0} = \delta(q_i^0) - q_i'^0\,\delta t_0 \quad \text{and} \quad [\delta q_i]_{t_1} = \delta(q_i^{(1)}) - q_i'^{(1)}\,\delta t_1.$$

Finally we thus have,

$$\left.\begin{aligned} \delta W = {} & \sum \left(\frac{\partial F}{\partial q'_i}\right) \delta\left(q_i^{(1)}\right) - \left(\sum q'_i \frac{\partial F}{\partial q'_i} - F\right)_1 \delta t_1 \\ & - \left[\sum \left(\frac{\partial F}{\partial q'_i}\right)_0 \delta\left(q_i^0\right) - \left(\sum q'_i \frac{\partial F}{\partial q'_i} - F\right)_0 \delta t_0\right] \\ & + \int_{t_0}^{t_1} \sum \left[\frac{\partial F}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial q'_i}\right)\right] \delta q_i\,dt. \end{aligned}\right\} \tag{10}$$

Hamilton's principle thus leads us to the following equations of motion, which are none other than *Lagrange's equations*,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_i}\right) - \frac{\partial T}{\partial q_i} - \frac{\partial U}{\partial q_i} = 0 \qquad (i = 1, 2, ..., n). \tag{11}$$

Comparison of the two values (8) and (10) found for δW consequently leads to the following expression for the tensor ω_δ

$$\omega_\delta = \frac{\partial T}{\partial q'_i}\delta q_i - H\,\delta t, \tag{12}$$

by putting

$$H = \sum q_i' \frac{\partial T}{\partial q_i'} - T - U. \tag{13}$$

The quantities $\dfrac{\partial T}{\partial q_i'}$ are the generalised momenta (relative to the chosen system of coordinates); the quantity H is the *generalised energy*.

10. A simple remark allows us in practice to simplify the calculation of the generalised energy H. In general, the kinetic energy T will contain terms of second degree, terms of first degree and terms of zero degree in $q_1', q_2', \dots, q_n'$; let

$$T = T_2 + T_1 + T_0;$$

application of Euler's formula for homogeneous functions then gives immediately

$$H = T_2 - T_0 - U;$$

in the generalised energy, the term T_2 can be considered kinetic in origin, while the term $-T_0 - U$ is dynamic in origin.

For example take the case of an unconstrained material point referred to axes that rotate about Oz with angular velocity r. We have

$$2T = m\left[(x' - ry)^2 + (y' + rx)^2 + z'^2\right]$$

and consequently the energy, *referred to the chosen system of reference*, is

$$H = \frac{1}{2}\, m(x'^2 + y'^2 + z'^2) - \frac{1}{2}\, mr^2(x^2 + y^2) - U;$$

the part of the energy of dynamic origin decomposes into two terms, one of which comes from the given forces and the other from the centrifugal forces. As for the components of the momentum, they are

$$m(x' - ry), \qquad m(y' + rx), \qquad mz',$$

that is, the projections of the absolute momentum onto the chosen coordinate axes.

11. *Hamilton's canonical variables.* — The equations of motion, considered as first order differential equations in q_i, q_i', t take an extremely simple form if we introduce the variables

$$p_i = \frac{\partial T}{\partial q_i'}; \tag{14}$$

the new variables, which we substitute for the q_i', are very simply the components of the momentum of the system. The tensor ω_δ then takes the simple form

$$\omega_\delta = \sum p_i\,\delta q_i - H\,\delta t, \tag{15}$$

where H must be regarded as a function of the q_i, the p_i and t.

We will now look directly for the equations of motion *by expressing the fact that they admit as an integral invariant the integral* $\int \omega_\delta$ *over any closed curve of states of the system.*

Let

$$\frac{dq_1}{Q_1} = \frac{dq_2}{Q_2} = \cdots = \frac{dp_n}{P_n} = \frac{dt}{T} \tag{16}$$

be any system of differential equations. To express the fact that it admits the integral invariant $\int \omega_\delta$, we need only repeat word for word the argument of n° **5**. We will consider a tube of integral curves of system (16); express the $2n+1$ coordinates p_i, q_i, t of a state on the tube as a function of two parameters α and u, where the first remains constant on an integral curve and varies in an interval (0 - ℓ) so that the integral curve $\alpha = \ell$ coincides with the integral curve $\alpha = 0$. Denoting by d a symbol of differentiation referring to the variable u, and putting

$$I = \int_{(C)} \omega_\delta,$$

we have, by an immediate integration by parts,

$$dI = \int_{(C)} \sum (dp_i\,\delta q_i - dq_i\,\delta p_i) - dH\,\delta t + dt\,\delta H.$$

For system (16) to admit the integral invariant $\int \omega_\delta$, it is necessary and sufficient that the coefficients of

$$\delta q_1,\ \delta q_2,$$
$$\ldots, \delta q_n,\ \delta t$$

in the quantity under the $\int$ sign are all zero when we take into account the equations of the system. Now, by setting these coefficients to zero, we obtain the $2n+1$ equations

$$\left.\begin{aligned} dp_i + \frac{\partial H}{\partial q_i}\,dt &= 0,\\ -dq_i + \frac{\partial H}{\partial p_i}\,dt &= 0,\\ -dH + \frac{\partial H}{\partial t}\,dt &= 0. \end{aligned}\right\} \tag{17}$$

This shows that *there is only one system of differential equations that admit the integral invariant* $\int \omega_\delta$, and this gives us at the same time the equations of motion in Hamilton's *canonical* form,

$$\left.\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}. \end{aligned}\right\} \tag{18}$$

The last equation

$$dH - \frac{\partial H}{\partial t}\, dt = 0$$

is the analytic translation of the *vis-viva*[8,9] theorem: it is a consequence of the first $2n$ equations.

12. We thus arrive at *the generalised principle of the conservation of energy-momentum* in the general case of the material systems of analytical mechanics:

If we assume that any motion of a system subject to given forces is a continuous sequence of states that satisfy a system of first order differential equations, these differential equations are characterised by the property that they admit the integral of the energy-momentum tensor over any closed contour of states as an integral invariant.

The energy momentum tensor takes any of the following forms

$$\begin{aligned} \omega_\delta &= \sum m(x'\delta x + y'\delta y + z'\delta z) - \left[\sum \frac{1}{2}\, m(x'^2 + y'^2 + z'^2) - U\right]\delta t, \\ \omega_\delta &= \sum \frac{\partial T}{\partial q_i'}\, \delta q_i - H\, \delta t \qquad\qquad \left(H = \sum q_i' \frac{\partial T}{\partial q_i'} - T - U\right), \\ \omega_\delta &= \sum p_i\, \delta q_i - H\, \delta t. \end{aligned}$$

If we move in the state space in such a way as to satisfy the relations

$$\delta q_i = q_i'\, \delta t,$$

the expression ω_δ reduces to Hamilton's elementary action $(T+U)\delta t$; if, on the contrary, we consider only a sequence of simultaneous states $(\delta t = 0)$, we obtain the expression

$$\sum p_i\, \delta q_i,$$

[8] Fr. *Forces vives*.

[9] Translator's Note. — See the Footnote 2 of n° **183** for more detail. Here, Cartan appears to be referring to a generalisation of the theorem known to English readers as the *work-enegy theorem* in which the potential is time dependent.

which is the element under the $\int$ sign in the integral invariant of H. Poincaré, properly so called.

13. The principle of the conservation of energy and of momentum allows us to form the equations of motion, irrespective of the way in which we chose the parameters $q_1, \ldots, q_n, t$ that serve to localise the system in space and in time. In other words, it gives to the laws of mechanics a form *that is independent of any particular way of charting space-time*,[10] as moreover is done *implicitly* by Hamilton's principle.[11] This property becomes analytically obvious if, instead of introducing the derivatives $q'_1, \ldots, q'_n$ of the spatial parameters with respect to the time parameter, we introduce $n+1$ quantities

$$\dot{q}_1,\ \dot{q}_2,\ \ldots,\ \dot{q}_n,\ \dot{t}$$

whose mutual ratios are defined by the equalities

$$\frac{\dot{q}_1}{q'_1} = \frac{\dot{q}_2}{q'_2} = \cdots = \frac{\dot{q}_n}{q'_n} = \frac{\dot{t}}{1}.$$

By putting

$$F = \dot{t}\ (T+U),$$

where the right hand side, which is homogeneous and of first degree in $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, \dot{t}$, is expressed in terms of the $q_i, t, \dot{q}_i, \dot{t}$, the energy-momentum tensor takes the form

$$\omega_\delta = \frac{\partial F}{\partial \dot{q}_1}\ \delta q_1 + \cdots + \frac{\partial F}{\partial \dot{q}_n}\ \delta q_n + \frac{\partial F}{\partial \dot{t}}\ \delta t.$$

In the theory of general relativity, the motion of a point subject to gravitational forces obeys the preceding principle: the function F is then of the form

$$F = \sqrt{\sum a_{ik} \dot{q}_i \dot{q}_k}$$

with four variables q_i which are used locate the point in space and in time.

III. — *Transformation of the Canonical Equations. Jacobi's Theorem.*

14. An important application of the preceding considerations relates to the transformation of the canonical equations and to Jacobi's method of integration of the equations of dynamics.

[10] Fr. *une forme indépendante de tout mode particulier de repérage de l'espace-temps*.

[11] Fr. *comme le fait du reste implicitement le principe d'Hamilton*.

The integral $\int \omega_\delta$ over a closed contour obviously does not change if we add to ω_δ an exact differential; conversely, if another linear differential form ϖ_δ has the property of giving the same integral as ω_δ when performed over any closed contour, ϖ_δ differs from ω_δ only by an exact differential.

Suppose then that we can find $2n$ new variables r_i, s_i and a function K such that the two expressions

$$\omega_\delta = \sum p_i\,\delta q_i - H\,\delta t,$$
$$\varpi_\delta = \sum r_i\,\delta s_i - K\,\delta t,$$

differ only by an exact differential. The differential equations of motion can be characterised by the property that they admit the integral invariant $\int \varpi_\delta$ and consequently they can be written as

$$\frac{ds_i}{dt} = \frac{\partial K}{\partial r_i}, \qquad \frac{dr_i}{dt} = -\frac{\partial K}{\partial s_i};$$

the canonical form of the equations will be conserved.

The assumption made translates into an identity of the form

$$\sum p_i\,\delta q_i - \sum r_i\,\delta s_i - (H-K)\delta t = \delta V. \tag{19}$$

And it is easy to produce such an identity. In fact, start from an arbitrary function V of $2n+1$ arguments q_i, s_i, t and put

$$p_i = \frac{\partial V}{\partial q_i}, \qquad r_i = -\frac{\partial V}{\partial s_i}, \qquad K = \frac{\partial V}{\partial t} + H; \tag{20}$$

if these equations define a change of variables, that is, if the first n can be solved for the $s1, s_2, \ldots, s_n$, the following n will give $r_1, r_2, \ldots, r_n$; the last will give the function K and the new variables obtained will preserve the canonical form of the equations of dynamics. It is important to note that if equations (20) can be solved with respect to the r_i and the s_i, conversely they are soluble with respect to the p_i and the q_i; in fact, in both cases the condition for this to be possible is that the determinant

$$\left| \frac{\partial^2 V}{\partial q_i \partial s_j} \right|$$

is not identically zero.

The solution thus obtained from identity (19) is not the most general solution; in, fact, it leaves out the cases where the $2n+1$ quantities q_i, s_i, t are related by one or more relations; besides, these singular cases are easy to treat directly by giving a priori the relations that exist between the q_i, s_i, and t.

15. The preceding general theory becomes especially interesting for applications in two cases.

The first is that where the function K is identically zero; the canonical equations become

$$\frac{ds_i}{dt} = 0, \qquad \frac{dr_i}{dt} = 0;$$

the equations of the trajectories reduce to

$$s_i = a_i, \qquad r_i = b_i,$$

where a_i and b_i are $2n$ arbitrary constants. According to equations (20), for this to be the case, it is sufficient to find a function $V(t; q_1, \ldots, q_n; a_1, \ldots, a_n)$ that satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + H\left(t, q_i, \frac{\partial V}{\partial q_i}\right) = 0; \tag{21}$$

if this function V, which contains n arbitrary constants $a_1, \ldots, a_n$, is such that the determinant

$$\left| \frac{\partial^2 V}{\partial q_i \partial a_j} \right|$$

is not identically zero, the equations of motion are

$$p_i = \frac{\partial V}{\partial q_i}, \qquad b_i = -\frac{\partial V}{\partial a_i};$$

this is Jacobi's theorem. The condition on the determinant comes down to saying that the function V is a *complete integral* of Jacobi's first order partial differential equation (21).

The second application to point out has to do with *perturbation* theory. Suppose that the function H is the sum of two terms H_1 and H_2, the second of which is very small compared with the first: this amounts to dividing the given forces into two groups, of which one, very small with respect to the other, will consist of what we call *perturbing forces*. The method used in celestial mechanics in this case consists of looking for a complete integral V of Jacobi's equation

$$\frac{\partial V}{\partial t} + H_1 = 0,$$

where we include only the principal term of the function H. The $2n$ new variables r_i, s_i which are thus introduced would be constants *if the perturbing forces did not exist*; these are thus the parameters of the undisturbed trajectories. *The introduction of these new variables preserves the canonical form of the equations with the new function $K = H_2$, that is, the part of H that relates only to the perturbing forces.*

We will not dwell further, at least for the moment, on the theory of canonical equations and Jacobi's theorems. In particular the relation between the integration of the equations of dynamics and the integration of a first order partial differential equation *that does not explicitly contain the unknown* *function* will be seen in a new light after we have shown that *we can associate a linear integral invariant* to any such partial differential equations of this type[12] — or more generally to any first order partial differential equation that admits a known infinitesimal transformation.

[12] Fr., *s'éclairera d'un jour nouveau quand nous aurons montré ...*; literally, *will be illuminated by a new day when we will have shown ...*.

Chapter II

The Two-dimensional Integral Invariant of Dynamics

I. — *Forming the Two-dimensional Integral Invariant of Dynamics.*

16. We have seen that Hamilton's elementary action can be obtained by assuming that in the expression

$$\omega_\delta = \sum p_i \,\delta q_i - H\,\delta t,$$

we have

$$\delta q_i = q_i'\,\delta t.$$

It is worth noting that *the trajectories of a material system still realise the extremum of the integral*

$$W = \int_{t_0}^{t_1} \sum p_i \;\delta q_i - H \;\delta t,$$

by assuming simply that the q_i and the q_i' are any functions of t subject only to the conditions that the q_i take values given in advance at the endpoints. We thus no longer assume, as in Hamilton's principle, that the q_i' are the derivatives with respect to time of the q_i. We can even assume more generally that the q_i, q_i' and t are functions of the same parameter u that varies from 0 to 1, where the quantities q_i and t take given values at the end points.

An easy calculation gives

$$\delta W = \left[\sum p_i\,\delta q_i - H\,\delta t\right]_{u=0}^{u=1} + \int_{u=0}^{u=1} \left(\sum(\delta p_i\,dq_i - \delta q_i\,dp_i) - \delta H\,dt + \delta t\,dH\right).$$

The fully integrated part is zero by hypothesis; the equations of the extremals are obtained by setting to zero the coefficients of

$$\delta q_1,\ \delta q_2,\ \ldots,\ \delta p_n,\ \delta t$$

in the quantity under the integral sign; now, this calculation was carried out in n° **11**, and gave us precisely the equations of motion in their canonical form.

17. The expression

$$\sum(\delta p_i\, dq_i - \delta q_i\, dp_i) - \delta H\, dt + \delta t\, dH,$$

which we have encountered twice, is linear with respect to two series of differentials; we can write it in the simpler form

$$d\omega_\delta - \delta\omega_d,$$

by assuming that the two symbols of differentiation commute with each other. This expression, which we shall denote by $\omega'(d,\delta)$, has the property that it is zero whenever the symbol d defines an elementary displacement in the direction of a trajectory in the state space, where the symbol δ defines an *arbitrary* elementary displacement. Moreover, it is by expressing this property that we obtained the relations between $dq_1, dq_2, \ldots, dp_n, dt$ which define the differential equations of the trajectories or, from another point of view, the differential equations which admit the integral invariant $\int \omega_\delta$.

Consider now, more generally, *any* two elementary displacements defined by two symbols of differentiation δ and δ' and let us investigate the meaning of the bilinear form $\omega'(\delta,\delta')$. For this, consider a two-dimensional continuous set of states; we can realise such a set by taking for the q_i, p_i and t functions of two parameters α and β; each state of the set can be represented on a plane by the point with coordinates (α,β), and the set will be represented by an area of the plane. The symbols δ and δ' will refer respectively to increments in α alone and in β alone. Consider then, in the state space, four states A,B,C,D that correspond respectively to values

$$\begin{array}{cc} \alpha & \beta \\ \alpha+\delta\alpha & \beta \\ \alpha & \beta+\delta'\beta \\ \alpha+\delta\alpha & \beta+\delta'\beta \end{array}$$

of the parameters, and form the integral $\int \omega$ over the closed contour $ABCD$. We have clearly

$$\int_{AB} = \omega_\delta, \qquad \int_{AC} = \omega_{\delta'}, \qquad \int_{CD} = \omega_\delta + \delta'\omega_\delta, \qquad \int_{BD} = \omega_{\delta'} + \delta\omega_{\delta'},$$

and consequently

$$\int_{ABDC} = \delta\omega_{\delta'} - \delta'\omega_\delta = \omega'(\delta,\delta').$$

18. The bilinear form $\omega'(\delta, \delta')$, which in summary concerns an arbitrary state and two states infinitely close to it, that according to the forgoing represents the value of the integral $\int \omega$ over a closed contour, is *invariant* for the system of differential equations of the trajectories, in the sense that its value does not change if we move each of the states along its corresponding trajectory. This form is also the element of a double integral: moreover, if for example we view p_1 and q_1 as the coordinates (that depend on two parameters α and β) of a point of a plane, the expression $\delta p_1 \, \delta' q_1 - \delta q_1 \, \delta' p_1$ is the element of area of this plane referred to curvilinear coordinates α and β; this is what we usually write as

$$dp_1 \, dq_1, \quad \text{or} \quad \delta p_1 \, \delta q_1$$

This leads us to the concept of a new integral invariant

$$\iint \omega' = \iint \sum \delta p_i \, \delta q_i - \delta H \, \delta t; \tag{1}$$

this double integral over a two-dimensional area in the state space reproduces itself if we move each of the states of this area along its corresponding trajectory. This double integral can also be obtained, using the generalised Stokes formula, as an expression of the curvilinear integral

$$\int \omega = \int \sum p_i \, \delta q_i - H \, \delta t$$

over the closed curve that bounds this area.

In H. Poincaré's approach, we consider only areas formed by *simultaneous* states. We can thus state the result obtained as follows:

Given a two-dimensional set of trajectories, if we take on each trajectory of the set the state corresponding to a given time t, the double integral

$$\iint \sum \delta p_i \, \delta q_i$$

over these states is independent of t.

As we see, this theorem expresses only one particular aspect of the property proved above.

19. The two-dimensional integral invariant $\iint \omega'$ is said by H. Poincaré to be *absolute*, in contrast with the invariant $\int \omega$, which is said to be *relative*; this means that the double integral $\iint \omega'$ has an invariant character whatever the domain of integration, *open* or *closed*, while the integral $\int \omega$ has an invariant character only over a *closed* curve.

Since the integral $\iint \omega'$ is nothing other than the integral $\int \omega$ over a closed curve, we can affirm that *the differential equations of motion are the only ones which admit the integral invariant* $\iint \omega'$. The invariance of the integral $\iint \omega'$ is thus a new analytic translation of the generalised principle of the conservation of energy-momentum.

II. — *Application to Vortex Theory.*

20. So far, we have considered sets of trajectories, but these sets were realised only in our imagination. There is one case where such sets have concrete existence. It is that of an ideal fluid subject to forces obtained from a force function U. In fact, we prove the following equations in hydrodynamics,

$$
\begin{aligned}
\gamma_x &= \frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x},\\
\gamma_y &= \frac{\partial U}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y},\\
\gamma_z &= \frac{\partial U}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z},
\end{aligned}
\tag{2}
$$

in which $\gamma_x, \gamma_y, \gamma_z$ denote the components of the acceleration of the molecule that occupies position (x, y, z) at time t, and p and ρ denote respectively the pressure and the density at this point.

Add the assumption that there is a relation between p and ρ, given beforehand, which is certainly the case if the motion is isothermal.

If we consider a specific motion of the fluid, we may regard p as a given function of x, y, z, t, and by putting

$$q = \int \frac{dp}{\rho}$$

we see that *each molecule behaves like a material point of mass 1 placed in a force field obtained from the force function* $U - q$.

We thus obtain a concrete realisation of an infinite number of trajectories of a moving point subject to given forces. We note that the part $-q$ of the force function represents the action of the surrounding molecules on the molecule considered.

21. The trajectory of each molecule can be regarded as a particular solution of the system of differential equations

$$\left.\begin{aligned}\frac{dx}{dt} &= u, & \frac{du}{dt} &= \frac{\partial(U-q)}{\partial x},\\ \frac{dy}{dt} &= v, & \frac{dv}{dt} &= \frac{\partial(U-q)}{\partial y},\\ \frac{dz}{dt} &= w, & \frac{dw}{dt} &= \frac{\partial(U-q)}{\partial z};\end{aligned}\right\} \tag{3}$$

if we thus consider a closed sequence of molecules in the fluid (each taken at any instant), the integral

$$\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t \tag{4}$$

over this closed sequence does not change its value if we move each molecule along its trajectory. In this expression, we have put

$$E = \frac{1}{2}(u^2+v^2+w^2) - U + q; \tag{5}$$

E is the energy (per unit mass) of the fluid; this energy is the sum of the kinetic energy $\frac{1}{2}(u^2 + v^2 + w^2)$, the potential energy $-U$, and the internal hydrodynamic energy q.

If in particular we consider a closed sequence of molecules, all considered at the same time t, that is, a closed fluid line, the integral $\int u\,\delta x + v\,\delta y + w\,\delta z$ *keeps the same value if we take the same fluid line (that is, the fluid line formed from the same molecules) at different instants of the motion*. This is the classic theorem of the *conservation of circulation*; in fact we call the integral $\int u\,\delta x + v\,\delta y + w\,\delta z$ the circulation.

22. Now adopt a slightly different point of view. Consider again a particular motion of the fluid bulk; in this motion, the components u, v, w of the velocity are specific functions of x, y, z, t and the trajectories of the various molecules can be regarded as solutions of the system of differential equations

$$\left.\begin{aligned}\frac{dx}{dt} &= u,\\ \frac{dy}{dt} &= v,\\ \frac{dz}{dt} &= w,\end{aligned}\right\} \tag{6}$$

on the right hand sides of which u, v, w are assumed replaced by their values as functions of x, y, z, t. The integral

$$\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t$$

is obviously still a relative integral invariant for these new differential equations. By transforming it into a double integral, we will obtain an absolute integral invariant for system (6).

By forming the expression $\delta\omega_{\delta'} - \delta'\omega_{\delta}$, we get

$$\omega'(\delta,\delta') = \delta u\,\delta' x - \delta x\,\delta' u + \delta v\,\delta' y - \delta y\,\delta' v + \delta w\,\delta' z - \delta z\,\delta' w - \delta E\,\delta' t + \delta t\,\delta' E$$

The right hand side is linear with respect to the six combinations

$$\begin{array}{lll} \delta y\,\delta' z - \delta z\,\delta' y, & \delta z\,\delta' x - \delta x\,\delta' z, & \delta x\,\delta' y - \delta y\,\delta' x, \\ \delta x\,\delta' t - \delta t\,\delta' x, & \delta y\,\delta' t - \delta t\,\delta' y, & \delta z\,\delta' t - \delta t\,\delta' z. \end{array}$$

A simple calculation, which is nothing more than the application of Stokes' formula, gives, for the coefficients of the first three terms,

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}:$$

these are the components of the vorticity vector. To calculate the three other coefficients, we can use the comment that, since the expression ω' is invariant for equations (6), the equations obtained by setting to zero the coefficients of $\delta x, \delta y, \delta z, \delta t$ in $\omega'(d,\delta)$ must be consequences of equations (6). Therefore put

$$\begin{aligned} \omega'(d,\delta) = {} & \xi(dy\,\delta y - dz\,\delta y) + \eta(dz\,\delta x - dx\,\delta z) + \zeta(dx\,\delta y - dy\,\delta x) \\ & + P(dx\,\delta t - dt\,\delta x) + Q(dy\,\delta t - dt\,\delta y) + R(dz\,\delta t - dt\,\delta z). \end{aligned}$$

The equations considered are

$$\left.\begin{aligned} \eta\ dz - \zeta\ dy - P\ dt &= 0, \\ \zeta\ dx - \xi\ dz - Q\ dt &= 0, \\ \xi\ dy - \eta\ dx - R\ dt &= 0, \\ P\ dx + Q\ dy + R\ dz &= 0. \end{aligned}\right\} \tag{7}$$

By expressing thay they are a consequences of equations (6), we obtain

$$\begin{aligned} P &= \eta w - \zeta v, \\ Q &= \zeta u - \xi w, \\ R &= \xi v - \eta u. \end{aligned}$$

Consequently, the double invariant integral that we seek is

$$\iint \xi\,\delta y\,\delta z + \eta\,\delta z\,\delta x + \zeta\,\delta x\,\delta y + (\eta w - \zeta v)\delta x\delta t + (\zeta u - \xi w)\delta y\delta t + (\xi v - \eta u)\delta z\delta t \qquad (8)$$

Applied to an area formed by molecules all taken at the same time t, this integral is the *vorticity flux* across this area: we again find the theorem of the conservation of the vorticity flux across a fluid surface.

23. We could have calculated the expression $\omega'(d,\delta)$ directly. In particular, the coefficient P of $dx\,\delta t$ is clearly

$$P = -\frac{\partial u}{\partial t} - \frac{\partial E}{\partial x} = -\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} - w\frac{\partial w}{\partial x} + \frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x};$$

by writing that this is equal to the value found previously,

$$P = \eta w - \zeta v = w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial U}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

which is none other than the first equation of hydrodynamics; in fact, the left hand side is the expanded expression for γ_x.

This result reminds us that the integral $\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t$ is invariant for the differential equations (6) only if u, v, w are the velocity components of a molecule of an ideal fluid subject to a force derived from a force function or, again, if there is a *potential for accelerations*.

24. Equations (7), which can also be written as

$$\begin{aligned} \eta(dz - w\,dt) - \zeta(dy - v\,dt) &= 0,\\ \zeta(dx - u\,dt) - \xi(dz - w\,dt) &= 0,\\ \xi(dy - v\,dt) - \eta(dx - u\,dt) &= 0, \end{aligned} \qquad (7^{\text{again}})$$

are a consequence of differential equations (6), but they are not *equivalent* to these differential equations; in other words, *the trajectory equations (6) are not the only ones to admit the integral invariant* $\int \omega$. In particular, the same is true of the equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{dt}{0} \qquad (9)$$

of which equations (7) are clearly a consequence. The solutions of these equations are what we call *vortex lines*. *The property of the differential equations of the trajectories and of the differ-*

ential equations of the vortex lines of admitting the same integral invariant will lead us to the fundamental theorems of vortex theory.

In fact, we can characterise an elementary displacement $(dx, dy, dz, 0)$ (in the four-dimensional universe x, y, z, t) in the direction of a vortex line by the property that the bilinear form $\omega'(d, \delta)$ is zero, *whatever the displacement* δ: this follows immediately from equations (7). That said, consider a vortex line (Γ) at a given time t; the molecules that compose it form a line (Γ') at another time t': *we shall show that* (Γ') *is a vortex line at the time* t'. In fact, let $(dx', dy', dz', 0)$ be an elementary displacement along (Γ'), and let us associate with it an *arbitrary* displacement $(\delta x', \delta y', \delta z', \delta t')$. Displace the three states

$$(x', y', z', t'), \qquad (x'+dx', y'+dy', z'+dz', t'), \qquad (x'+\delta x', y'+\delta y', z'+\delta z', t'+\delta t')$$

along their respective trajectories, the first two up to time t, the last up to time $t+\delta t$; we obtain a two-dimensional element for which $(dx, dy, dz, 0)$ represents a displacement along the *vortex line* (Γ); the form $\omega'(d, \delta)$ thus has a value zero; so it is also zero for the original element and consequently (Γ') is a vortex line: this is Helmholtz's famous theorem.

25. Consider a vortex tube at time t and two closed curves (C) and (C_1) that go around the tube: the *circulation* along these two closed curves is the same, since $\int \omega$ is an integral invariant for the differential equations (9) of the vortex lines. At another time t', the vortex tube will have taken up another position in space, but the circulation along any closed line that goes around the new tube will not have changed either, since $\int \omega$ is an integral invariant for the differential equations of the trajectories. We find again the concept of what is called in hydrodynamics the *moment* or the *intensity* of a vortex tube, a quantity that is conserved for the entire motion. This property is only a particular aspect of the invariance of the integral

$$\int u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t$$

for the differential equations of the trajectories and for those of the vortex lines.

Incidentally, we will rediscover all these results as a special case of a general theorem on differential forms that are simultaneously invariant for several systems of differential equations.

It is needless to point out that all the above essentially assumes that ξ, η, ζ are not all zero, that is, that the motion of the fluid is *rotational*.

Chapter III

Integral Invariants and Invariant Differential Forms

I. — *General concept of an integral invariant.*

26. The preceding chapters have shown us the importance of the concept of an integral invariant for mechanics. We will now discuss this concept in all its generality.

Consider any system of first-order ordinary differential equations (we know that we can always reduce it to this case) which we will write as

$$\left.\begin{aligned} \frac{dx_1}{dt} &= X_1, \\ \frac{dx_2}{dt} &= X_2, \\ &\vdots \\ \frac{dx_n}{dt} &= X_n. \end{aligned}\right\} \tag{1}$$

We have distinguished the independent variable t and the dependent variables $x_1, x_2, \ldots, x_n$ but, as we will see, this distinction is not essential. We will continue to say that t represents time: the set of values $x_1, x_2, \ldots, x_n, t$ which correspond to a solution will be said to form a *trajectory*, which we can regard as a curve in the $(n+1)$-dimensional space $(x_1, x_2, \ldots, x_n, t)$.

That said, H. Poincare gave the name of *integral invariant* to an integral (simple or multiple) which, when over any set of *simultaneous* points (that is, all corresponding to the same value of t), does not change its value when we move the points of this set along their corresponding trajectories to any other instant t'. The integral invariant is said to be *absolute* if the property of invariance holds whatever the domain of integration; it is said to be *relative* if the property of invariance holds only for a *closed* domain of integration. The linear integral invariant

$$\int \sum p_i \, \delta q_i$$

of mechanics is relative; the double integral invariant

$$\iint \sum \delta p_i \, \delta q_i$$

of mechanics is absolute.

The simplest forms of integral invariants are

$$\int a_1 \, \delta x_1 + a_2 \, \delta x_2 + \cdots + a_n \, \delta x_n$$
$$\int \sqrt{a_{11} \, \delta x_1^2 + a_{22} \, \delta x_2^2 + \cdots + 2a_{12} \, \delta x_1 \, \delta x_2 + \cdots}$$
$$\iint a_{12} \, \delta x_1 \, \delta x_2 + a_{13} \, \delta x_1 \, \delta x_3 + \cdots + a_{n-1,n} \, \delta x_{n-1} \, \delta x_n$$
$$\iiint a_{123} \, \delta x_1 \, \delta x_2 \, \delta x_3 + \cdots$$

27. The quantity under the summation sign in an integral invariant is a differential form into which enter the variables, dependent and independent, and their differentials (or even several series of differentials). This form F can be considered in itself and has the property that, calculated for any point and one or more infinitely close but *simultaneous* points, it does not change its value if we move these points along their respective trajectories, *but always leaving them simultaneous*. It is clear that from this point of view that we could consider more general forms F than those which are likely to enter under an integral sign, for example any (homogeneous) rational function of $\delta x_1, \ldots, \delta x_n$.

As the examples treated in the first two chapters have shown us, *it is in our interest not to restrict ourselves to considering simultaneous points*. We shall see that any elementary integral invariant in the sense of H. Poincare can be regarded as resulting from the suppression all terms which contain the differential or differentials of the independent variable t in a more complete elementary integral invariant.

But, to arrive at this important result, which will give us the key to almost all the properties of integral invariants, it is necessary to recall briefly the classic properties of the first integrals of a system of differential equations.

II. — *First Integrals*

28. As we know, we call a function $u(x_1, \ldots, x_n, t)$ a *first integral* of system (1) if it has the property that, when $x_1, \ldots, x_n$ are replaced by their values as functions of t corresponding to *any* trajectory, the function u of t thus obtained reduces to a constant. These first integrals are solutions of the first-order linear partial differential equation

$$\frac{\partial u}{\partial t} + X_1 \frac{\partial u}{\partial x_1} + X_2 \frac{\partial u}{\partial x_2} + \cdots + X_n \frac{\partial u}{\partial x_n} = 0. \tag{2}$$

Suppose that we have integrated equations (1) and that we have expressed the dependent variables $x_1, \ldots, x_n$ as functions of time t and of their initial values $x_1^0, \ldots, x_n^0$ at $t = 0$, say,

$$\begin{aligned} x_1 &= f_1(t; x_1^0, \ldots, x_n^0) \\ &\vdots \\ x_n &= f_n(t; x_1^0, \ldots, x_n^0); \end{aligned}$$

these equations, solved with respect to $x_1^0, \ldots, x_n^0$, give for these n quantities functions of $x_1, \ldots, x_n, t$ which clearly are first integrals; we thus obtain a system of n first integrals, clearly *independent*, that is, which are not related by any identical relation in $x_1, \ldots, x_n, t$.

It is clear that any function of the first integrals $x_1^0, \ldots, x_n^0$ is a first integral *and conversely*; because if u is any first integral, its numerical value for *any* trajectory is, by this same property, equal to $u(x_1^0, \ldots, x_n^0, 0)$.

The total differential of any function u of $x_1, \ldots, x_n, t$ can obviously be put into the form

$$du = \lambda_1(dx_1 - X_1 dt) + \lambda_2(dx_2 - X_2 dt) + \cdots + \lambda_n(dx_n - X_n dt) + \lambda\, dt;$$

the necessary and sufficient condition that it be a first integral is that the coefficient λ be identically zero; we can easily understand this by a direct argument; we can also verify it by noting that λ is nothing other than the left hand side of equation (2). Thus *the differential of any first integral is a linear combination of the n linear differential forms*

$$dx_1 - X_1 dt, \quad dx_2 - X_2 dt, \quad \ldots, \quad dx_n - X_n dt,$$

and, conversely, each of these forms is a linear combination of the differentials of n given independent first integrals.

III. — *Absolute Integral Invariants and Invariant Differential Forms.*

29. That said, we will first deal with *absolute* integral invariants. The element of any absolute integral invariant is a differential form $F(x_1, \ldots, x_n, t; \delta x_1, \ldots, \delta x_n)$ which does not change in value if we move the point $(x_1, \ldots, x_n, t)$ and the infinitely close point $(x_1 + \delta x_1, \ldots, x_n + \delta x_n, t)$ along their respective trajectories, but always considering them at the same time. In particular, consider them at time $t = 0$. We will have

$$F(x_1,\ldots,x_n,t;\delta x_1,\ldots,\delta x_n) = F(x_1^0,\ldots,x_n^0,0;\delta x_1^0,\ldots,\delta x_n^0).$$

Now consider the x_i^0 on the right hand side, as first integrals of system (1) and replace them by their values as functions of $x_1,\ldots,x_n,t$; we will obtain a new identity

$$F(x_1^0,\ldots,x_n^0,0;\delta x_1^0,\ldots,\delta x_n^0) = \Phi(x_1,\ldots,x_n,t;\delta x_1,\ldots,\delta x_n,\delta t).$$

The right hand side of this identity is clearly a quantity whose numerical value concerns only the trajectory defined by the initial values $x_1^0,\ldots x_n^0$ and the infinitely close trajectory. Its value is thus independent of the particular point $x_1,\ldots,x_n,t$ taken on the first trajectory and of the particular point $x_1+\delta x_1,\ldots,x_n+\delta x_n,t+\delta t$ taken on the infinitely close trajectory; it is thus also an element of an integral invariant, but of a more complete integral invariant than the one that served as our starting point, *since now we are no longer obliged to restrict ourselves to the consideration of simultaneous points*.

Note now that it is very easy to go from the initial form F to the final form Φ. In fact, were we to regard t as a constant in the calculation of $x_1^0,\ldots,x_n^0,\delta x_1^0,\ldots,\delta x_n^0$, we would obviously return to the form F; we thus have

$$\Phi(x_1,\ldots,x_n,t;\delta x_1,\ldots,\delta x_n,0) = F(x_1,\ldots,x_n,t;\delta x_1,\ldots,\delta x_n)$$

Now, δt enters only through $\delta x_1^0,\ldots,\delta x_n^0$, and these n differentials are linear combinations of

$$\delta x_1 - X_1\delta t, \qquad \delta x_2 - X_2\delta t, \qquad \ldots, \qquad \delta x_n - X_n\delta t;$$

consequently, Φ depends only on these n linear combinations and when we have its expression for $\delta = 0$, we immediately get its expression for any δt by replacing δx_1 by $\delta x_1 - X_1\delta t$, etc...

Finally we have

$$\Phi(x_1,\ldots,x_n,t;\delta x_1,\ldots,\delta x_n,\delta t) = F(x_1,\ldots,x_n,t;\delta x_1 - X_1\delta t,\ldots,\delta x_n - X_n\delta t). \tag{3}$$

30. Let us summarise the results that we have just obtained. They are two in number.

1° The form F, which is the *element* of an absolute integral invariant in the sense of H. Poincare, and in which only the differentials of the dependent variables enter, is associated with a more complete form Φ in which the differential (or differentials) of the independent variable t also enters at the same time. We go from the form Φ to the form F by deleting the terms that contain δt, and conversely we go from the form F to the form Φ by replacing the

$$\delta x_1, \delta x_2, \ldots, \delta x_n$$

by the

$$\delta x_1 - X_1 \delta t, \qquad \delta x_2 - X_2 \delta t, \qquad \ldots, \qquad \delta x_n - X_n \delta t$$

respectively.

2° The form Φ can be expressed by means of the first integrals of system (1) and their differentials.

This last property shows clearly the invariant character of the form Φ.

A simple example will make us better understand the relation between the two forms F and Φ. If we begin from any integral u, the total differential δu is clearly a form Φ; the corresponding form F is

$$F = \frac{\partial u}{\partial x_1}\,\delta x_1 + \frac{\partial u}{\partial x_2}\,\delta x_2 + \cdots + \frac{\partial u}{\partial x_n}\,\delta x_n,$$

and we indeed have

$$\Phi = \delta u = \frac{\partial u}{\partial x_1}\,(\delta x_1 - X_1 \delta t) + \frac{\partial u}{\partial x_2}\,(\delta x_2 - X_2 \delta t) + \cdots + \frac{\partial u}{\partial x_n}\,(\delta x_n - X_n \delta t).$$

31. We will agree to say that a differential form which can be expressed by means of first integrals of system (1) and of their differentials is an *invariant* form for system (1). The quantity under the integral sign in an absolute integral invariant is obtained by setting δt equal to zero in an invariant form. This is how the double integral invariant of dynamics corresponds to the invariant form

$$\sum \delta p_i\, \delta q_i - \delta H\, \delta t,$$

or, if we prefer, by introducing two series of differentials,

$$\Phi = \sum (\delta p_i\, \delta' q_i - \delta q_i\, \delta' p_i) - \delta H\, \delta' t + \delta t\, \delta' H.$$

Expressed by means of the first integrals p_i^0, q_i^0 is clearly

$$\Phi = \sum (\delta p_i^0\, \delta' q_i^0 - \delta q_i^0\, \delta' p_i^0).$$

IV. — *Relative Integral Invariants. Hamilton's Function.*

32. Part of the preceding results apply also to the theory of relative integral invariants. This is how the linear integral invariant of dynamics, as considered by H. Poincare,

$$\int \sum p_i\, \delta q_i$$

which does not change in value when we move each state along its trajectory from time t to any other time t', is equal to the integral

$$\int \sum p_i^0 \, \delta q_i^0.$$

Any relative integral invariant can thus assume an expression which involves only first integrals and their differentials and, in this form, it can be extended without loosing its invariant character to any closed domain formed from simultaneous or non-simultaneous states.

But if in the new expression we replace the first integrals by their expressions as functions of the dependent and independent variables, we obtain a form Φ under the summation sign *which can no longer be deduced from the initial form F by the same procedure* as in the case of the absolute invariants. In fact, the equality

$$\int F(x_1, \ldots, x_n, t;\, \delta x_1, \ldots, \delta x_n) = \int \Phi(x_1, \ldots, x_n, t;\, \delta x_1, \ldots, \delta x_n, 0)$$

does indeed hold for all *closed* domains of integration formed by simultaneous points, but term-by-term equality of the two sums does not follow and we no longer necessarily have the identity

$$F(x_1, \ldots, x_n, t;\, \delta x_1, \ldots, \delta x_n) = \Phi(x_1, \ldots, x_n, t;\, \delta x_1, \ldots, \delta x_n, 0)$$

which would be necessary for us to be able to deduce, in accordance with formula (4), that

$$\Phi(x_1, \ldots, x_n, t;\, \delta x_1, \ldots, \delta x_n, \delta t) = F(x_1, \ldots, x_n, t;\, \delta x_1 - X_1 \delta t, \ldots, \delta x_n - X_n \delta t).$$

This is how, in the simple case of an unconstrained material point, the element

$$F = m(x' \, \delta x + y' \, \delta y + z' \, \delta z),$$

which enters under the summation sign into the expression of the linear integral invariant of H. Poincare, would lead to the form

$$m(x' \, \delta x + y' \, \delta y + z' \, \delta z) - m(x'^2 + y'^2 + z'^2)\delta t$$

which is not at all an element of a complete integral invariant, and which does not differ by a simple exact differential from the form,

$$\omega_\delta = m(x' \delta x + y' \delta y + z' \delta z) - \left[\frac{1}{2}\, m(x'^2 + y'^2 + z'^2) - U\right] \delta t$$

as would be necessary.

It should be noted that the difficulty that presents itself here in going from a relative integral invariant in the sense of H. Poincare to the complete integral invariant is not of great practical importance, because any relative integral invariant reduces to an absolute integral invariant. In fact, we know that an integral over a closed contour, a closed surface, etc., reduces to an integral over an area bounded by the closed contour, a volume bounded by the closed surface, etc.

33. It is useful to illustrate the preceding considerations with some examples.

Return to the (complete) linear integral invariant of dynamics, that is, the energy-momentum tensor,

$$\omega_\delta = \sum p_i \delta q_i - H\delta t.$$

We have the equality

$$\int_{(C)} \sum p_i \delta q_i - H\delta t = \int_{(C_0)} \sum p_i^0 \delta q_i^0,$$

where we assume that the closed contour (C_0) is formed from states that make up (C), but moved along their trajectories to time $t = 0$. We can also consider the integral on the right hand side as over the same contour (C) as the integral on the left hand side, provided that we regard the p_i^0 and the q_i^0 as functions of the p_i, q_i and t. From this point of view, the two expressions

$$\sum p_i \delta q_i - H\delta t \qquad \text{and} \qquad \sum p_i^0 \delta q_i^0$$

which give the same integral along any closed contour, differ only by an exact differential, and we have

$$\sum p_i \delta q_i - H\delta t = \delta S + \sum p_i^0 \delta q_i^0 \tag{4}$$

We call the function S Hamilton's function, and it has a simple concrete interpretation. In fact, if we refer to formula (10) of Chapter I which gives the variation of the action along a variable trajectory, we see that *S can be interpreted as representing the action between time 0 and time t along the trajectory which ends at the state (p,q,t).*

This function S was considered by Hamilton and, from the historical point of view, it is of some importance, since it was Hamilton's remarks about it which set Jacobi on the path to his discoveries concerning the integration of the equations of dynamics. In fact, Hamilton remarked that if we knew how to express the function S, not as a function of the p_i, q_i and t, but as a function of q_i, q_i^0 and t, we would thereby also have integrated the equations of motion. Identity (4), put into in the form

$$\delta S = \sum p_i \delta q_i - H\delta t - \sum p_i^0 \delta q_i^0$$

in fact would give

$$p_i = \frac{\partial S}{\partial q_i}, \qquad -p_i^0 = \frac{\partial S}{\partial q_i^0}, \qquad \frac{\partial S}{\partial t} + H = 0. \tag{5}$$

The second equations would give the p_i as functions of t and of $2n$ initial values, the first would give the momenta p_i. Finally, the last shows that the function S is a solution of the partial differential equation

$$\frac{\partial S}{\partial t} + H\left(t, q_i, \frac{\partial S}{\partial q_i}\right) = 0. \tag{6}$$

In this approach, the difficulty was not so much to integrate this partial differential equation, but to find a solution for which *the arbitrary constants* q_i^0 *were precisely the initial values of the* q_i. Jacobi resolved this difficulty by showing that this condition was not useful for pressing into service the integration of the partial differential equation (6) for the integration of the equations of motion: we have already explained this briefly in n°14.

34. It is quite instructive to calculate Hamilton's function S in a simple case, for example that of an unconstrained point of mass 1 which is not subject to any force. Here the equations of motion are

$$\begin{aligned} x &= {x_0}' t + x_0, \qquad x' = {x_0}', \\ y &= {y_0}' t + y_0, \qquad y' = {y_0}', \\ z &= {z_0}' t + z_0, \qquad z' = {z_0}'. \end{aligned}$$

The difference

$$\delta S = \omega_\delta - (\omega_\delta)_0 = x'\,\delta x + y'\,\delta y + z'\,\delta z - \frac{1}{2}(x'^2 + y'^2 + z'^2)\delta t - ({x_0}'\,\delta x_0 + {y_0}'\,\delta y_0 + {z_0}'\delta z_0)$$

is equal to

$$\begin{aligned} \delta S &= x'\,\delta x + y'\,\delta y + z'\,\delta z - \frac{1}{2}(x'^2 + y'^2 + z'^2)\delta t - x'\,\delta(x - tx') - y'\,\delta(y - ty') - z'\,\delta(z - tz') \\ &= \frac{1}{2}(x'^2 + y'^2 + z'^2)\delta t + t(x'\,\delta x' + y'\,\delta y' + z'\,\delta z'), \end{aligned}$$

from which, taking into account that S must vanish with t,

$$S = \frac{1}{2}(x'^2 + y'^2 + z'^2)t.$$

By expressing S by means of $x, y, z, x_0, y_0, z_0, t$, we get

$$S = \frac{1}{2}\,\frac{(x - x_0)^2 + (y - y_0)^2 + (y - y_0)^2}{t}.$$

From this function, Hamilton's formulae (5) allow us to deduce the equations of motion,

$$x' = \frac{\partial S}{\partial x} = \frac{x - x_0}{t}, \qquad -x'_0 = \frac{\partial S}{\partial x_0} = -\frac{x - x_0}{t},$$
$$y' = \frac{\partial S}{\partial y} = \frac{y - y_0}{t}, \qquad -y'_0 = \frac{\partial S}{\partial y_0} = -\frac{y - y_0}{t},$$
$$z' = \frac{\partial S}{\partial z} = \frac{z - z_0}{t}, \qquad -z'_0 = \frac{\partial S}{\partial z_0} = -\frac{z - z_0}{t}.$$

V. — *Examples. The "element of mass" form.*

35. After the preceding parenthesis, let us return to absolute integral invariants.

In the simplest cases, it is good to appreciate directly the invariant character of the differential forms Φ which are deduced from the forms F by replacing the δx_i by $\delta x_i - X_i \delta t$ as stated above.

To simplify, take a system of two differential equations in two unknown functions

$$\frac{dx}{dt} = X, \qquad \frac{dy}{dt} = Y,$$

and start from an absolute linear integral invariant

$$I = \int a(x,y,t)\, \delta x + b(x,y,t)\, \delta y;$$

the associated complete integral invariant is

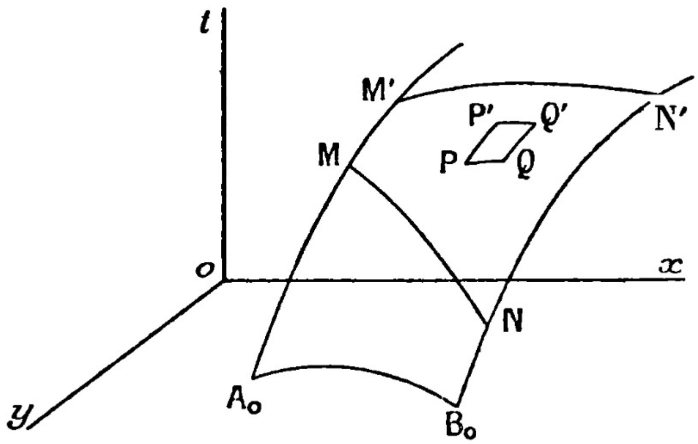
$$J = \int a(x,y,t)(\delta x - X\, \delta t) + b(x,y,t)(\delta y - Y\, \delta t).$$

Start from a curvilinear arc $A_0 B_0$ in the xy plane and lay down the corresponding trajectories through the various points of this arc of curve; we thus get a kind of cylindrical surface whose (non rectilinear) generators would be the trajectories. On this surface, draw two arcs of curve MN and $M'N'$ joining the trajectory through A_0 to the trajectory through B_0. We want to show that we have

$$J_{MN} = J_{M'N'}$$

The two arcs of curve MN and $M'N'$, with the trajectory arcs MM' and NN', bound a closed area on the surface; on the other hand, the integral J over each of these last two arcs is clearly zero, since by moving along one of these arcs, we always have

$$\delta x = X\, \delta t, \qquad \delta y = Y\, \delta t.$$



Consequently, the integral J over the closed contour $MNN'M'$ is

$$J_{MNN'M'} = J_{MN} - J_{M'N'}$$

and so it all comes down to proving that this integral is zero. Now, according to the Stokes formula, this integral reduces to a surface integral over the area $MNN'M'$. *We will show that the element of this surface integral is zero.* In fact, for this decompose the surface into surface elements by small parallelograms formed, on the one hand by trajectory arcs, on the other *of sections by planes* $t=$ constant. Let $PQQ'P'$ be one of these elements of surface. The corresponding element of the surface integral is equal to

$$J_{PQ} - J_{P'Q'},$$

but, *since the points of PQ are simultaneous*, as well as those of $P'Q'$, J_{PQ} reduces to I_{PQ}, and $J_{P'Q'}$ to $I_{P'Q'}$. Now, these two integrals I_{PQ} and $I_{P'Q'}$ are equal, by the property that I is an integral invariant.

Thus the element of the surface integral is indeed zero, and the theorem is proved.

36. A similar argument could be made in the case of a double integral invariant

$$I = \iint a(x,y,t)\,\delta x \delta y.$$

Here going from the form F to the form Φ is a little more difficult than in the preceding case. We get there by likening the surface element $\delta x \delta y$ to a bilinear force $\delta x \delta' y - \delta y \delta' x$; for this, it is sufficient to imagine any system of curvilinear coordinates α, β and to regard $\delta x, \delta y$ as the elementary displacement with respect to an increase $\delta\alpha$ of the first coordinate α, and $\delta' x, \delta' y$ as the elementary displacement with respect to an increase $\delta'\beta$ of the second β. We then have

$$F = a\begin{vmatrix} \delta x & \delta y \\ \delta' x & \delta' y \end{vmatrix};$$

from this we deduce

$$\Phi = a\begin{vmatrix} \delta x - X\delta t & \delta y - Y\delta t \\ \delta' x - X\delta' t & \delta' y - Y\delta' t \end{vmatrix} = a\begin{vmatrix} \delta x & \delta y \\ \delta' x & \delta' y \end{vmatrix} + aX\begin{vmatrix} \delta y & \delta t \\ \delta' y & \delta' t \end{vmatrix} + aY\begin{vmatrix} \delta t & \delta x \\ \delta' t & \delta' x \end{vmatrix},$$

or, returning to the notation in use in the theory of surface integrals,

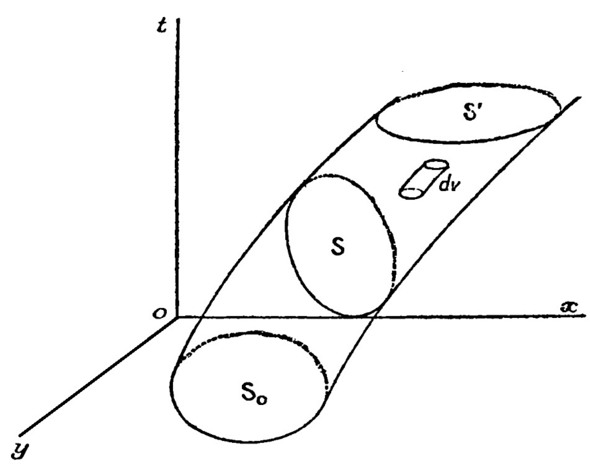
$$\Phi = a\,\delta x\delta y + aX\,\delta y\delta t + aY\,\delta t\delta x.$$

Consider then the surface integral

$$J = \iint a\,\delta x\delta y + aX\,\delta y\delta t + aY\,\delta t\delta x$$

and let us try to appreciate directly its invariant character. For this, imagine any area S_0 in the xy plane and construct the trajectories through the various points of this area. We thus obtain an indefinite volume bounded by a kind of cylindrical lateral surface generated by the trajectories which start from the contour of S_0. Cut this volume by any two surfaces: in this way we get two areas (plane or curved) S and S' in the interior of the volume, but extending as far as the lateral surface. We want to prove that we have

$$J_S = J_{S'}.$$



Together with a portion of the lateral surface of the cylinder, the areas S and S' define a volume V; on the other hand, the integral J over the lateral area which bounds this volume is clearly zero since, calling the element of area $d\sigma$ and the direction cosines of the normal α, β, γ, we have

$$J = \iint a(\gamma + X\alpha + Y\beta)\, d\sigma$$

and that the direction $(X, Y, 1)$, which is that of the tangents to the trajectories that generate the lateral surface considered, is normal to the direction (α, β, γ). It follows from this that the difference $J_{S'} - J_S$ can be regarded as the surface integral J over the closed area that bounds the volume V. It all comes down to showing that the integral of the equivalent volume is identically zero. Now, the element of this volume integral is clearly zero: to see this, it is sufficient to take for the elementary volume the volume bounded laterally by small arcs of the trajectory and two plane areas *parallel to the xy plane* at the ends, because then the surface integral J over each of the bases reduces to the integral I, and the value of the integral I is, by hypothesis, the same for the two bases.

37. The kinematics of continuous media provides us with a concrete illustration of the ideas developed in this Chapter.

In a continuous medium in motion, the trajectory of each molecule can be regarded as a solution of the system of differential equations

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w,$$

where u, v, w, the components of the velocity, are assumed to be expressed as a function of x, y, z, t. On the other hand, let $\rho(x, y, z, t)$ be the density at time t at the point (x, y, z). The mass which at time t fills any volume V is given by the triple integral

$$I = \iiint_V \rho\, \delta x \delta y \delta z;$$

this integral is clearly an absolute integral invariant in the sense of H. Poincare: it is in fact the first example of an integral invariant given by H. Poincare. If the molecules which fill the volume V at time t fill the volume V' at another time t', we have clearly

$$\iiint_V \rho(x, y, z, t)\, \delta x \delta y \delta z = \iiint_{V'} \rho(x', y', z', t')\, \delta x' \delta y' \delta z'.$$

The form Φ associated with the form $F = \rho\, \delta x \delta y \delta z$ will be calculated, as in the preceding example, by writing F in the form

$$F = \rho \begin{vmatrix} \delta x & \delta y & \delta z \\ \delta' x & \delta' y & \delta' z \\ \delta'' x & \delta'' y & \delta'' z \end{vmatrix};$$

from which we deduce

$$\Phi = \rho \begin{vmatrix} \delta x - u\delta t & \delta y - v\delta t & \delta z - w\delta t \\ \delta' x - u\delta' t & \delta' y - v\delta' t & \delta' z - w\delta' t \\ \delta'' x - u\delta'' t & \delta'' y - v\delta'' t & \delta'' z - w\delta'' t \end{vmatrix},$$

whence, by a simple calculation,

$$\Phi = \rho\,(\delta x \delta y \delta z - u \delta y \delta z \delta t - v \delta z \delta x \delta t - w \delta x \delta y \delta t)\,.$$

This form Φ represents the *mass element* considered in its complete kinematic form. If we consider any three-dimensional set of molecules, and if we take each molecule of the set at any time t of its motion, we get a three-dimensional domain in the four-dimensional universe (x,y,z,t); the triple integral of Φ over this domain will be equal to the total mass of the set of molecules considered. If the molecules are all taken at the same time t, they fill a certain volume V at this instant and the integral of Φ reduces to the integral $\iiint_V \rho\,\delta x \delta y \delta z$. But this is a very special case.

For definiteness, consider for example an area S in space and the set of all the molecules that cross this area S between time t_0 and time t_1. Take each of these molecules *at the moment that they cross the area S*. We have here a three-dimensional domain in the universe (x,y,z,t). The states of this domain are easily expressed in terms of three parameters α,β,γ: for this, it is sufficient to express the coordinates of a point of S as a function of two parameters α,β and to take $t=\gamma$. We will then have formulae such as

$$\begin{aligned} x &= f(\alpha,\beta), \\ y &= g(\alpha,\beta), \\ z &= h(\alpha,\beta), \\ t &= \gamma, \end{aligned}$$

where the parameters α,β take all values corresponding to the various points of the area S and the parameter γ takes all possible values in the interval (t_0,t_1). The integral Φ over this domain, not taking signs into account, will clearly be

$$\int_{t_0}^{t_1} \delta t \left[\iint_{(S)} \rho u\,\delta y \delta z + \rho v\,\delta z \delta x + \rho w\,\delta x \delta y \right].$$

The surface integral in square brackets represents the *flow of mass* at the time t across the surface S; multiplied by δt, it represents the quantity of mass that crosses the surface S in the interval $(t,t+\delta t)$. The total integral thus represents the total mass that crosses S in the interval (t_0,t_1), as we should have expected.

38. Similar remarks apply to the double integral invariant that we encountered in hydrodynamics (Chapter II, formula (8))

$$J = \iint \xi \delta y \delta z + \eta \delta z \delta x + \zeta \delta x \delta y + (\eta w - \zeta v) \delta x \delta t + (\zeta u - \xi w) \delta y \delta t + (\xi v - \eta u) \delta z \delta t.$$

We have seen (n° 25) that this integral, over a two-dimensional set of molecules taken at the same time t, represented the moment or the intensity of the vortex tube formed by the vortex lines that start from these molecules. Consider then the set of molecules that cross an arc of curve C in a time interval (t_0, t_1). Instead of taking these molecules at the same time t, take each of them at the time that they cross the arc of curve C. The moment of the vortex tube of which they are a part at any instant t will be equal to the integral

$$\int_{t_0}^{t_1} \delta t \int_{(C)} \begin{vmatrix} \delta x & \delta y & \delta z \\ \xi & \eta & \zeta \\ u & v & w \end{vmatrix}.$$

Chapter IV

The Characteristic System of a Differential Form.

I. — *The class of a differential form.*

39. Throughout this chapter, we consider systems of differential equations in n variables $x_1, x_2, \ldots, x_n$ without distinguishing the independent variable by a special notation: it will be any one of the variables $x_1, \ldots, x_n$. In other words, we will consider systems of differential equations of the type

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n}\,. \tag{1}$$

One of the first problems that arise in the theory of integral invariants is the following: *recognise if a given differential form is invariant for a given system of differential equations, and more generally, determine all systems of differential equations that admit a given differential form as an invariant form.*

Before solving this problem for the differential forms that arise most frequently in applications, we make some general remarks that will lead us to an extremely important theorem.

For a form Φ to be invariant for system (1), it is necessary and sufficient that it can be expressed by means of first integrals of (1) and their differentials. Thus a *necessary* condition for a given form Φ to be an invariant form for a suitably chosen system of differential equations is that this form can be expressible in terms of *at most* $n-1$ *quantities* and their differentials.

40. Suppose then that the given form Φ can be expressed by means of $r < n$ quantities $y_1, \ldots, y_r$ (functions of the x_i) and their differentials; suppose also that *it cannot be* expressed similarly by means of *less than* r quantities. Under these conditions, we will now prove the following theorem:

For a system of differential equations to admit Φ as an invariant form, it is necessary and sufficient that $y_1, \ldots, y_r$ are first integrals of this system.

The condition is clearly sufficient. To prove that it is necessary, consider a system of differential equations that admits Φ as an invariant form, and write down the equations of this system by taking as new variables $y_1, \ldots, y_r$ and $n-r$ other independent quantities $y_{r+1}, \ldots, y_n$. Let the equations of this system be

$$\frac{dy_1}{Y_1} = \frac{dy_2}{Y_2} = \cdots = \frac{dy_r}{Y_r} = \frac{dy_{r+1}}{Y_{r+1}} = \cdots = \frac{dy_n}{Y_n} \tag{2}$$

Were $y_1, \ldots, y_r$ not all first integrals, the first r denominators $Y_1, \ldots, Y_r$ would not all be zero; suppose for example that $Y_r \neq 0$. We could then take y_r as the independent variable and the form Φ would not change its value if we were to replace y_r and δy_r everywhere by zero, and then

$$y_1,\ \ldots,\ y_{r-1},\ y_{r+1},\ \ldots,\ y_n$$

by their initial values

$$y_1^0,\ \ldots,\ y_{r-1}^0,\ y_{r+1}^0,\ \ldots,\ y_n^0$$

considered as first integrals of system (2), and finally the differentials

$$\delta y_1,\ \ldots,\ \delta y_n$$

by

$$\delta y_1^0,\ \ldots,\ \delta y_n^0.$$

But then, since Φ contains neither $y_{r+1}, \ldots, y_n$ nor their differentials, the new form Ψ obtained would depend only on $y_1^0, \ldots, y_{r-1}^0$ and their differentials: in other words, we could find $r-1$ functions $z_1, \ldots, z_{r-1}$ of the x_i, such that Φ can be expressed in terms of these $r-1$ functions and their differentials. This result is contrary to the hypothesis. The number r will be called the *class* of the form Φ.

II. — *The characteristic system of a differential form.*

41. This extremely general theorem has important consequences which will help us better to understand its scope.

The most general system of differential equations that admits the form Φ as an invariant form, written using the variables $y_1, \ldots, y_n$, is, according to the preceding,

$$\frac{dy_1}{0} = \frac{dy_2}{0} = \cdots = \frac{dy_r}{0} = \frac{dy_{r+1}}{Y_{r+1}} = \cdots = \frac{dy_n}{Y_n}\,, \tag{3}$$

where $Y_{r+1}, \dots, Y_n$ are *arbitrary* functions. We deduce immediately that any first integral common to these systems is a function of $y_1, \dots, y_r$. *If therefore the form Φ can be expressed in a second way by means of r quantities $z_1, \dots, z_n$ and their differentials, the z_i will be functions of the y_i and conversely*, since the z_i are first integrals common to all differential systems which admit Φ as an invariant form. This comes down to saying that *there is essentially only one way to express the form Φ by means of the minimum number of variables and their differentials*, in the sense that when we have one expression involving the minimum number r of quantities $y_1, \dots, y_r$, all others are obtained by performing an arbitrary change of variables on the y. — This conclusion would obviously be false if r were not the minimum number of variables.

42. Another consequence is the following. Let us agree to say that a certain number, three for example, of systems of differential equations in n variables,

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n},$$
$$\frac{dx_1}{X_1'} = \frac{dx_2}{X_2'} = \cdots = \frac{dx_n}{X_n'},$$
$$\frac{dx_1}{X_1''} = \frac{dx_2}{X_2''} = \cdots = \frac{dx_n}{X_n''},$$

are *linearly independent* if it is impossible to find three coefficients $\lambda, \lambda', \lambda''$ *not all zero* such that we have

$$\begin{aligned}
\lambda X_1 + \lambda' X_1' + \lambda'' X_1'' &= 0,\\
\lambda X_2 + \lambda' X_2' + \lambda'' X_2'' &= 0,\\
&\vdots\\
\lambda X_n + \lambda' X_n' + \lambda'' X_n'' &= 0.
\end{aligned}$$

Otherwise, we will say that they are *linearly dependent*.

The property of several systems of being linearly independent, or not, clearly persists with any changes of variables.

Among systems (3) which admit Φ as an invariant form, we can obviously find $n-r$ linearly independent systems, namely those obtained by making *all but one* of the denominators $Y_{r+1}, \dots, Y_n$ zero. Moreover, all the systems (3) depend linearly on these $n-r$ particular systems.

We thus see that *if a form Φ is invariant for $n-r$, and only $n-r$, linearly independent systems of differential equations, it is invariant for any system that depends linearly on them, and moreover, all these systems have in common r independent first integrals.*

43. Suppose for example that $n-r=2$. There exist two systems of differential equations, say

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n},$$
$$\frac{dx_1}{X_1'} = \frac{dx_2}{X_2'} = \cdots = \frac{dx_n}{X_n'},$$

that admit Φ as an invariant form, and any other system that has this property depends linearly on these two. Call the trajectories of the first system (C), and those of the second (Γ). At any point M in the n-dimensional space, take the trajectory (C) and the trajectory (Γ) which pass through this point; take any point P on (C), and any point Q on (Γ); finally, construct the trajectory (Γ') that passes through P, and the trajectory (C') that passes through Q. *These two new trajectories intersect.* In fact, if $y_1, \ldots, y_{n-2}$ are the first integrals common to the two systems considered, and if $a_1, \ldots, a_{n-2}$ are the numerical values of these integrals at M, their numerical values at point P and at point Q are still the same, consequently the curves $(C), (\Gamma), (\Gamma'), (C')$ all lie on the same two-dimensional manifold,

$$y_1 = a_1, \quad y_2 = a_2, \quad \ldots, \quad y_{n-2} = a_{n-2},$$

so, finally, the last two intersect.

44. The preceding case arises precisely for the double integral invariant of vortex theory, which corresponds to the differential form

$$\Phi = \xi\,\delta y\delta z + \eta\,\delta z\delta x + \zeta\,\delta x\delta y + (\eta w - \zeta v)\delta x\delta t + (\zeta u - \xi w)\delta y\delta t + (\xi v - \eta u)\delta z\delta t. \tag{4}$$

We have seen (n° **24**) that the systems of differential equations which admit Φ as an invariant form are those which have as a consequence the three equations

$$\begin{aligned} \eta(dz - w\,dt) - \zeta(dy - v\,dt) &= 0,\\ \zeta(dx - u\,dt) - \xi(dz - w\,dt) &= 0,\\ \xi(dy - v\,dt) - \eta(dx - u\,dt) &= 0. \end{aligned} \tag{5}$$

The most general of these systems can be written as

$$\frac{dx}{\lambda u + \mu\xi} = \frac{dy}{\lambda v + \mu\eta} = \frac{dz}{\lambda w + \mu\zeta} = \frac{dt}{\lambda}$$

and it is deduced linearly from two systems

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dt}{1},$$
$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{dt}{0},$$

which define the trajectories of the fluid molecules and the vortex lines. The first are the curves (C), and the second the curves (Γ) from earlier, and the properties obtained in the general case can be expressed here by saying that *the molecules that form a vortex line* (Γ) *at time* t *again form a vortex line* (Γ') *at time* t'. The Helmholtz theorem is thus a very special consequence of the general theorem proved at the beginning of this Chapter.

45. In the two preceding Paragraphs we assumed that $n-r=2$. Analogous geometric considerations could be developed whatever may be the values of n and r; they would be based on the existence of manifolds defined by r equations of the form

$$y_1 = a_1, \quad y_2 = a_2, \quad \ldots, \quad y_r = a_r$$

and such that any trajectory of a differential system (3) which has one of its points there is entirely contained in it. Each of these manifolds, which is $n-r$-dimensional, could be obtained by beginning from any point M and constructing through this point a trajectory of any of the systems which admit Φ as an invariant form, and by constructing a trajectory of any other of these systems through any point P on this trajectory, and so on; by these operations, we could reach the entire $(n-r)$-dimensional manifold and never exit it.

We call these manifolds *characteristic manifolds* of the form Φ.

The characteristic manifolds can be regarded as the result of the integration of the equations

$$dy_1 = 0, \quad dy_2 = 0, \quad \ldots, \quad dy_r = 0;$$

now, returning to the original variables $x_1, \ldots, x_n$, *these equations are formed by a set of linear relations in* $dx_1, ..., dx_n$ *which are consequences of the equations of any of the differential systems that admit* Φ *as an invariant form.*

We can say even more simply: *the necessary and sufficient condition for the elementary displacement* $(dx_1, \ldots, dx_n)$ *to take place in the direction of a trajectory of a differential system that admits* Φ *as an invariant form is translated analytically by a certain number of equations linear in* $dx_1, \ldots, dx_n$. *These equations, assumed to be independent and* r *in number, define* $(n-r)$*-dimensional manifolds that depend on* r *arbitrary constants such that one and only one passes through any point of the space: these are the characteristic manifolds. The system of linear total differential equations itself is called the characteristic system of the form* Φ.

46. For brevity, call an equation that is linear in $dx_1, \ldots, dx_n$ a *Pfaffian equation*, and a system of Pfaffian equations a *Pfaffian system*. A Pfaffian system of r equations in n variables can always be regarded as defining r of these variables, considered as dependent variables, as functions of $n-r$ others considered as independent variables. *In general, such a system is impossible*. For example, a classic result is that a Pfaffian equation in three variables

$$P\,dx + Q\,dy + R\,dz = 0,$$

where we regard z as an unknown function of x and y, admits a solution corresponding to arbitrarily given initial values only if a certain integrability condition is satisfied, namely

$$P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=0;$$

in this case we say that it is *completely integrable*.

Similarly, we say that a Pfaffian system of r equations with r unknown functions of $n-r$ variables is *completely integrable* if it always admits a solution corresponding to arbitrarily given initial values of the variables. *This is the case for the characteristic Pfaffian system of a form* Φ.

The fundamental theorem of this chapter can thus be stated as follows:

The characteristic Pfaffian system of any differential form Φ *is always completely integrable.*

47. Return one last time to the form Φ of vortex theory. The characteristic Pfaffian system of this form is defined by equations (5) or, equivalently,

$$\frac{dx-u\,dt}{\xi}=\frac{dy-v\,dt}{\eta}=\frac{dz-w\,dt}{\zeta};$$

if we knew how to express the fact that such a system is completely integrable, we would necessarily arrive at the analytic translation of the Helmholtz theorem. As regards the characteristic manifolds, they are formed by the set of all the *states* of the molecules that make up the same vortex line.

For the double integral invariant of dynamics, the characteristic Pfaffian system reduces to the equations of motion, and the characteristic manifolds to the trajectories.

It could be different if we considered, as we did in the theory of vortices, only some of the trajectories, for example all those that satisfy the same system of relations between the variables.

Chapter V

Invariant Pfaffian Systems and their Characteristic Systems

I. — *The Concept of an Invariant Pfaffian System.*

48. Instead of invariant *forms* for a system of differential equations, we can consider invariant *equations*. In particular, H. Poincaré has used *finite* systems of invariant equations:[1] these have the property that if a point satisfies such a system, all points deducible from it by moving along the corresponding trajectory still satisfy this system. In geometric language, *the manifold represented by a system of invariant equations is generated by the trajectories.*

We can also consider invariant differential equations. Consider first, from a narrower perspective, the simple case of two differential equations

$$\frac{dx}{dt} = X, \qquad \frac{dy}{dt} = Y. \tag{1}$$

The equation

$$\delta y - m(x,y,t)\,\delta x = 0 \tag{2}$$

will be said to be *invariant in the sense of H. Poincare* if, given any two infinitely close *simultaneous* points (x,y,t), $(x+\delta x, y+\delta y, t)$ that satisfy relation (2), the points (x',y',t') and $(x'+\delta x', y'+\delta y', t')$ obtained by displacing them along their respective trajectories to any other time t' still satisfy relation (2), that is, if we also have

$$\delta y' - m(x',y',t')\,\delta x' = 0.$$

If equation (2) is invariant in the sense just specified, it will be equivalent to the equation

$$\delta y_0 - m(x_0,y_0,0)\,\delta x_0 = 0, \tag{3}$$

[1] Fr. *des systèmes d'équations invariantes finies*.

where we denote by x_0, y_0 the initial values of x, y on the trajectory which passes through the point x, y, t. If we now regard x_0, y_0 in equation (3) as functions of (x, y, t), and if we replace x_0, y_0 by their values, this equation clearly takes the form

$$\delta y - Y\,\delta t - m(\delta x - X\,\delta t) = 0. \tag{4}$$

By virtue of its origin, the new equation (4) has an *invariant* meaning in the complete sense of this term, because it expresses an *intrinsic* property of the two trajectories that correspond to points (x, y, t) and $(x + \delta x, y + \delta y, t + \delta t)$.

Geometrically, equation (4) associate with each point $M(x, y, t)$ of the space, a plane (P) that passes through that point. The property of invariance means that the line MM_1, which joins a point M to an infinitely close point M_1, is in the plane (P) corresponding to the point M, and if we move M and M_1 along their respective trajectories to M' and M'_1 (always keeping them infinitely close to each other), the line $M'M'_1$ is in the plane (P') which corresponds to the point M'. It should be noted that *the plane (P) is tangent to the trajectory which passes through the point M.*

According to the above, it is clear that if a curve (C) satisfies equation (4) at each of its points, that is, if it is *an integral curve*, the surface generated by the trajectories which pass through the various points of (C) is also an integral surface of equation (4). This also follows analytically from the form (3) of equation (4).

49. The above considerations generalise easily. Given a system of differential equations

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, ..., n), \tag{5}$$

a Pfaffian system

$$\left.\begin{array}{l} a_{11}\,\delta x_1 + \cdots + a_{1n}\,\delta x_n + a_1\,\delta t = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{h1}\,\delta x_1 + \cdots + a_{hn}\,\delta x_n + a_h\,\delta t = 0. \end{array}\right\} \tag{6}$$

will be said to be invariant for system (5) if equations (6) can be expressed only by means of first integrals of (5) and of their differentials, for example in the form

$$\left.\begin{array}{l} a^0_{11}\,\delta x^0_1 + \cdots + a^0_{1n}\,\delta x^0_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a^0_{h1}\,\delta x^0_1 + \cdots + a^0_{hn}\,\delta x^0_n = 0. \end{array}\right\} \tag{7}$$

First, this requires that equations (6) be identically satisfied when we replace δx_i by $X_i \delta t$; but this condition is clearly not sufficient. Be that as it may, if the Pfaffian system (6) is invariant, it has one important geometric property, which is that *given any integral manifold of system (6), the manifold obtained by laying down through each point of the given manifold the corresponding trajectory of* *equations (5) is still integral*; in fact, it follows from this that, if we move along this new manifold, equations (7) are still satisfied at any point.

(It is understood that we call a manifold and integral manifold with the property that, if we move in any direction $(\delta x_1, ..., \delta t)$ *on this manifold*, equations (6) are satisfied.)

From this it follows that *any integral manifold of the invariant Pfaffian system (6) has the property, either of indeed being generated by the trajectories of the given equations (5), or of being part of an integral manifold a with greater number of dimensions which is itself generated by trajectories.*

II. — *The Characteristic System of a Pfaffian System.*

50. Given any Pfaffian system in n variables $(x_1, \ldots, x_n)$

$$\left.\begin{array}{c} a_{11}\delta x_1 + \cdots + a_{1n}\delta x_n = 0, \\ \dots\dots\dots\dots\dots\dots \\ a_{h1}\delta x_1 + \cdots + a_{hn}\delta x_n = 0, \end{array}\right\} \tag{8}$$

we can propose to determine all systems of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \tag{9}$$

for which system (8) is invariant. We will solve this problem later, but without solving it we can prove an important theorem regarding all these systems, identical moreover to that proved in the previous chapter concerning a given differential form.

Suppose that the given equations (8) can be written using r quantities $y_1, \ldots, y_r$, functions of the x's and their differentials, in the form

$$\left.\begin{array}{c} b_{11}(y)\,\delta y_1 + \cdots + b_{1r}(y)\,\delta y_r = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ b_{h1}(y)\,\delta y_1 + \cdots + b_{hr}(y)\,\delta y_r = 0, \end{array}\right\} \tag{10}$$

and assume furthermore that they cannot be written using less than r quantities and their differentials. The number r will be called the *class* of the system. *The necessary and sufficient condition for a Pfaffian system (8) to be invariant for equations (9) is that these equations (9) admit $y_1, \ldots, y_r$ as first integrals.*

The proof is precisely the same as in the previous chapter and the consequences that we draw from it are also the same. In particular, the equations that express the fact that the given Pfaffian system (8) is invariant for the system of differential equations (9) reduce to r equations linear in $X_1, \ldots, X_n$ or, which amounts to the same thing, to r equations linear in $dx_1, \ldots, dx_n$, and these r equations form a Pfaffian system equivalent to

$$dy_1 = 0, \quad \dots, \quad dy_r = 0,$$

that is, they are *completely integrable*. This Pfaffian system is called the *characteristic system* of the given Pfaffian system (8); the equations of the characteristic system can moreover be obtained by adding to the h equations (8) of the given system $r-h$ other equations.

The necessary and sufficient condition for a Pfaffian system (8) to be completely integrable is clearly that it coincide with its characteristic system, so that if we know how to form the characteristic system of any Pfaffian system, we will be able to express by this very fact that it is completely integrable.

51. It is clear that a Pfaffian system (8) can be regarded as invariant for its characteristic system: *any integral manifold of system (8) either is generated by characteristic manifolds, or else it forms part of an integral manifold with a larger number of dimensions itself generated by characteristic manifolds.*

If we consider any differential form, and if this form is invariant for a certain system of differential equations, the characteristic Pfaffian system of the form is invariant for this same system of differential equations.

So, in hydrodynamics, the Pfaffian system

$$\frac{\delta x - u\,\delta t}{\xi} = \frac{\delta y - v\,\delta t}{\eta} = \frac{\delta z - w\,\delta t}{\zeta}$$

is invariant for the differential equations of the trajectories of the fluid molecules (as well as for the differential equations of vortex lines).

All these theorems, and others that we could easily conceive, are immediate consequences of the characteristic property of an invariant system of involving only first integrals of the differential equations for which it is invariant.

52. Consider either a differential form, or a Pfaffian system, or even a set of several differential forms and a Pfaffian system, and denote by $y_1, \dots, y_r$ the first integrals of the characteristic Pfaffian system, or of the given differential form, or of the given Pfaffian system, etc. It is clear that *if we focus only on the way in which the differentials* $\delta x_1, \dots, \delta x_n$ *enter* into the differential form, or into the Pfaffian system, etc, ignoring the coefficients, *these differentials enter only in the combinations* $\delta y_1, \dots, \delta y_r$. But it could also be that they enter as less than r linear combinations. In any case, if we know the minimum number of linear combinations of the δx_i by means of which we can express the form (or the Pfaffian system, etc.), *the equations obtained by setting these linear combinations to zero are part of the characteristic system.*

III. — *The rank of an algebraic form and its associated system.*

53. The preceding considerations will gain clarity if we prove for algebraic forms a theorem similar to that which led us to the concept of a characteristic system:

If an algebraic form in n variables $u_1, \dots, u_n$ can be expressed in terms of r independent linear combinations $v_1, \dots, v_r$ of the variables, without being expressible in terms of a smaller number, and if moreover we have found another expression of the form in terms of r other linear combinations $w_1, \dots, w_r$ of the variables, the w_i are independent linear combinations of the v_i.

In fact, consider the $2r$ linear forms

$$v_1, \dots, v_r; \quad w_1, \dots, w_r,$$

of the given variables. Suppose that among these forms there are $2r - \rho$ independent ones ($0 \leq \rho \leq r$); this amounts to saying that there are ρ independent linear combinations of the v's which are at the same time linear combinations of the w's; call them $t_1, \dots, t_\rho$. Suppose also, which is allowed, that $t_1, \dots, t_\rho$ are independent linear combinations at the same time of $v_1, \dots, v_\rho$ and of $w_1, \dots, w_\rho$. We thus have a double equality of the type

$$F(x_1, \dots, x_n) = \Phi(t_1, \dots, t_\rho; v_{\rho+1}, \dots, v_r) = \Psi(t_1, \dots, t_\rho; w_{\rho+1}, \dots, w_r).$$

Since the quantities $t_1, \dots, t_\rho, v_{\rho+1}, \dots, v_r, w_{\rho+1}, \dots, w_r$ are independent, this is possible only if Φ, for example, does not depend on $v_{\rho+1}, \dots, v_r$. This is consistent with the hypothesis only if $\rho = r$, and so the theorem is proved.

The system of linear equations

$$v_1 = v_2 = \cdots = v_r = 0$$

will be called the *associated system* of the given form. The concept of the associated system clearly generalises to a set of forms, or also to a system of algebraic equations. We call the integer r the *rank* of the form.

According to this, the *characteristic* system of a differential form always contains the *associated* system of this form, considered as an algebraic form in $\delta x_1, \dots, \delta x_n$. *But it may contain equations other than those of the associated system.*

Chapter VI

Exterior Forms

I. — *The associated system of a quadratic form.*

54. We have a few comments to make[1] on ordinary algebraic forms, quadratic forms, cubic, etc.

As we know, a quadratic form

$$F(u) = \sum_{i,j}^{1,\dots,n} a_{ij} u_i u_j = a_{11} u_1^2 + a_{22} u_2^2 + \cdots + 2a_{12} u_1 u_2 + \cdots \tag{1}$$

is reducible to a sum of squares; there are n independent squares if the discriminant of the form is not zero. We propose to determine the minimum number of variables by means of which the form can be expressed (by a suitable linear substitution). To obtain these variables, it is sufficient to consider the system of linear equations

$$\frac{\partial F}{\partial u_1} = 0, \qquad \frac{\partial F}{\partial u_2} = 0, \qquad \dots, \qquad \frac{\partial F}{\partial u_n} = 0. \tag{2}$$

First of all, it is clear that this system is independent of the choice of these variables. Suppose that it reduces to r independent equations, which we can always assume to be

$$x_1 = 0, \qquad x_2 = 0, \qquad \dots, \qquad x_r = 0.$$

That said, *the form F can be expressed in terms of the r variables $x_1, \dots, x_r$ and it cannot be expressed in terms of less than r variables.*

In fact, express F in terms of $x_1, \dots, x_r$ and of $n - r$ other independent forms $x_{r+1}, \dots, x_n$: the variable x_{r+1} for example will not enter F because, if it entered through a term such as $A x_{r+1} x_\alpha$,

[1] Fr. *Nous n'avons qu'un mot à dire*.

the equation

$$\frac{\partial F}{\partial x_\alpha} = 0$$

would contain x_{r+1}, contrary to hypothesis.

Conversely, suppose that the form F could be expressed by means of $\rho \leq r$ variables $y_1, y_2, ..., y_\rho$; system (2) formed by starting from variables $y_1, \ldots, y_\rho, ..., \ldots, y_n$, clearly would contain only the variables $y_1, \ldots, y_\rho$; it is necessary then that $\rho = r$ and system (2) will thus reduce to

$$y_1 = y_2 = \cdots = y_r = 0;$$

the $y_1, \ldots, y_r$ are thus independent linear combinations of $x_1, \ldots, x_r$.

The last part of the proof shows, as we already knew, that expressing of F in terms of the minimum number of variables is possible *essentially* in only one way, up to a linear substitution on this smallest number of variables.

System (2) is the *associated system* of the form F.

The above generalises to any integral and homogeneous form of any degree. For example, if F is a cubic form, the associated system of linear equations will be obtained by equating to zero all the *second* derivatives of F

$$\frac{\partial^2 F}{\partial u_i \partial u_j} = 0;$$

this system gives the minimum number of variables in terms of which F can be expressed.

II. — *Alternating bilinear forms and quadratic exterior forms.*

55. The forms that we will now consider are those which are found under a multiple integral sign when we consider differentials as variables. These are forms that have special rules of calculation which are worth emphasising.

We begin with a bilinear form

$$f(u, v) = \sum a_{ij} u_i v_j$$

in two series of variables

$$u_1, \ldots, u_n; \quad v_1, \ldots, v_n.$$

Such a form is said to be *symmetric* if it is conserved when we swap the two series of variables:

$$f(u,v) = f(v,u),$$

and *alternating* if it is conserved *with a change of sign* under the same conditions:

$$f(u,v) = -f(v,u)$$

The conditions that the coefficients must satisfy for the form to be symmetric are

$$a_{ij} = a_{ji};$$

the conditions for it to be alternating are

$$a_{ij} + a_{ji} = 0, \qquad a_{ii} = 0.$$

If we subject the two series of variables u_i and v_i to the *same* linear substitution, the form $f(u,v)$ changes into a new bilinear form $F(U,V)$ of new variables U_i, V_i, and it is clear that the form $F(U,V)$ is still symmetric if f was, and alternating if f was: this is due to the fact that swapping the two new series of variables U and V comes down to swapping the two original series of variables u and v.

Any symmetric bilinear function can be made to correspond to a quadratic form, namely $f(u,u)$, and the correspondence is mutual. If we put

$$f(u,u) = F(u),$$

we have

$$f(u,v) = \frac{1}{2}\left(v_1 \frac{\partial F}{\partial u_1} + v_2 \frac{\partial F}{\partial u_2} + \cdots + v_n \frac{\partial F}{\partial u_n}\right).$$

A similar correspondence can no longer be established for alternating forms, because in this case $f(u,u)$ is identically zero. We can overcome this disadvantage as follows.

56. First note that, in an alternating bilinear form, the coefficients of the terms $u_i v_i$ are all zero and that the coefficients of the terms $u_i v_j$ and that the coefficients of the terms $u j v_i$ have opposite signs. We can thus write

$$f(u,v) = \sum_{(ij)} a_{ij}(u_i v_j - u_j v_i),$$

where the the sum on the right hand side is over all two by two *combinations* of n indices, so that there are $\frac{n(n-1)}{2}$ terms on the right hand side. Since the expression $u_i v_j - u_j v_i$ is the determinant

$$\begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix},$$

we can, *by a notational convention*, denote it by the notation

$$u_i v_j - u_j v_i = [u_i\, u_j],$$

where we write the two entries of the first line one after the other and enclose them in square brackets. With this notation, we have

$$f(u,v) = \sum a_{ij}[u_i\, v_j].$$

We agree similarly to denote by the notation $[f(u)\, f'(u)]$ the alternating bilinear form defined by the determinant

$$\begin{vmatrix} f(u) & f'(u) \\ f(v) & f'(v) \end{vmatrix}$$

where f and f' denote any two linear forms

$$f(u) = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n,$$
$$f'(u) = a'_1 u_1 + a'_2 u_2 + \cdots + a'_n u_n.$$

If we expand the above determinant, we find immediately

$$[f(u)\, f'(u)] = \begin{vmatrix} f(u) & f'(u) \\ f(v) & f'(v) \end{vmatrix} = \sum_i \sum_j a_i a'_j \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} = \sum_i \sum_j a_i a'_j [u_i\ u_j].$$

Compariing the left and right hand sides shows that *the expansion of* $[f(u)\, f'(u)]$ *can be obtained by regarding this expression as a product, and expanding this product according to the ordinary rules of algebra, but taking care not to change the order of the factors in the partial products and agreeing that every partial product that contains two identical variables is zero and that any partial product of two different variables changes sign when we change the order of the factors*.

The multiplication whose rules have just been stated is due to H. Grassmann who called it *exterior multiplication*.

Using this operation, we see that *any alternating bilinear form corresponds to a form of second degree in a single series of variables, but with exterior multiplication, and conversely, any quadratic form with exterior multiplication corresponds to an alternating bilinear form.*

To abbreviate, we will say "exterior form" instead of "a form with exterior multiplication".

57. If we perform a linear substitution on the variables in an exterior form $F(u)$, the new form is obtained simply by expanding each partial product $[u_i\, u_j]$ as a function of the new variables.

The partial derivative $\dfrac{\partial F}{\partial u_1}$ of an exterior quadratic form is defined simply as the sum of the partial derivatives of its terms; a term which does not contain u_1 will naturally have a zero derivative; as for a term which contains u_1, we can always assume u_1 has been brought into first position in the partial product; the derivative of $A[u_1\, u_i]$ will then be Au_i. We have for example

$$\frac{\partial [u_1\, u_2]}{\partial u_1} = u_2, \qquad \frac{\partial [u_1\, u_2]}{\partial u_2} = -u_1, \qquad \frac{\partial [u_1\, u_2]}{\partial u_3} = 0, \qquad \ldots, \qquad \frac{\partial [u_1\, u_2]}{\partial u_n} = 0.$$

With these conventions we have

$$2F(u) = \left[u_1 \frac{\partial F}{\partial u_1}\right] + \left[u_2 \frac{\partial F}{\partial u_2}\right] + \cdots + \left[u_n \frac{\partial F}{\partial u_n}\right],$$

where the partial products on the right hand side are exterior products.

If $F(u)$ corresponds to the alternating form $f(u,v)$, we obviously have

$$f(u,v) = -\left(v_1 \frac{\partial F}{\partial u_1} + v_2 \frac{\partial F}{\partial u_2} + \cdots + v_n \frac{\partial F}{\partial u_n}\right),$$

where the partial products follow the rules of ordinary multiplication.

Finally, note that if we perform on the u_i a linear substitution

$$u_i = h_{i1}U_1 + h_{i2}U_2 + \cdots + h_{in}U_n \qquad (i = 1, 2, \ldots, n)$$

and if $F(u)$ becomes $\Phi(U)$ by this substitution, we have

$$\frac{\partial \Phi}{\partial U_k} = h_{1k} \frac{\partial F}{\partial u_1} + h_{2k} \frac{\partial F}{\partial u_2} + \cdots + h_{nk} \frac{\partial F}{\partial u_n},$$

as if F were an ordinary algebraic form.

58. The system of linear equations

$$\frac{\partial F}{\partial u_1} = 0, \qquad \frac{\partial F}{\partial u_2} = 0, \qquad \ldots, \qquad \frac{\partial F}{\partial u_n} = 0, \tag{3}$$

where F is a given exterior quadratic form, is clearly independent of the choice of variables. We can thus assume that it reduces to the equations

$$u_1 = 0, \qquad u_2 = 0, \qquad \ldots, \qquad u_r = 0 \qquad (r \leq n).$$

In this case, *the form F does not depend on $u_{r+1}, \ldots, u_n$*; in fact, if it contained a term such as $A[u_{r+1}\, u_\alpha]$, the equation

$$\frac{\partial F}{\partial u_\alpha} = 0$$

would not be a consequence of equations (3). The form F can thus be expressed uniquely in terms of the left hand sides of equations (3).

Conversely, suppose that the form F could be expressed in terms of $\rho \leq r$ variables $v_1, v_2, \ldots, v_\rho$. The left hand sides of the equations of the system

$$\frac{\partial F}{\partial v_1} = 0, \quad \ldots, \quad \frac{\partial F}{\partial v_n} = 0$$

would depend only on $v_1, \ldots, v_\rho$; this system would thus contain ρ independent equations at most. Consequently, we have $\rho = r$, and the v_i are linear combinations of the u_i.

The associated system of an exterior quadratic form is thus obtained by equating to zero all its partial derivatives of first order.

59. This result can be made more precise; we will show that *the rank r is necessarily even*, and at the same time find a reduced form for exterior quadratic forms that plays the same role as the sum of squares for ordinary quadratic forms.

For definiteness, suppose that the coefficient a_{12} of $F(u)$ is not zero, and consider the form

$$\frac{1}{a_{12}}\left[\frac{\partial F}{\partial u_1}\,\frac{\partial F}{\partial u_2}\right] = \frac{1}{a_{12}}\left[(a_{12}u_2 + a_{13}u_3 + \cdots + a_{1n}u_n)(a_{21}u_1 + a_{23}u_3 + \cdots + a_{2n}u_n)\right];$$

this form has the same coefficients as F for the terms in

$$[u_1\,u_2],\ [u_1\,u_3],\ \ldots,\ [u_1\,u_n],\quad [u_2\,u_3],\ \ldots,\ [u_2\,u_n],$$

that is, for terms which contain at least one of the variables u_1 and u_2. Consequently, the form

$$F(u) - \frac{1}{a_{12}}\left[\frac{\partial F}{\partial u_1}\,\frac{\partial F}{\partial u_2}\right] = F'(u)$$

contains only the variables $u_3, u_4, \ldots, u_n$. Suppose then that the coefficient a'_{34} of this form is not zero; similarly, we see that the form

$$F'(u) - \frac{1}{a'_{34}}\left[\frac{\partial F'}{\partial u_3}\,\frac{\partial F'}{\partial u_4}\right] = F''(u)$$

now contains only the variables $u_5, u_6, \ldots, u_n$. We can continue like this step by step until we arrive at a form that is identically zero. Suppose for example that we have

$$F(u) = \frac{1}{a_{12}}\left[\frac{\partial F}{\partial u_1}\,\frac{\partial F}{\partial u_2}\right] + \frac{1}{a'_{23}}\left[\frac{\partial F'}{\partial u_3}\,\frac{\partial F'}{\partial u_4}\right] + \frac{1}{a''_{56}}\left[\frac{\partial F''}{\partial u_5}\,\frac{\partial F''}{\partial u_6}\right].$$

The six linear forms

$$\frac{\partial F}{\partial u_1}, \quad \frac{\partial F}{\partial u_2}, \quad \frac{\partial F'}{\partial u_3}, \quad \frac{\partial F'}{\partial u_4}, \quad \frac{\partial F''}{\partial u_5}, \quad \frac{\partial F''}{\partial u_6},$$

are clearly independent. By putting

$$\begin{aligned} \frac{\partial F}{\partial u_1} &= U_1, & \frac{\partial F}{\partial u_2} &= a_{12} U_2, \\ \frac{\partial F'}{\partial u_3} &= U_3, & \frac{\partial F'}{\partial u_4} &= a'_{34} U_4, \\ \frac{\partial F''}{\partial u_5} &= U_5, & \frac{\partial F''}{\partial u_6} &= a''_{56} U_6, \end{aligned}$$

the form F reduces to the desired canonical form,

$$F(U) = [U_1 U_2] + [U_3 U_4] + [U_5 U_6].$$

The reasoning is clearly general and leads to the canonical form

$$F(U) = [U_1 U_2] + [U_3 U_4] + \cdots + [U_{2s-1} U_{2s}] \qquad (2s \leq n).$$

The associated system is obviously

$$U_1 = U_2 = \cdots = U_{2s} = 0.$$

This result will be very important later.

60. The reduction of an exterior quadratic form to its canonical form is obviously possible in an infinite number of ways; the set of linear substitutions which take us from one canonical form to another is an important group which depends on $s(2s+1)$ arbitrary parameters. If $s = 1$, these substitutions in two variables are characterised by the condition of having unit determinant.

III. — *Exterior forms of degree greater than two.*

61. We can imagine exterior forms of any degree. We get there most naturally by starting from a linear form in p series of variables $u_i, v_i, \ldots, w_i$

$$f(u, v, \ldots, w)$$

that satisfy the condition that swapping two series of variables with each other reproduces the form, but changed in sign. In the case $p = 3$, for example, the consequence of this hypothesis is that any term where the same index enters twice has zero coefficient and that the set of terms with three distinct indices, for example 1,2,3, has the form

$$a_{123}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The same notational convention as above leads to a distributive multiplication law, but *non commutative*, since each partial product changes sign if we swap two variables that enter it with each other. Consequently, we will have

$$[u_1\,u_2\,u_3] = -[u_2\,u_1\,u_3] = -[u_1\,u_3\,u_2] = -[u_3\,u_2\,u_1] = [u_2\,u_3\,u_1] = [u_3\,u_1\,u_2].$$

We can then define an exterior product like

$$[F\,\Phi\,\Psi],$$

where F, Φ, Ψ are exterior forms of any degree; the degree of the product is the sum of the degrees of the factors. The product is necessarily zero if the sum of degrees is greater than n. We establish easily that if we swap two factors of the product with each other, the product does not change if at least one of these factors is of even degree, and it reproduces with changed sign if both are of odd degree. We define similarly a sum of products of this kind.

In particular the product of a form with itself is zero if this form is of odd degree, but it is not necessarily zero if it is of even degree. Take for example a quadratic form F reduced to its canonical form

$$F = [u_1\,u_2] + [u_3\,u_4] + \cdots + [u_{2s-1}\,u_{2s}];$$

we have

$$\begin{aligned} \frac{1}{2}\,[F^2] &= [u_1\,u_2\,u_3\,u_4] + [u_1\,u_2\,u_5\,u_6] + \cdots + [u_{2s-3}\,u_{2s-2}\,u_{2s-1}\,u_{2s}], \\ \frac{1}{3!}\,[F^3] &= [u_1\,u_2\,u_3\,u_4\,u_5\,u_6] + \cdots, \\ \frac{1}{s!}\,[F^s] &= [u_1\,u_2\,u_3\,u_4\,u_5\,u_6\ldots u_{2s-1}\,u_{2s}], \\ \frac{1}{(s+1)!}\,[F^{s+1}] &= 0. \end{aligned}$$

The rank $2s$ of a quadratic form F is thus twice the largest power to which we can raise F without it vanishing.

The following is a simple application to the theory of determinants. Let

$$F = a_{12}[u_1\, u_2] + a_{13}[u_1\, u_3] + a_{14}[u_1\, u_4] + a_{23}[u_2\, u_3] + a_{24}[u_2\, u_4] + a_{34}[u_3\, u_4]$$

be a form in four variables; we have

$$\frac{1}{2}\,[F^2] = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})[u_1\, u_2\, u_3\, u_4];$$

on the other hand the associated system of F is

$$\begin{aligned} a_{12}u_2 + a_{13}u_3 + a_{14}u_4 &= 0,\\ a_{21}u_1 \qquad\quad + a_{23}u_3 + a_{24}u_4 &= 0,\\ a_{31}u_1 + a_{32}u_2 \qquad\quad + a_{34}u_4 &= 0,\\ a_{41}u_1 + a_{42}u_2 + a_{43}u_3 \qquad\quad &= 0. \end{aligned}$$

The condition for the form to be expressible in terms of fewer than four variables is, on the one hand, that we have $[F^2] = 0$, that is

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0,$$

and on the other that the determinant of the associated system is zero, that is

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = 0.$$

These two equations are equivalent, in spite of appearances; in fact, we show that the determinant, which is skew symmetric of even degree, is the square of the expression

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

62. Any exterior form of degree n (equal to the number of variables) has the form

$$A[u_1\, u_2 \dots u_n].$$

We can obtain canonical forms when the degree is $n-1$ or $n-2$. We get there easily by the concept of the *adjoint form* of a given form.

Consider a form F of degree p and denote by $\xi, \xi', \dots, \xi^{(n-p-1)}$ the linear forms with undetermined coefficients

$$\begin{aligned} \xi &= \xi_1 u_1 + \cdots + \xi_n u_n, \\ \xi' &= \xi'_1 u_1 + \cdots + \xi'_n u_n, \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

The exterior product $[F\,\xi\,\xi' \ldots \xi^{(n-p-1)}]$ has degree n and consequently has the form

$$\Phi[u_1\,u_2 \ldots u_n];$$

the coefficient Φ is linear with respect to each series of coefficients ξ, and is also *alternating*; so, to it there corresponds an exterior form of degree $n-p$ with variables $\xi_1, \ldots, \xi_n$: this is, by definition, the *adjoint form* of F.

If we perform a linear substitution on the variables u and if we perform at the same time a linear substitution on the ξ which preserves the expression $\xi_1 u_1 + \cdots + \xi_n u_n$, the expression $\Phi[u_1 \ldots u_n]$ is clearly conserved; in other words, *the adjoint form is reproduced multiplied by the determinant of the substitution performed on the variables u.*

The adjoint form of a form $F = F_1 + F_2$ is clearly the sum of the adjoint forms Φ_1 and Phi_2. Similarly, the adjoint form of aF, where a is a numerical coefficient, is $a\Phi$. According to this, to calculate the adjoint form of any form, it is sufficient to know how to calculate the adjoint form of a monomial form such as

$$F = [u_{\alpha_1}\,u_{\alpha_2} \ldots u_{\alpha_p}].$$

By applying the given definition, we find

$$\Phi = [\xi_{\alpha_{p+1}} \ldots \xi_{\alpha_n}],$$

where the indices $\alpha_{p+1}, \ldots, \alpha_n$ are the indices $1, 2, \ldots, n$ which do not appear in the sequence $\alpha_1, \alpha_2, \ldots, \alpha_p$; these indices are assumed arranged in an order such that the complete sequence

$$\alpha_, \ldots, \alpha_p, \alpha_{p+1}, \ldots, \alpha_n$$

is even.

63. Assume then that F is a form of degree $n-1$; the adjoint form will be of first degree; we can thus assume it reduced to ξ_n for example, so that F *can always be reduced to the expression*

$$F = [u_1\,u_2 \ldots u_{n-1}].$$

Suppose now that F is of degree $n-2$; the form Φ will be of second degree; we can thus always suppose it given by the formula

$$\Phi = [\xi_1\,\xi_2] + \cdots + [\xi_{2s-1}\,\xi_{2s}];$$

consequently, we will have

$$F = [u_3\, u_4\, u_5 \dots u_n] + [u_1\, u_2\, u_5\, u_6 \dots u_n] + \cdots + [u_1\, u_2 \dots u_{2s-2}\, u_{2s+1} \dots u_n].$$

If $s = 1$, F reduces to a monomial form.

For example if $n = 5$, any form F of degree $5 - 2 = 3$ is reducible to one of the canonical forms

$$F = [u_3\, u_4\, u_5]$$
$$F = [u_1\, u_2\, u_5] + [u_3\, u_4\, u_5];.$$

if $n = 6$, any form F of degree 4 is reducible to one of the forms

$$F = [u_3\, u_4\, u_5\, u_6]$$
$$F = [u_3\, u_4\, u_5\, u_6] + [u_1\, u_2\, u_5\, u_6] = [([u_1\, u_2] + [u_3\, u_4])u_5\, u_6]$$
$$F = [u_3\, u_4\, u_5\, u_6] + [u_1\, u_2\, u_5\, u_6] + [u_1\, u_2\, u_3\, u_4] = \frac{1}{2}\ [[u_1\, u_2] + [u_3\, u_4] + [u_5\, u_6]]^2\,.$$

The concept of the adjoint form allows us to define the product of two forms whose sum of degrees exceeds n: this is the operation H. Grassmann called *regressive exterior multiplication*, but we will not use it.

64. We point out some more applications of exterior multiplication.

Suppose $f_1, f_2, \dots, f_h$ are h *independent* linear forms.

The equation

$$[F\, f_1\, f_2 \dots f_h] = 0$$

where F is any exterior form, gives the necessary and sufficient condition for F to be zero when we establish between the variables the relations

$$f_1 = 0, \qquad f_2 = 0, \qquad \dots, \qquad f_h = 0.$$

In fact, we can first reduce it to the case where we have $f_i = u_i$. If then any term of F contains at least one of the variables $u_1, \dots, u_h$, it is clear that the product $[F\, u_1 \dots u_h$ is zero. Conversely, if this product is zero, every term of F contains one at least of the variables $u_1, \dots, u_h$ as a factor, otherwise multiplying this term by $[u_1 \dots u_h]$ would in fact give a non-zero product, which cannot be cancelled by any other.

IV. — *The associated system of an exterior form.*

65. Determination of the associated system is as easy for an exterior form of any degree as for a quadratic form. In fact, if the form is of degree p, the associated system is obtained in fact by setting to zero all partial derivatives of order $p-1$ of F. We define a derivative of first order such as $\dfrac{\partial F}{\partial u_1}$ as the coefficient of u_1 in the set of the terms of F which contain this variable, taking care beforehand to move u_1 to first position in each of these terms. Note that this derivative $\dfrac{\partial F}{\partial u_1}$ no longer depends on u_1. By definition, the derivative $\dfrac{\partial^2 F}{\partial u_1 \partial u_2}$ will be the derivative with respect to u_2 of $\dfrac{\partial F}{\partial u_1}$: we thus get it by taking the set of terms of F which contain at the same time the two variables u_1 and u_2, moving in each of these terms the variable u_1 to first position and the variable u_2 to the second, and finally deleting the variables u_1 and u_2 in all these terms. According to this, we have

$$\frac{\partial^2 F}{\partial u_1 \partial u_2} = -\frac{\partial^2 F}{\partial u_2 \partial u_1}\,.$$

The partial derivatives of higher order are defined in the same way: they are necessarily taken with respect to variables which are all different.

The rank of a form of degree n which is not identically zero is clearly equal to n. The rank of a form of degree $n-1$ is equal to $n-1$. The rank of a form of degree $n-2$ is equal to $n-2$ if it is reducible to a monomial, and to n in all other cases. If the degree is less than $n-2$, we can say nothing *a priori* about the rank.

V. — *Formulae for exterior quadratic forms.*

66. Revisit the case of an exterior quadratic form F in n variables $u_1, u_2, \dots, u_n$. It can happen that the variables related by a linear relation

$$f \equiv a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0.$$

The form F, where we will assume for example that u_n is expressed as a function of $u_1, u_2, \dots, u_{n-1}$ using the given relation, will have a certain rank that corresponds to the number of linearly independent equations of its associated system. The latter clearly has as equations

$$\frac{\partial F}{\partial u_1} - \frac{a_1}{a_n}\frac{\partial F}{\partial u_n} = 0, \qquad \dots, \qquad \frac{\partial F}{\partial u_{n-1}} - \frac{a_{n-1}}{a_n}\frac{\partial F}{\partial u_n} = 0, \qquad f = 0,$$

or again

$$\frac{\dfrac{\partial F}{\partial u_1}}{a_1} = \frac{\dfrac{\partial F}{\partial u_2}}{a_2} = \cdots = \frac{\dfrac{\partial F}{\partial u_n}}{a_n}, \qquad f = 0.$$

More generally, we could assume that the variables are related by any number of relations

$$\begin{aligned} f &\equiv a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0 \\ g &\equiv b_1 u_1 + b_2 u_2 + \cdots + b_n u_n = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h &\equiv \ell_1 u_1 + \ell_2 u_2 + \cdots + \ell_n u_n = 0. \end{aligned}$$

The associated system of F will then be defined by the formulae

$$\left\| \begin{matrix} \dfrac{\partial F}{\partial u_1} & \dfrac{\partial F}{\partial u_2} & \cdots & \dfrac{\partial F}{\partial u_n} \\ a_1 & a_2 & & a_n \\ b_1 & b_2 & & b_n \\ \vdots & & & \vdots \\ \ell_1 & \ell_2 & \cdots & \ell_n \end{matrix} \right\| = 0, \qquad f = 0, \qquad g = 0, \qquad \ldots, \qquad h = 0;$$

equating the matrix we have just written to zero means setting to zero all determinants formed with the rows of this matrix and the same number of columns.

We note that *the rank $2s'$ of the form F, when we assume the variables related by the given relations, is twice the greatest exponent such that the form*

$$[f g \ldots h F^{s'}]$$

is not zero.

67. Suppose in particular that $n = 2s$ and that the form F has rank n. If we relate the variables by just one relation, it is clear that the rank of the form cannot exceed $n - 1 = 2s - 1$ and, *since this rank is even*, it is at most equal to $2s - 2$. Moreover, it is easy to see that it cannot fall below this limit.

It follows from this that if we relate the variables by p independent linear relations, the rank of F will decrease by *at most* $2p$ units. We investigate in which case the maximum reduction will be attained. If the relations are

$$f_1 = 0, \qquad f_2 = 0, \qquad \ldots, \qquad f_p = 0,$$

it will be necessary and sufficient that we have

$$[f_1\, f_2 \dots f_p\, F^{s-p+1}] = 0. \tag{4}$$

This condition can be replaced by other simpler conditions. In fact, note that if we take any two of the p given relations, these two relations necessarily reduce the rank of F by 4 units; we thus have

$$[f_i\, f_j\, F^{s-1}] = 0 \qquad (i, j = 1, 2, \dots, n). \tag{5}$$

We will now show that *these* $\dfrac{p(p-1)}{2}$ *necessary equations are also sufficient.*

In fact, suppose these conditions are satisfied and make a change of variables so as to reduce f_i to u_i. We will then have

$$[u_i\, u_j\, F^{s-1}] = 0,$$

which shows that, *in the adjoint form* $\Phi(\xi)$ *of* $F^{s-1}(u)$, there are no terms in $[\xi_i\, \xi_j]$. Now, the adjoint form of F^{s-q} is Φ^q: this is easily seen by assuming that F is reduced to its canonical form. Consequently, each term of the adjoint form of F^{s-p+1}, which is Φ^{p-1}, contains *at least* $p-1$ of the variables $\xi_{p+1}, \dots, \xi_n$, because each term of Φ contains at least one of these variables. Each of the terms of Φ^{p-1} thus contains *at most* $p-1$ of the variables $\xi_1, \dots, \xi_p$. Consequently, the adjoint form of Φ^{p-1} contains *at least* one of the variables $u_1, \dots, u_p$. This is the same as saying that we have

$$[u_1\, u_2 \dots u_p\, F^{s-p+1}] = 0.$$

Q.E.D.

The relevance of the above theorem is easy to see. Since the forms $[f_i\, f_j\, F^{s-1}]$ are of degree n, there are $\dfrac{p(p-1)}{2}$ equations to write; whereas, since the form $[f_1\, f_2 \dots f_p\, F^{s-p+1}]$ is of degree $n-p+2$, the number of equations expressing that it is zero is C_n^{p-2}, and moreover each of them contains the coefficients of *all* of the given relations.

If for example we have

$$\begin{aligned} F &= [u_1\, u_2] + [u_3\, u_4] + [u_5\, u_6] \\ f_1 &= a_1 u_1 + \cdots + a_6 u_6, \\ f_2 &= b_1 u_1 + \cdots + b_6 u_6, \\ f_3 &= c_1 u_1 + \cdots + c_6 u_6, \end{aligned}$$

the condition that F be of rank $6-6=0$, taking into account the three relations $f_1 = f_2 = f_3 = 0$, is, by the first method,

$$[f_1 f_2 f_3 F] = 0,$$

which gives

$$\begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_5 & a_6 \\ b_1 & b_5 & b_6 \\ c_1 & c_5 & c_6 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} + \begin{vmatrix} a_2 & a_5 & a_6 \\ b_2 & b_5 & b_6 \\ c_2 & c_5 & c_6 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_3 & a_5 & a_6 \\ b_3 & b_5 & b_6 \\ c_3 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 & a_2 \\ b_3 & b_1 & b_2 \\ c_3 & c_1 & c_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_1 & a_2 \\ b_4 & b_1 & b_2 \\ c_4 & c_1 & c_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_5 & a_1 & a_2 \\ b_5 & b_1 & b_2 \\ c_5 & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} a_5 & a_3 & a_4 \\ b_5 & b_3 & b_4 \\ c_5 & c_3 & c_4 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_6 & a_1 & a_2 \\ b_6 & b_1 & b_2 \\ c_6 & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} a_6 & a_3 & a_4 \\ b_6 & b_3 & b_4 \\ c_6 & c_3 & c_4 \end{vmatrix} = 0.$$

On the other hand, the above theorem puts the required conditions into a much simpler form

$$\begin{aligned} b_1c_2 - c_1b_2 + b_3c_4 - c_3b_4 + b_5c_6 - c_5b_6 &= 0, \\ c_1a_2 - a_1c_2 + c_3a_4 - a_3c_4 + c_5a_6 - a_5c_6 &= 0, \\ a_1b_2 - b_1a_2 + a_3b_4 - b_3a_4 + a_5b_6 - b_5a_6 &= 0. \end{aligned}$$

68. There is a theorem that is still more precise than the previous one and which allows us to find in the simplest way the rank of the form to which F reduces when we assume that the variables are related by p given relations. For this, define the alternating bilinear form

$$\Phi(\xi,\xi') = \sum a_{ij}\xi_i\xi'_j$$

by the equality

$$s[F^{s-1}(\xi_1 u_1 + \cdots + \xi_n u_n)(\xi'_1 u_1 + \cdots + \xi'_n u_n)] = \Phi(\xi,\xi')[F^s];$$

The exterior quadratic form

$$\Phi(\xi) = \sum a_{ij}[\xi_i\xi_j]$$

is (up to a factor) the adjoint form of $\dfrac{F^{s-1}}{(s-1)!}$; it is an *absolute covariant* of F in the sense that if we perform any linear substitution on the variables $u_1,\ldots,u_n$, and the linear substitution which preserves $\xi_1 u_1 + \cdots + \xi_n u_n$ on the variables $\xi_1,\ldots,\xi_n$, if finally by these two substitutions the two forms $F(u)$ and $\Phi(\xi)$ become respectively $\overline{F}(\overline{u})$ and $\overline{\Phi}(\overline{\xi})$, we again have

$$s[\overline{F}^{s-1}(\overline{\xi}_1\overline{u}_1 + \cdots + \overline{\xi}_n\overline{u}_n)(\overline{\xi}'_1\overline{u}_1 + \cdots + \overline{\xi}'_n\overline{u}_n)] = \overline{\Phi}(\overline{\xi},\overline{\xi}')[\overline{F}^s].$$

In particular, if F has been reduced to its canonical form

$$F = [u_1\, u_2] + \cdots + [u_{2s-1}\, u_{2s}],$$

we find immediately for Φ the canonical form

$$\Phi = [\xi_1\, \xi_2] + \cdots + [\xi_{2s-1}\, \xi_{2s}].$$

From this, the general identity

$$\left[\frac{F^{s-p}}{(s-p)!}(\xi_1 u_1 + \cdots + \xi_n u_n)\ (\xi_1' u_1 + \cdots + \xi_n' u_n) \cdots (\xi_1^{(2p-1)} u_1 + \cdots + \xi_n^{(2p-1)} u_n)\right] = \frac{\Phi^{(p)}(\xi, \xi', \ldots, \xi^{(2p-1)})}{p!} \left[\frac{F^s}{s!}\right] \qquad (6)$$

follows easily, where the exterior form of degree p corresponding the the alternating multilinear form $\Phi^{(p)}$ is equal to $\Phi^p(\xi)$: this identity is obvious when F has been reduced to its canonical form, and it is thus true in the general case. This basically reduces to the property, invoked in the preceding number, that the adjoint form of $[F^{s-p}]$ is equal to $[\Phi^p]$ up to a scalar factor.

In particular, setting $p = 2$, and taking in identity (6) the terms in $[\xi_i\, \xi_j'\, \xi_k''\, \xi_\ell''']$, we get

$$\left[\frac{F^{s-2}}{(s-2)!}\ u_i\, u_j\, u_k\, u_\ell\right] = (a_{ij}a_{k\ell} + a_{ik}a_{\ell j} + a_{i\ell}a_{jk}) \left[\frac{F^s}{s!}\right], \qquad (7)$$

where the coefficients a_{ij} are defined by

$$\left[\frac{F^{s-1}}{(s-1)!}\, u_i\, u_j\right] = a_{ij} \left[\frac{F^s}{s!}\right].$$

Finally, we can deduce another identity which will be useful to us later. Consider the form

$$\left[\frac{F^{s-2}}{(s-2)!}\ u_i\, u_j\, u_k\right] - \left[\frac{F^{s-1}}{(s-1)!}(a_{ij}u_k + a_{jk}u_i + a_{ki}u_j)\right];$$

it is of degree $2s-1$; if we form its exterior product with any one of the variables $u_1, \ldots, u_{2s}$, say u_ℓ, we see immediately from (7) that the product is zero. Consequently the form itself is identically zero. Since u_i, u_j, u_k can be replaced by any three linear forms of the variables, we arrive at the following theorem:

If we consider any number of linear forms $f_1, f_2, \ldots, f_p$, and if we put

$$\left[\frac{F^{s-1}}{(s-1)!}\ f_i\, f_j\right] = a_{ij} \left[\frac{F^s}{s!}\right], \qquad (i, j = 1, 2, \ldots, p)$$

we have the identities

$$\left[\frac{F^{s-2}}{(s-2)!}\, f_i f_j f_k\right] = \left[\frac{F^{s-1}}{(s-1)!}(a_{ij}f_k + a_{jk}f_i + a_{ki}f_j)\right]. \tag{8}$$

69. We turn now to the problem stated above, which is to find the rank of the form to which F reduces when we assume that the variables are related by p independent linear equations

$$f_1 = 0, \quad f_2 = 0, \quad \dots, \quad f_p = 0.$$

We can assume that these relations are

$$u_1 = 0, \quad u_2 = 0, \quad \dots, \quad u_p = 0;$$

we will be allowed to perform any linear substitution on the u's, subject only to the condition that the first p variables $u_1, \dots, u_p$ are exchanged with each other. It follows that we will be able to perform any linear substitution on the variables ξ, subject only to the condition that the *last* $2s-p$ *variables* $\xi_{p+1}, \dots, \xi_{2s}$ are exchanged with each other. Put then

$$\left[\frac{F^{s-1}}{(s-1)!}\, u_i u_j\right] = a_{ij}\left[\frac{F^s}{s!}\right] \qquad (i, j = 1, 2, \dots, p).$$

If in Φ we delete the terms in $\xi_{p+1}, \dots \xi_{2s}$, we clearly get

$$\overline{\Phi} = \sum_{(ij)}^{1,\dots,p} a_{ij}\,[\xi_i\,\xi_j]\,.$$

Let $2q$ be the rank of the form $\overline{\Phi}$; by an appropriate linear substitution on the $\xi_1, \dots, \xi_p$, we will be able to reduce $\overline{\Phi}$ to

$$\overline{\Phi} = [\xi_1\xi_2] + \cdots + [\xi_{2q-1}\xi_{2q}];$$

consequently, by subtracting as necessary from $\xi_1, \dots, \xi_p$ linear combinations of $\xi_{p+1}, \dots, \xi_{2s}$, *which is allowed*, we will be able to reduce Φ to

$$\begin{aligned}\Phi = [\xi_1\,\xi_2] + \cdots + [\xi_{2q-1}\,\xi_{2q}] + [\xi_{2q+1}\,\xi_{p+1}] + \cdots + [\xi_p\,\xi_{2p-2q}] \\ + [\xi_{2p-2q+1}\,\xi_{2p-2q+2}] + \cdots + [\xi_{2s-1}\,\xi_{2s}].\end{aligned}$$

But then the form F will become

$$F = [u_1\,u_2] + \cdots + [u_{2q-1}\,u_{2q}] + [u_{2q+1}\,u_{p+1}] + \cdots + [u_p\,u_{2p-2q}] + \cdots + [u_{2s-1}\,u_{2s}].$$

We see that, if we take into account the relations

$$u_1 = 0, \quad u_2 = 0, \quad \dots, \quad u_p = 0,$$

the rank of F is reduced by $2p - 2q$ units.

We thus arrive at the following theorem:

Consider the independent linear forms $f_1, f_2, \dots, f_p$, the $\dfrac{p(p-1)}{2}$ quantities a_{ij} defined by the equalities

$$\left[\frac{F^{s-1}}{(s-1)!}\, f_i f_j\right] = a_{ij} \left[\frac{F^s}{s!}\right]$$

and the exterior quadratic form in p variables $\xi_1, \dots, \xi_p$,

$$\Phi(\xi) = \sum_{(ij)}^{1,\dots,p} a_{ij}\, [\xi_i\, \xi_j]\,.$$

If this form is of rank $2q$, the rank of the form F reduces by $2p - 2q$ units when we assume that the variables are related by the p equations

$$f_1 = 0, \quad f_2 = 0, \quad \dots, \quad f_p = 0.$$

Furthermore, if we perform on the p given linear forms a linear substitution such that Φ reduces to its canonical form

$$\Phi = [\xi_1\, \xi_2] + \cdots + [\xi_{2q-1}\, \xi_{2q}],$$

the form F reduces to the canonical form

$$F = [f_1\, f_2] + \cdots + [f_{2q-1}\, f_{2q}] + [f_{2q+1}\, f_{p+1}] + \cdots + [f_p\, f_{2p-2q}] + \cdots + [f_{2s-1}\, f_{2s}],$$

where we denote new suitably chosen linear forms that are independent of each other and independent of the given forms by $f_{p+1}, \dots, f_{2s}$.

In particular, if $q = 0$, we recover the theorem previously stated and proved (n°**67**).

Chapter VII

Exterior Differential Forms and their Derived Forms

I. — *The bilinear covariant of a Pfaffian form.*

70. Consider now a linear differential form (a Pfaffian form)

$$\omega_\delta = a_1 \delta x_1 + a_2 \delta x_2 + \cdots + a_n \delta x_n.$$

We can derive from this form an alternating bilinear form in two series of differentials, namely

$$\delta \omega_{\delta'} - \delta' \omega_\delta = \sum a_i (\delta \delta' x_i - \delta' \delta x_i) + \sum (\delta a_i \delta' x_i - \delta x_i \delta' a_i).$$

Assume that the two differentiation symbols commute, that is, that we have

$$\delta \delta' x_i = \delta' \delta x_i\,;$$

an exterior quadratic differential form corresponds to the right hand side, which we call the *bilinear covariant* of the form ω, which with the conventions made above we will write as

$$\omega'_\delta = \sum_i [\delta a_i \, \delta x_i] = \sum_{(i\,j)} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) [\delta x_i \, \delta x_j];$$

this form is called the *exterior derivative* of the form ω.

This method of derivation has a significance that is *independent of the choice of variables*; furthermore, it is the one that takes a curvilinear integral over a closed contour into a double integral over a surface bounded by the contour.

For example, if we have three variables x, y, z and if we put

$$\omega_\delta = P\,\delta x + Q\,\delta y + R\,\delta z,$$

we have

$$\begin{aligned}\omega'_\delta = [\delta P\,\delta x]+[\delta Q\,\delta y]+[\delta R\,\delta z] &= \frac{\partial P}{\partial x}[\delta x\,\delta x]+\frac{\partial P}{\partial y}[\delta y\,\delta x]+\frac{\partial P}{\partial z}[\delta z\,\delta x]\\ &+\frac{\partial Q}{\partial x}[\delta x\,\delta y]+\frac{\partial Q}{\partial y}[\delta y\,\delta y]+\frac{\partial Q}{\partial z}[\delta z\,\delta y]\\ &+\frac{\partial R}{\partial x}[\delta x\,\delta z]+\frac{\partial R}{\partial y}[\delta y\,\delta z]+\frac{\partial R}{\partial z}[\delta z\,\delta z]\\ =\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)[\delta y\,\delta z]&+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)[\delta z\,\delta x]+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)[\delta x\,\delta y]\end{aligned}$$

and the Stokes formula is

$$\int_C \omega_\delta = \iint_S \omega'_\delta\,,$$

where S denotes a surface bounded by the contour C.

The necessary and sufficient condition for ω' to be zero is that the form ω is an exact differential.

NOTE. — The two symbols of differentiation δ and δ' must commute when the differentiations are applied to any function y of independent variables, *otherwise the operation just defined would not have a covariant character.* This is easy to verify; if we put

$$\omega_\delta = \delta y = \frac{\partial y}{\partial x_1}\,\delta x_1+\cdots+\frac{\partial y}{\partial x_n}\,\delta x_n.$$

ω_δ is an exact differential and we have

$$\delta\omega_{\delta'}-\delta'\omega_\delta = 0,$$

that is,

$$\delta\delta' y = \delta'\delta y.$$

II. — *The exterior derivative.*

71. The same derivation process applies to an exterior differential form of any degree. For example, let

$$\Omega = \sum a_{ij}[\delta x_i\,\delta x_j]$$

be a quadratic form; consider the alternating bilinear form

$$\Omega(\delta,\delta') = \sum a_{ij}(\delta x_i\,\delta' x_j - \delta x_j\,\delta' x_i)$$

which corresponds to it, and introduce *three* mutually commuting[1] differentiation symbols $\delta, \delta', \delta''$. Finally, consider the expression

$$\delta\Omega(\delta',\delta'') - \delta'\Omega(\delta,\delta'') + \delta''\Omega(\delta,\delta'),$$

which clearly has an intrinsic significance independent of the choice of variables. We establish easily, by performing the calculation, that it reduces to an alternating trilinear expression

$$\begin{aligned}\Omega'(\delta,\delta',\delta'') = \sum\big[\delta a_{ij}(\delta' x_i\,\delta'' x_j - \delta' x_j\,\delta'' x_i) - \delta' a_{ij}(\delta x_i\,\delta'' x_j - \delta x_j\,\delta'' x_i)\\ +\delta'' a_{ij}(\delta x_i\,\delta' x_j - \delta x_j\,\delta' x_i)\big]\,.\end{aligned}$$

To this trilinear form there now corresponds an exterior cubic differential form

$$\Omega'_\delta = \sum[\delta a_{ij}\,\delta x_i\,\delta x_j] = \sum\left(\frac{\partial a_{ij}}{\partial x_k} + \frac{\partial a_{jk}}{\partial x_i} + \frac{\partial a_{ki}}{\partial x_j}\right)[\delta x_i\,\delta x_j\,\delta x_k],$$

which we will call the *derived* form of Ω.

72. In the case considered, it is important to be aware of the relationship that there is between the operation of derivation of an exterior quadratic form and the operation which consists of passing from a double integral over a *closed* surface to a triple integral over the volume bounded by the surface.

For this, imagine that $x_1,\ldots,x_n$ are functions of three parameters α,β,γ and consider an elementary parallelepiped in n-dimensional space whose edges are portions of coordinate lines, where the vertices A,B,C,D,E,F,G,H of this parallelepiped correspond respectively to the curvilinear coordinates

$$\begin{array}{cccc}(\alpha,\beta,\gamma), & (\alpha+\delta\alpha,\beta,\gamma), & (\alpha,\beta+\delta'\beta,\gamma), & (\alpha,\beta,\gamma+\delta''\gamma),\\ (\alpha+\delta\alpha,\beta+\delta'\beta,\gamma), & (\alpha+\delta\alpha,\beta,\gamma+\delta''\gamma), & (\alpha,\beta+\delta'\beta,\gamma+\delta''\gamma), & (\alpha+\delta\alpha,\beta+\delta'\beta,\gamma+\delta''\gamma).\end{array}$$

As we see, the symbols δ,δ',δ'' refer respectively to differentiations with respect to three parameters α,β,γ.

Consider now the curvilinear integral $\iint \Omega$ over the surface which bounds this parallelepiped.

The integrals over the three faces that start from A are, up to a sign,

$$\Omega(\delta',\delta''),\qquad \Omega(\delta'',\delta),\qquad \Omega(\delta,\delta'),$$

[1] Fr. *échangeables*.

and, for these integrals to be entirely over the internal face or entirely over the external face, it is necessary to take them equal to the three preceding expressions, or equal and opposite. If we take them to be equal and opposite, the sum of the integrals over the six faces will be

$$\begin{aligned}&-\Omega(\delta',\delta'')-\Omega(\delta'',\delta)-\Omega(\delta,\delta')+\left[\Omega(\delta',\delta'')+\delta\Omega(\delta',\delta'')\right]\\&\quad+\left[\Omega(\delta'',\delta)+\delta'\Omega(\delta'',\delta)\right]+\left[\Omega(\delta,\delta')+\delta''\Omega(\delta,\delta')\right]\\&\quad=\delta\Omega(\delta',\delta'')+\delta'\Omega(\delta'',\delta)+\delta''\Omega(\delta,\delta')=\Omega'((\delta,\delta',\delta'').\end{aligned}$$

The surface integral $\iint \Omega$ is thus indeed transformed into the volume integral $\iiint \Omega'$.

In the simple case of three variables, if we put

$$\Omega = P[\delta y\,\delta z] + Q[\delta z\,\delta x] + R[\delta x\,\delta y],$$

we have

$$\Omega' = [\delta P\,\delta y\,\delta z] + [\delta Q\,\delta z\,\delta x] + [\delta R\,\delta x\,\delta y] = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)[\delta x\,\delta y\,\delta z].$$

73. These considerations generalise to exterior forms of any degree. Any exterior form admits a derived form whose degree is greater by one unit and whose calculation is extremely easy, because each term of the form

$$A[\delta x_1\,\delta x_2 ... \delta x_\ell]$$

gives rise to a derived term

$$[\delta A\,\delta x_1\,\delta x_2 ... \delta x_\ell].$$

Note some useful formulae that are easy to prove. *If m is a coefficient that is a finite function of the variables, and Ω is any exterior form, we have*

$$(m\Omega)' = [dm\,\Omega] + m\Omega'.$$

If Ω and Π are any two exterior differential forms, we have

$$[\Omega\,\Pi]' = [\Omega'\,\Pi] \pm [\Omega\,\Pi'],$$

where the $+$ sign refers to the case where Ω is of even degree and the $-$ sign to the case where Ω is of odd degree. In particular if Ω is of even degree, the derived form of $[\Omega^p]$ is given by the ordinary formula

$$[\Omega^p] = p[\Omega^{p-1}\Omega'].$$

74. In the above, we have assumed that the coefficients of the forms considered were continuous functions admitting first order partial derivatives. *However, there are cases where the coefficients of a form Ω do not admit derivatives, nevertheless we can define an exterior derived form Ω'.* A classic example is provided by potential theory.

Consider a material volume V bounded by a surface S; let ρ be the density at a point of V; we will assume that the function ρ is continuous. The potential U of this mass is a function that is continuous in all space, which everywhere admits continuous first order derivatives. There is a theorem for this function (Gauss's theorem) translated by the formula

$$\iint \frac{\partial U}{\partial x}\, dy\,dz + \frac{\partial U}{\partial y}\, dz\,dx + \frac{\partial U}{\partial z}\, dx\,dy = \iiint -4\pi\rho\, dx\,dy\,dz,$$

where the integral on the left hand side is over *any* closed surface and that on the right hand side is over the volume bounded by this surface. It follows from this that by putting

$$\Omega = \frac{\partial U}{\partial x}[dy\,dz] + \frac{\partial U}{\partial y}[dz\,dx] + \frac{\partial U}{\partial z}[dx\,dy],$$

we can define the exterior derivative Ω' of Ω by

$$\Omega' = -4\pi\rho\,[dx\,dy\,dz].$$

If *the function U admits second order partial derivatives*, this is the classic Poisson formula, because the method of derivation just defined gives immediately

$$\Omega' = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)[dx\,dy\,dz];$$

but if the function U does not admit second order partial derivatives, *which is the general case when we make no additional hypothesis about the function ρ, we can still define the derivative Ω'.*

We thus conceive the possibility of defining the exterior derivation as an *autonomous* operation, independent of the classical derivation. It would then be necessary to prove directly the formula of the previous n°

$$[\Omega\,\Pi]' = [\Omega'\,\Pi] \pm [\Omega\,\Pi'], \tag{1}$$

where we simply assume that Ω and Π have an exterior derivative.

75. Take the simplest case of a linear form in two variables

$$\omega = P\,\delta x + Q\,\delta y$$

which has an exterior derivative

$$\omega' = R[\delta x\,\delta y].$$

Suppose that the functions P and Q are continuous and consider finally a function m that has continuous first order partial derivatives. The formula

$$(m\omega)' = m\omega' + [\delta m\,\omega]$$

here comes down to

$$\int m(P\,\delta x + Q\,\delta y) = \iint \left(mR + Q\frac{\partial m}{\partial x} - P\frac{\partial m}{\partial y}\right)\delta x\,\delta y.$$

The proof of this formula is very easy. Let A be the area of integration, and C its boundary contour. Partition the area A into a large number of partial areas, for example by parallels to the axes. In each of these partial areas, take a point (x_0, y_0) and let

$$m_0, \qquad P_0, \qquad Q_0, \qquad \left(\frac{\partial m}{\partial x}\right)_0, \qquad \left(\frac{\partial m}{\partial y}\right)_0.$$

be the values of the functions $m, P, Q, \dfrac{\partial m}{\partial x}, \dfrac{\partial m}{\partial y}$ at this point. In the interior or on the contour of this area, we can put

$$m = m_0 + (x - x_0)\left[\left(\frac{\partial m}{\partial x}\right)_0 + \varepsilon_1\right] + (y - y_0)\left[\left(\frac{\partial m}{\partial y}\right)_0 + \varepsilon_2\right]$$
$$P = P_0 + \varepsilon_3, \qquad Q = Q_0 + \varepsilon_4;$$

the integral $\int m(P\,\delta x + Q\,\delta y)$ over the contour of this partial area will be equal to

$$\int m_0(P\,\delta x + Q\,\delta y) + \int \left[(x - x_0)\left(\frac{\partial m}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial m}{\partial y}\right)_0\right](P_0\,\delta x + Q_0\,\delta y)$$

plus a quantity less than $\varepsilon M \Delta\,\ell$, where we denote by ε an upper bound of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, by M a fixed number, by Δ the diameter of the area and by ℓ the length of its contour. The sum of all these additional quantities can obviously be made as small as we like, because $\Sigma\,\Delta\,\ell$ is of the order of the total area A. As for the sum of the two integrals above, it is equal to

$$\sum \iint \left[m_0 R + Q_0\left(\frac{\partial m}{\partial x}\right)_0 - P_0\left(\frac{\partial m}{\partial y}\right)_0\right]\delta x\,\delta y.$$

We deduce easily the proof of the formula in question.

This proof will generalise with similar hypotheses to the case of a quadratic form

$$\Omega = P[\delta y\,\delta z] + Q[\delta z\,\delta x] + R[\delta x\,\delta y];$$

the existence of the equality

$$\iint P\,\delta y\,\delta z + Q\,\delta z\,\delta x + R\,\delta x\,\delta y = \iint H\,\delta x\,\delta y\,\delta z$$

leads to

$$\iint m(P\,\delta y\,\delta z + Q\,\delta z\,\delta x + R\,\delta x\,\delta y) = \iiint \left(mH + P\frac{\partial m}{\partial x} + Q\frac{\partial m}{\partial y} + R\frac{\partial m}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

The proof is more difficult in the case of two linear forms in three variables

$$\begin{aligned} \omega &= A\,\delta x + B\,\delta y + C\,\delta z, \\ \omega_1 &= A'\,\delta x + B'\,\delta y + C'\,\delta z. \end{aligned}$$

Suppose that these two forms are derivable and that we have for example

$$\begin{aligned} \int A\,\delta x + B\,\delta y + C\,\delta z &= \iint P\,\delta y\,\delta z + Q\,\delta z\,\delta x + R\,\delta x\,\delta y, \\ \int A'\,\delta x + B'\,\delta y + C'\,\delta z &= \iint P'\,\delta y\,\delta z + Q'\,\delta z\,\delta x + R'\,\delta x\,\delta y; \end{aligned}$$

here formula (1) would become

$$\begin{aligned} &\iint (BC' - CB')\delta y\,\delta z + (CA' - AC')\delta z\,\delta x + (AB' - BA')\delta x\,\delta y \\ &\qquad = \iiint (PA' + QB' + RC' - P'A - Q'B - R'C)\delta x\,\delta y\,\delta z. \end{aligned}$$

It does not seem possible to prove this by the same method as in the previous cases, unless we add further hypotheses, for example that the functions A, B, C, A', B', C' satisfy a condition akin to the Lipschitz condition. It would be interesting to study this issue and to see if in fact *the derivability of an exterior product always leads to the derivability of its factors.*

As regards the issue of knowing under what conditions an exterior differential form is derivable, this is related to the theory of additive set functions of M. C. de la Valles-Poussin[2], at least for forms of degree $n-1$ in n variables. For example, the form $P[\delta y\,\delta z] + Q[\delta z\,\delta x] + R[\delta x\,\delta y]$ is derivable if the sum of integrals $\iint \Omega$ over surfaces that bound a finite number of cubes formed by planes parallel to the coordinate planes tends to zero when the sum of the volumes of these cubes tends to zero; the function H which enters into the expression of the derivative

[2] See the work titled: *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire*; Paris, Gauthier-Villars, 1916.

$$\Omega' = H[\delta x\,\delta y\,\delta z]$$

is of course not continuous in general.

In what follows, we will always assume the legitimacy of the operations performed.

III. — *Exact differential exterior forms.*

76. Here now is an important theorem:

The derivative of the derivative Ω' of any exterior differential form Ω is identically zero.

In fact, take any term in Ω, say

$$a[\delta x_1\,\delta x_2 \ldots \delta x_p];$$

the corresponding term of Ω' is

$$[\delta a\,\delta x_1\,\delta x_2 \ldots \delta x_p].$$

If a depends only on $x_1, \ldots, x_p$, this last term is zero, and so is its derivative; if on the contrary a is independent of $x_1, \ldots, x_p$, we can implement a change variables such that a becomes equal to x_{p+1}; the derivative of the term

$$[\delta x_{p+1}\,\delta x_1\,\delta x_2 \ldots \delta x_p]$$

is then zero because the coefficient of this term being unity, its derivation will contribute nothing in the formation of the exterior derivative.

This theorem has a converse, namely:

If the derivative of a differential form Ω is zero, the form Ω can be regarded as the derivative of a form Π whose degree is less than that of Ω by one unit.

To prove this theorem, we will rely on the following lemma, which we will find useful later:

If the derivative of a form Ω is zero and if this form does not contain the differential δx_n, its coefficients are all independent of x_n.

In fact, take a term in Ω such as

$$A[\delta x_1\,\delta x_2 \ldots \delta x_p];$$

in the derivation it will contribute the term

$$[\delta A\, \delta x_1\, \delta x_2 \dots \delta x_p];$$

from which, expanding, we get many terms one of which will be

$$\frac{\partial A}{\partial x_n}[\delta x_n\, \delta x_1\, \delta x_2 \dots \delta x_p];$$

this last term clearly cannot be cancelled by any other, because no term in Ω contains δx_n. Since $\Omega' = 0$, we necessarily have

$$\frac{\partial A}{\partial x_n} = 0.$$

With the lemma thus proved, let us return to our theorem. Call Ω_0 what Ω becomes when we put $x_n = x_n^0$ and $\delta x_n = 0$. The derivative of Ω_0 is clearly zero if that of Ω is zero. Suppose then that the theorem is proved for $n-1$ variables: it will be possible to find a form Π_0 constructed with the variables $x_1, \dots, x_{n-1}$ and whose derivative is Ω_0:

$$\Pi_0' = \Omega_0.$$

This being so, separate the terms in the given form Ω and in the unknown form Π which do not contain δx_n from those that contain it; we can write

$$\Omega = \Omega_1 + [\delta x_n\, \Omega_2], \qquad \Pi = \Pi_1 + [\delta x_n\, \Pi_2];$$

if we calculate in Π' the terms which contain δx_n, we will find

$$\Pi' = \left[\delta x_n\, \frac{\partial \Pi_1}{\partial x_n}\right] - [\delta x_n\, \Pi_2'] + \cdots .$$

Choose the form Π_2 *arbitrarily* and determine Π_1 by the conditions

1°. that for $x_n = x_n^0$, Π_1 reduces to Π_0;

2°. that we have

$$\frac{\partial \Pi_1}{\partial x_n} = \Pi_2' + \Omega_2;$$

we thus get Π_1 by quadratures.

With the form Π chosen as we have just said, it has the following properties:

1°. the difference $\Pi' - \Omega$, when we have reduced similar terms, no longer contains δx_n;

2°. it reduces to zero when in its coefficients we put $x_n = x_n^0$.

Note now that the derivative of this form is zero and that consequently, according to the lemma, all its coefficients have a value independent of x_n; it is thus identically zero, and the theorem is proved.

The proof itself shows that in the form Π we can arbitrarily choose the terms which contain δx_n, arbitrarily choose the values for $x_n = x_n^0$ of terms which do not contain δx_n but contain δx_{n-1}^0; arbitrarily choose, for $x_n = x_n^0, x_{n-1} = x_{n-1}^0$, the values of terms which contain neither δx_n nor δx_{n-1}, but contain δx_{n-2}, etc.

It is quite clear moreover that if we have a solution of the problem, all the others will be deduced from this by adding to Π the derivative of an arbitrary form (of degree smaller by two units than that of Ω).

77. If Ω is a linear form, the hypothesis that its exterior derivative is zero thus leads, according to the previous theorem, to the conclusion already indicated that Ω is an exact differential. If Ω is a quadratic form in three variables

$$\Omega = P[\delta y\,\delta z] + Q[\delta z\,\delta x] + R[\delta x\,\delta y],$$

the condition

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

is necessary and sufficient for Ω to be regarded as the derivative of a linear form, that is, such that we can find three functions A, B, C that satisfy

$$\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} = P,$$
$$\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} = Q,$$
$$\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = R.$$

Note. — If the coefficients of the form Ω are uniform in a certain domain, the condition $\Omega' = 0$ is not always sufficient to ensure the existence of a form Π that is *uniform in this domain* and of which Ω is the exterior derivative. Consider for example the two dimensional domain (closed and without boundary) formed by the points of a sphere Σ, and let Ω be a form of degree 2 uniform in this domain (and with coefficients that admit continuous first order partial derivatives). The derivative Ω' is clearly zero. Nevertheless, were there to exist a linear form ω whose derivative ω' were equal to Ω, we would have, by integrating $\int \omega$ twice along the same great circle of the sphere in two different directions,

$$\iint_{\Sigma} \Omega = 0,$$

where the integral is over the entire surface of the sphere. *The previous equation gives an additional condition for* Ω *to be considered as the exact derivative of a form* ω *that is uniform over the entire sphere*.

Chapter VIII

The Characteristic System of an Exterior Differential Form. Forming Integral Invariants.

I. — *The Characteristic System of an Exterior Differential Form.*

78. The results of the previous Chapter allow us easily to form the *characteristic* Pfaffian system of a given exterior differential form Ω.

For this, note that if Ω is invariant for the system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \tag{1}$$

Ω is expressible by means of $n-1$ independent first integrals $y_1, \ldots, y_{n-1}$ and of their differentials; *consequently, the same is true for its derivative* Ω'. Hence the system of linear (total differential) equations associated with two exterior forms Ω and Ω' will be a consequence of the equations

$$dy_1 = 0, \qquad ..., \qquad dy_n = 0 \tag{2}$$

and consequently of equations (1).

Said differently, *for system (1) to admit $\Omega(\delta)$ as an invariant form, it is necessary that the associated system of Ω and of Ω' be satisfied when we replace the variables $\delta x_1, \ldots, \delta x_n$ by $X_1, \ldots, X_n$.*

Conversely, suppose that this condition is satisfied. Since the associated system of $\Omega(d)$ is satisfied when we take into account equations (1), they will be satisfied when we take into account the equivalent equations (2); thus $\Omega(d)$, considered as an exterior form of the quantities dx_i, will be expressible solely by means of the $dy_1, \ldots, dy_{n-1}$, where the coefficients are functions of the x's, which can always be assumed to be expressed by means of the $y_1, \ldots, y_{n-1}$ and x_n (if $X_n \neq 0$). We then have

$$\Omega = \sum A_{i_1 \ldots i_p} [dy_{i_1} \ldots dy_{i_p}].$$

By forming Ω', we find that the only term in $[dx_n dy_{i_1} \dots dy_{i_p}]$ has as coefficient $\frac{\partial A_{i_1 \dots i_p}}{\partial x_n}$; now, by hypothesis, Ω' must also be expressible by means of the dy_i alone; we thus have

$$\frac{\partial A_{i_1 \dots i_p}}{\partial x_n} = 0;$$

consequently Ω' is expressed in terms of first integrals of the given system and their differentials: it is therefore an invariant form.

From this, it follows immediately *that the equations of the characteristic system of Ω reduce to equations of the associated system of Ω joined to equations of the associated system of Ω'.*

79. Let us look at some important special cases.

Suppose that Ω is an exact derivative, that is, $\Omega' = 0$. In this case, *the characteristic system of the differential form Ω is the same as the associated system of the form Ω.*

As an application, let us look for the characteristic system of a (complete) *relative* invariant integral $\int \Omega$. This relative invariant reduces to the absolute invariant $\int \Omega'$; now, Ω' is an exact derivative. So *the characteristic system of a relative integral invariant $\int \Omega$ is the same as the associated system of the derived form Ω'.*

This is the case for the linear integral invariant of dynamics

$$\int \omega_\delta = \int \sum p_i \; \delta q_i - H \; \delta t.$$

Here we have

$$\omega'_\delta = \int \sum [\delta p_i \; \delta q_i] - [\delta H \; \delta t].$$

The associated system of the form ω' is

$$\frac{\partial \omega'}{\partial (\delta q_i)} = 0, \qquad \frac{\partial \omega'}{\partial (\delta p_i)} = 0, \qquad \frac{\partial \omega'}{\partial (\delta t)} = 0,$$

that is, by taking the symbol d instead of δ,

$$-dp_i - \frac{\partial H}{\partial q_i}\, dt = 0,$$
$$dq_i - \frac{\partial H}{\partial p_i}\, dt = 0,$$
$$dH - \frac{\partial H}{\partial t}\, dt = 0:$$

*this is the same calculation that we performed in Chapter 1 (^{0} **11**).*

Here the differential form ω' is quadratic; consequently we know in advance that the number of independent equations of the associated system is *even*: which explains why the $2n+1$ equations of the characteristic system reduce to $2n$. We also have the explanation of what happens in hydrodynamics with respect to the invariant form

$$\xi[\delta y\delta z] + \eta[\delta z\delta x] + \zeta[\delta x\delta y] + (\eta w - \zeta v)[\delta x\delta t] + (\zeta u - \xi w)[\delta y\delta t] + (\xi v - \eta u)[\delta z\delta t].$$

Here $n=4$; the characteristic system thus contains 4 or 2 or 0 independent equations: now it cannot be 4 because the form is invariant for the differential equations of the trajectories of the molecules: so it is 2 or 0. It is 0 if $\xi = \eta = \zeta = 0$, that is, if the motion is irrotational. Otherwise one could predict *a priori* that the trajectories would not be the only characteristic curves of the form.

80. A final important case is that where the form Ω is of degree $n-1$. If it is invariant for a system of differential equations, this system is necessarily unique, because the associated Pfaffian system of Ω is formed by $n-1$ independent equations. For the associated system of Ω' to contain no more than $n-1$ independent equations, it is clearly necessary that Ω' be zero. Consequently, *for a form Ω of degree $n-1$ to be invariant for a system of differential equations, it is necessary and sufficient that its derivative be identically zero.*

A simple example is provided by the integral invariant of the kinematics of continuous media

$$\iiint \rho(\delta x - u\,\delta t)(\delta y - v\,\delta t)(\delta z - w\,\delta t).$$

Here the form Ω is

$$\Omega = \rho[\delta x\delta y\delta z] - \rho u[\delta y\delta z\delta t] - \rho v[\delta z\delta x\delta t] - \rho w[\delta x\delta y\delta t].$$

The condition $\Omega' = 0$ gives

$$\Omega' = \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right][\delta t\delta x\delta y\delta z] = 0;$$

this is the condition of continuity, or the law of the conservation of mass

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0.$$

We see that *this law of the conservation of mass is expressed by the simple condition that the derivative* Ω' *of the form which defines the elementary quantity of mass is zero*.

81. Conservation laws in Physics are frequently expressed by similar conditions. The law of conservation of force flux for a force field X, Y, Z is expressed by the condition that the *divergence* of this force field is zero, that is

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0;$$

now this simply expresses the fact that the derivative of the elementary force flux

$$\Omega = X[\delta y\,\delta z]+Y[\delta z\,\delta x]+Z[\delta x\,\delta y]$$

is zero.

All (static) magnetic fields satisfy this condition. The electromagnetic field, defined by means of the exterior form

$$\Omega = H_x[\delta y\,\delta z]+H_y[\delta z\,\delta x]+H_z[\delta x\,\delta y]+E_x[\delta x\,\delta t]+E_y[\delta y\,\delta t]+E_z[\delta z\,\delta t],$$

also satisfies the condition that Ω' is zero. We have

$$\begin{aligned}\Omega' = &\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)[\delta x\,\delta y\,\delta z]+\left(\frac{\partial H_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)[\delta y\,\delta z\,\delta t]\\ &+\left(\frac{\partial H_y}{\partial t}+\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)[\delta z\,\delta x\,\delta t]+\left(\frac{\partial H_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)[\delta x\,\delta y\,\delta t].\end{aligned}$$

By equating to zero the four coefficients of Ω' we obtain the four classic equations which in vector notation can be written as

$$\begin{aligned}\operatorname{div} H &= 0,\\ \frac{\partial H}{\partial t}+\operatorname{curl} E &= 0.\end{aligned}$$

Since the hydrodynamic form, already considered so often,

$$\Omega = \xi[\delta y\,\delta z]+\eta[\delta z\,\delta x]+\zeta[\delta x\,\delta y]+(\eta w-\zeta v)[\delta x\,\delta t]+(\zeta u-\xi w)[\delta y\,\delta t]+(\xi v-\eta u)[\delta z\,\delta t]$$

also has a zero derivative because Ω is the derivative of the linear energy-momentum form, the vectors (ξ, η, ζ) and $(\eta w-\zeta v, \zeta u-\xi v, \xi v-\eta u)$ satisfy the same relations as the magnetic field and the electric field: these two vectors are first the vorticity, which plays the role of the magnetic force, and the other the vector product of the vorticity and the velocity which plays the role of the electric force.

It should be noted that the electromagnetic field (or rather the form Ω which represents it) cannot be invariant for any system of differential equations, because $[\Omega^2]$ is in general not zero. The only exception would be if the magnetic field were perpendicular to the electric field; the characteristic system would then be defined by the equations

$$\begin{aligned} H_z\, dy - H_y\, dz + E_x\, dt &= 0, \\ H_x\, dz - H_z\, dx + E_y\, dt &= 0, \\ H_y\, dx - H_x\, dy + E_z\, dt &= 0, \\ -E_x\, dx - E_y\, dy - E_z\, dz &= 0, \end{aligned}$$

which reduce to three. One of the systems of differential equations which admits Ω as an invariant form would then be

$$\frac{dx}{H_x} = \frac{dy}{H_y} = \frac{dz}{H_z} = \frac{dt}{0} :$$

at each instant, it defines the lines of force of the magnetic field; another is

$$\frac{dx}{E_y H_z - E_z H_y} = \frac{dy}{E_z H_x - E_x H_z} = \frac{dz}{E_x H_y - E_y H_x} = \frac{dt}{H_x^2 + H_y^2 + H_z^2}.$$

If the magnetic field were zero, the characteristic manifolds would be defined by the equations

$$dt = 0, \qquad E_x\, dx + E_y\, dy + E_z\, dz = 0;$$

these would be the equipotential surfaces considered at each time t.

II. — *Forming Integral Invariants.*

82. It is obvious that the exterior product of two invariant exterior forms is also an invariant form.

From this, knowledge of an invariant exterior form Ω leads to knowledge of an entire series of other invariant forms, namely Ω' and all the forms deduced by exterior multiplication of Ω and Ω'.

First, suppose that Ω is an *absolute* invariant form of *even* degree: we then get the two series of absolute invariant forms

$$[\Omega^p], \qquad [\Omega^{p-1}\,\Omega'], \qquad (p = 1, 2, \ldots);$$

the derivative of a form of the first series is a form of the second series, the derivative of a form of the second series is zero.

Secondly, suppose Ω is an *absolute* invariant form of *odd* degree: we get the two series

$$[\Omega'^p], \qquad [\Omega\,\Omega'^{p-1}], \qquad (p = 1, 2, ...);$$

the derivative of a form of the first series is zero; the derivative of a form of the second series is equal to a form of the first series.

Now suppose that we have a *relative* integral invariant $\int \Omega$ and suppose first that Ω is of even degree; we can only deduce one new invariant, the absolute invariant $\int \Omega'$.

If, on the contrary, Ω is of odd degree, we get a series of relative integral invariants $\int \Omega\Omega'^{p-1}$ and a series of absolute integral invariants $\int \Omega'^p$. Moreover, the relative invariant $\int \Omega\Omega'^{p-1}$ leads by derivation to the absolute invariant $\int \Omega'^p$.

For example, it is so for the relative invariant of Dynamics

$$\int \omega = \int \sum_{i=1}^{i=n} p_i\,\delta q_i - H\,\delta t;$$

the relative integral invariants that can be derived from it are

$$\int \omega\omega'^{p-1} \qquad (p = 1, ...n);$$

the absolute integral invariants are

$$\int \omega'^p \qquad (p = 1, ...n).$$

There thus exists an invariant (relative or absolute) of any given degree less than or equal to $2n$.

83. Ir would be wrong to believe that the new invariants that we have just indicated are always the only ones that can be deduced (without integration) from a given invariant. For example, suppose that we know an invariant form Ω is reducible to the form

$$\Omega = [\omega_1\,\omega_2\,\omega_3] + [\omega_4\,\omega_5\,\omega_6],$$

where $\omega_1, \omega_2, ..., \omega_6$ are six independent linear (Pfaffian) forms. Introduce six indeterminates $\xi_1, ..., \xi_6$ and consider the auxiliary quadratic form

$$\Pi = \xi_1 \frac{\partial \Omega}{\partial \omega_1} + \xi_2 \frac{\partial \Omega}{\partial \omega_2} + \cdots + \xi_6 \frac{\partial \Omega}{\partial \omega_6}.$$

It is clear that if we look at the ξ as quantities covariant with ω, the form Π is a covariant with of Ω. Express the fact that this form is of rank 2; we obtain the conditions

$$\xi_i \xi_\alpha = 0, \qquad (i = 1,2,3;\ \alpha = 4,5,6);$$

there are two possible solutions provided either by

$$\xi_1 = \xi_2 = \xi_3 = 0,$$

or by

$$\xi_4 = \xi_5 = \xi_6 = 0.$$

From this it follows that there there exist two systems of three covariant Pfaffian equations, namely

$$\omega_1 = \omega_2 = \omega_3 = 0,$$
$$\omega_4 = \omega_5 = \omega_6 = 0.$$

Consequently also the form $[\omega_1\, \omega_2\, \omega_3]$ and the form $[\omega_4\, \omega_5\, \omega_6]$ are also covariant: the first is obtained by taking into account in Ω the equations of the second covariant system; the second, by taking into account the equations of the first covariant system.

Now suppose that Ω is expressed by means of the first integrals of the system of equations for which Ω is an invariant form, and of their differentials. The two covariant Pfaff systems will be formed in the same way as for the reduced form, and each of them will contain only the first integrals and their differentials: the same will apply to the two forms $[\omega_1\, \omega_2\, \omega_3]$ and $[\omega_4\, \omega_5\, \omega_6]$, *which are consequently invariant forms.*

The existence of the integral invariant $\int \Omega$ *thus leads to the existence of each of the integral invariants* $\int [\omega_1\, \omega_2\, \omega_3]$, $\int [\omega_4\, \omega_5\, \omega_6]$.

By a similar argument, we would see that *the existence of an invariant form of degree $p > 2$, reducible to a sum of h monomial terms such that the hp factors which enter into these terms are linearly independent, leads to the property that each of these monomial terms is an invariant form.*

This theorem is not true if $p = 2$.

84. In some cases, the existence of an invariant form leads to the existence of an invariant *equation*. Consider, for example, the form

$$\Omega = [\omega_1\, \omega_2\, \omega_5] + [\omega_3\, \omega_4\, \omega_5],$$

where $\omega_1, \ldots, \omega_5$ denote five independent Pfaffian forms. The only linear relation between these forms which makes Ω equal to zero is obviously

$$\omega_5 = 0;$$

this last equation is thus *invariant*: it can be expressed by means of first integrals of differential equations for which Ω is an invariant form.

Generally, if Ω is an invariant form and *if the associated system of Ω is not the same as its characteristic system*, this associated system is an invariant Pfaffian system.

We could vary these considerations in many ways.

85.[1] Take again the case of two quadratic invariant forms Ω_1 and Ω_2 that have the same associated system; let $2s$ be their common rank. The equation of degree s in λ

$$[(\Omega_1 - \lambda\Omega_2)^s] = 0,$$

which expresses the fact that the rank of the form $\Omega_1 - \lambda\Omega_2$ is less than $2s$, clearly has an invariant meaning. *The roots of the equation in λ are thus first integrals of differential equations which admit Ω_1 and Ω_2 as first integrals*. We can show that, *in the general case*, Ω_1 and Ω_2 are reducible to the forms

$$\begin{aligned}\Omega_1 &= \lambda_1[\omega_1\,\omega_2] + \lambda_2[\omega_3\,\omega_4] + \cdots + \lambda_s[\omega_{2s-1}\,\omega_{2s}],\\ \Omega_2 &= [\omega_1\,\omega_2] + [\omega_3\,\omega_4] + \cdots + [\omega_{2s-1}\,\omega_{2s}].\end{aligned}$$

Each of the monomial forms $[\omega_1\,\omega_2], [\omega_3\,\omega_4], \cdots, [\omega_{2s-1}\,\omega_{2s}]$ is invariant.

[1] TRANSLATOR'S NOTE. This Paragraph is labelled **85** in Cartan's book. However, Chapter IX begins with another Paragraph **85**, so there is clearly a mismatch in the numbering of Paragraphs here. Cartan probably included this Paragraph as an afterthought after the book had already been typeset. I have resisted the temptation to call this Paragraph n° ***84a*** and have retained the numbering as it appeared originally.

Chapter IX

Differential Systems that Admit an Infinitesimal Transformation.

I. — *The notion of an infinitesimal transformation.*

85. A *transformation* in n variables is defined by a system of equations

$$x_i' = f_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{1}$$

solvable with respect to $x_1, \ldots, nx_n$. Geometrically, if in the space of n-dimensions we regard $x_1, \ldots, x_n$ as coordinates of a point M, the transformation (1) takes us from any point M of the space to another point M' according to a well defined law. Transformations are commonly used in geometry (homothety, similarity, inversion, or, simpler still, rotation, translation, etc.).

Transformation (1) is said to be the *identity* when the right hand sides reduce respectively to $x_1, \ldots, x_n$; any point is then transformed into itself.

Given a system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n}, \tag{2}$$

this system is said to *admit transformation (1)* when this transformation, applied to different points of *any* integral curve of (2), gives points that all belong to the same new integral curve.

Consider a transformation that depends on a parameter a and reduce to the identity transformation for a certain value a_0 of this parameter. Put $a - a_0 = \varepsilon$ and suppose that the right hand sides can be expanded in powers of ε

$$x_i' = x_i + \varepsilon \xi_i(x_1, \ldots, x_n) + \cdots$$

We will have what we call an *infinitesimal transformation* by focusing only on terms of first order in ε. An infinitesimal transformation is thus completely defined by n functions ξ_i of $x_1, \ldots, x_n$;

we obtain the same infinitesimal transformation by multiplying all these functions by a common constant factor. We will say that the function ξ_i represents the increase of the variable x_i by the infinitesimal transformation (in reality the increase is $\varepsilon\xi_i$, but the coefficient ε plays only a secondary role).

Given a function $f(x_1,\ldots,x_n)$, the increase that the infinitesimal transformation causes this function to undergo is, up to a factor ε, the first terms in the expansion of

$$f(x'_1,\ldots,x'_n) - f(x_1,\ldots,x_n) = f(x_1+\varepsilon\xi_1,...,x_n+\varepsilon\xi_n) - f(x_1,\ldots,x_n);$$

it is thus

$$\xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n};$$

We will denote this expression by the symbol Af:

$$Af = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n}; \tag{3}$$

we will agree to say that Af *is the symbol of the infinitesimal transformation considered*.

86. Formula (3) is similar to that which gives the total differential of a function f:

$$\delta f = \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \delta x_n \frac{\partial f}{\partial x_n}.$$

The only difference is that δ is the symbol of an *indeterminate* operation, while A is the symbol of a *definate* operation. The symbol for differentiation becomes the symbol of an infinitesimal transformation as soon as we give to $\delta x_1,\ldots,\delta x_n$ definite values (given functions of the variables).

The operation symbolised by A can be applied not only to finite functions, but also to differential forms. For example, we shall understand by $A(dx_i)$ the principal part (divided by ε) of the increase of dx_i. Now, we have

$$dx'_i - dx_i = \varepsilon d\xi_i + \cdots;$$

we are thus led to put

$$A(dx_i) = d\xi_i = d(Ax_i).$$

From this, we see that *the operation A must be considered as commuting with the operation of differentiation* (indeterminate).

87. Let us return to system (2) of differential equations. This system will be said to admit infinitesimal transformation (3) if this transformation, applied to different points of any integral curve, gives points that all lie on the same new integral curve, *up to infinitesimals of second order*.

It is very clear that if equations (2) admit a transformation that depends on a parameter a, whatever the numerical value of this parameter, they will admit the infinitesimal transformation that corresponds to the values of a infinitely close to the value a_0 (if it exists) which gives the identity transformation.

If y is a first integral of equations (2) and if these equations (2) admit the infinitesimal transformation Af, it is clear that $A(y)$ will also be a first integral. In fact, at each point M of any integral curve (C), y keeps the same numerical value c; at point M', the transform of M, the function y increases by $\varepsilon A(y)$; this increase must be the same whatever the point M of (C); it is necessary therefore that $A(y)$ has the same numerical value at all points of (C); in other words, $A(y)$ is a first integral.

Conversely, if the operation A, applied to any first integral, again gives a first integral, then system (2) admits the infinitesimal transformation Af. In fact, if

$$c_1, c_2, \dots, c_{n-1}$$

are the constant numerical values which $n-1$ independent first integrals

$$y_1, y_2, \dots, y_{n-1}$$

take at the different points M of an integral curve (C), the values which these integrals take at the transformed points M' are those which the functions

$$y_1 + \varepsilon A(y_1), y_2 + \varepsilon A(y_2), \dots, y_{n-1} + \varepsilon A(y_{n-1})$$

take at the points M themselves; these are thus *constants*. Consequently, the points M' indeed generate an integral curve.

II. — *Forming integral invariants starting from infinitesimal transformations.*

88. The preceding property shows us that *knowledge of an infinitesimal transformation Af admitted by the differential equations (2) allows us to deduce from any invariant differential form Ω another invariant form, namely* $A(\Omega)$. If the form Ω is exterior, so is the form $A(\Omega)$, and the new form has the same degree as the old one.

There is a second operation which allows to deduce from the invariant exterior form Ω another invariant form. For definiteness, assume that Ω is of third degree and consider the corresponding trilinear differential form $\Omega(\delta,\delta',\delta'')$. In this form, replace the symbol δ of indeterminate differentiation by the symbol of the infinitesimal transformation; we obtain an alternating linear form $\Omega(A,\delta',\delta'')$ in two series of differentials δ',δ'' to which there corresponds finally an exterior quadratic form which we will denote by $\Omega(A,\delta)$. This new form is deduced from the first by an operation which has a meaning independent of the choice of variables. If Ω is expressed by means of the first integrals y_i of equations (2) and their differentials, the expression $\Omega(A,\delta)$ is also expressed by means of the y_i and the dy_i. As a result, *the operation just defined allows us to deduce from any invariant form another invariant form whose degree is reduced by one.*

We have moreover

$$\Omega(A,\delta) = \xi_1 \frac{\partial\Omega}{\partial(\delta x_1)} + \xi_2 \frac{\partial\Omega}{\partial(\delta x_2)} + \cdots + \xi_n \frac{\partial\Omega}{\partial(\delta x_n)}. \tag{4}$$

89. The two new operations just defined are not independent of one another. For definiteness, suppose that Ω is of second degree, and return the formula that defines the exterior derivative Ω'. We have (n° **71**)

$$\Omega'(\delta,\delta',\delta'') = \delta\Omega(\delta',\delta'') - \delta'\Omega(\delta,\delta'') + \delta''\Omega(\delta,\delta')$$

with the sole condition that the three symbols δ,δ',δ'' commute with one another. Replace the symbol δ with that of the infinitesimal transformation Af. We will have

$$\Omega'(A,\delta',\delta'') = A(\Omega(\delta',\delta'')) - \delta'\Omega(A,\delta'') + \delta''\Omega(A,\delta')$$

that is, returning to exterior forms,

$$\Omega'(A,\delta) = A(\Omega(\delta)) - [\Omega(A,\delta)]'$$

or finally,

$$A(\Omega(\delta)) = \Omega'(A,\delta) + [\Omega(A,\delta)]' \tag{5}$$

This fundamental formula contains on the left hand side the result of the first operation performed on Ω. As for the two terms on the right hand side, the first is obtained by performing on Ω, first the operation of exterior differentiation, then the second operation associated with Af; as for the second term it is deuced from Ω by the same operations, but performed in reverse order.

In conclusion, *knowledge of an infinitesimal transformation Af admitted by equations (2) equips us with a new essential operation, defined by formula (4), which allows us to deduce from an invariant form $\Omega(\delta)$ an new invariant form $\Omega(A,\delta)$.*

Note in particular that if y is a first integral, the first integral $A(y)$ can be obtained, first by differentiation, which gives $\omega(\delta) = \delta y$, then by application of operation (4), which gives

$$\omega(A) = A(y).$$

III. — *Examples.*

90. Consider a continuous material medium in motion, where the density is ρ and the components of the velocity are u, v, w. The differential equations of the motion of a molecule

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w, \tag{6}$$

admit, as we have seen (n° **37**), the integral invariant

$$\iiint \rho\,(\delta x \delta y \delta z - u\,\delta y \delta z \delta t - v\,\delta z \delta x \delta t - w\,\delta x \delta y \delta t)$$

which corresponds to the invariant form

$$\Omega = \rho\,[\delta x \delta y \delta z] - \rho u\,[\delta y \delta z \delta t] - \rho v\,[\delta z \delta x \delta t] - \rho w\,[\delta x \delta y \delta t].$$

Suppose the motion is *permanent*, that is, ρ, u, v, w are independent of t. Equations (6) do not contain time explicitly, that is, they do not change when we change t into $t + \varepsilon$, admitting the infinitesimal transformation

$$Af = \frac{\partial f}{\partial t}.$$

Consequently, it admits the invariant form

$$\Omega(A, \delta) = \frac{\partial \Omega}{\partial(\delta t)} = -\rho u\,[\delta y \delta z] - \rho v\,[\delta z \delta x] - \rho w\,[\delta x \delta y].$$

The property of this form of being invariant is obvious physically. In fact, consider a *tube* of trajectories and cut this tube by any two surfaces, which determine two areas S and S' inside the tube. The mass that fills the volume contained between the lateral surface of the tube and the two surfaces S and S' is always the same, consequently the algebraic mass flux across the surface that bounds this volume is zero. Now, the flux across the lateral surface is zero. We thus have

$$\iint_S \rho\,(u\,\delta y \delta z + v\,\delta z \delta x + w\,\delta x \delta y) = \iint_{S'} \rho\,(u\,\delta y \delta z + v\,\delta z \delta x + w\ , \delta x \delta y).$$

Note that the invariant form $\Omega(A, \delta)$ is an exact derivative; in fact, were its derivative not zero, it could only differ from Ω by a finite factor; now, this derivative does not contain δt; we thus have

$$[\Omega(A, \delta)]' = 0.$$

The characteristic system of $\Omega(A,\delta)$ consequently reduces to the associated system; it is thus given by the equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w};$$

it defines the trajectories of the molecules, but independently of how these trajectories are traversed over time.

Formula (5) shows also the property of the form $\Omega(A,\delta)$ that its derivative is zero; in fact, here the form Ω' is identically zero. On the other hand, since Ω does not contain t explicitly, does not change when we change t to $t+\varepsilon$, consequently $A(\Omega)$ is zero. This remark will be applied to the following examples.

91. Consider now a perfect fluid in motion under the action of forces derived from a potential. We have seen (n° **22**) the existence of an absolute invariant form

$$\begin{aligned}\omega' = \xi[\delta y\delta z] + \eta[\delta z\delta x] + \zeta[\delta x\delta y] + (\eta w - \zeta v)[\delta x\delta t] \\ +(\zeta u - \xi w)[\delta y\delta t] + (\xi v - \eta u)[\delta z\delta t];\end{aligned}$$

it came from the exterior derivative of a linear form

$$\omega = u\,\delta x + v\,\delta y + w\,\delta z - E\,\delta t,$$

where the coefficient E, the energy per unit mass, was expressed as

$$E = \frac{1}{2}(u^2+v^2+w^2) - U + \int \frac{dp}{\rho}.$$

Suppose the motion is *permanent*, that is, u,v,w,p,ρ are independent of t. Here again, we will have a new invariant form

$$\begin{aligned}\omega'(A,\delta) = \frac{\partial\omega'}{\partial(\delta t)} &= (v\zeta - w\eta)\delta x + (w\xi - u\zeta)\delta y + (u\eta - v\xi)\delta z \\ &= \begin{vmatrix} \delta x & \delta y & \delta z \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix}.\end{aligned}$$

Starting from the expression

$$\omega' = [\delta u\ \delta x] + [\delta v\ \delta y] + [\delta w\ \delta z] - [\delta E\ \delta t],$$

we find on the other hand

$$\omega'(A,\delta) = \delta E.$$

Consequently E is a first integral of the equations of motion: we find again *Bernoulli's theorem*, according to which, in a perfect fluid in permanent motion, the quantity

$$\frac{1}{2}(u^2+v^2+w^2)-U+\int\frac{dp}{\rho}.$$

remains constant along each streamline.

But the form δE is not only invariant for the differential equations of motion of the fluid molecules: it is also for the differential equations of the vorticity lines which also admit the invariant form ω'; consequently *the quantity E remains constant not only along each streamline, but also along each line of vorticity.*

If the motion is irrotational, the form $\omega'(A,\delta)$, as originally written, is clearly identically zero: in this case the energy is constant throughout the fluid mass at all times.

The equality

$$\delta E = \begin{vmatrix} \delta x & \delta y & \delta z \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix}$$

allows us to represent at each point M the (spatial) variation of the energy by means of a vector MH having this point as origin and which will be the vector product of the velocity vector (u,v,w) and the vorticity vector (ξ,η,ζ); the derivative of the energy in a given direction will be equal to the projection of the vector MH onto this direction.

92. Another very general application is to a problem in dynamics, when the given constraints and forces are time-independent. The infinitesimal transformation $Af=\dfrac{\partial f}{\partial t}$ admitted by the equations of motion allows us to deduce from the fundamental integral invariant of dynamics

$$\iint\omega' = \iint\sum\delta p_i\delta q_i - \delta H\delta t$$

the new integral invariant

$$\int\delta H$$

obtained by partial differentiation with respect to δt. We thus obtain the *integral of the generalised energy*

$$H=h,$$

under the sole condition that the function H be time independent.

More generally, suppose that the function H does not contain one of the variables p_i *and* q_i, for example let it be q_n. The equations of motion then admit the infinitesimal transformation $\frac{\partial f}{\partial q_n}$, from which we deduce the linear invariant form

$$\frac{\partial \omega'}{\partial(\delta q_n)} = -\delta p_n;$$

thus if the function H does not contain one of the canonical variables, the conjugate variable is a first integral of the equations of motion.

IV. — *Applications to the* n*-body problem.*

93. Consider n material points which are mutually attracted by forces proportional to their masses and inversely proportional to a given power of their separation. There is a force function

$$U = f \sum_{i,j} \frac{m_i m_j}{r_{ij}^p},$$

where the exponent p is given (equal to 1 in the case of celestial mechanics), and where the quantity r_{ij} denotes the separation of the two points M_i and M_j with masses m_i and m_j.

The equations of motion of the system admit a certain number of obvious infinitesimal transformations. First, time does not enter explicitly into these equations. Furthermore, from any solution of the problem we deduce another by displacement of the ensemble in space, and also by giving each of the n points an additional rectilinear and uniform motion (the same for all the points). We deduce immediately from this the existence of the infinitesimal transformations

$$A_0 f = \frac{\partial f}{\partial t},$$

$$A_1 f = \sum_i \frac{\partial f}{\partial x_i}, \qquad A_2 f = \sum \frac{\partial f}{\partial y_i}, \qquad A_3 f = \sum \frac{\partial f}{\partial z_i},$$

$$A_4 f = \sum y_i \frac{\partial f}{\partial z_i} - z_i \frac{\partial f}{\partial y_i} + y_i' \frac{\partial f}{\partial z_i'} - z_i' \frac{\partial f}{\partial y_i'}, \qquad A_5 f = \cdots, \qquad A_6 f = \cdots$$

$$A_7 f = \sum \left(\frac{\partial f}{\partial x_i'} + t \frac{\partial f}{\partial x_i} \right), \quad A_8 f = \sum \left(\frac{\partial f}{\partial y_i'} + t \frac{\partial f}{\partial y_i} \right), \quad A_9 f = \sum \left(\frac{\partial f}{\partial z_i'} + t \frac{\partial f}{\partial z_i} \right).$$

The transformation $A_1 f$ corresponds to a translation parallel to Ox, the transformation $A_4 f$ to a rotation about Ox, the transformation $A_7 f$ to an additional motion of constant velocity ε parallel to Ox.

Finally, we can point out a last infinitesimal transformation based on considerations of homogeneity. The equations

$$m_i \frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i}$$

remain unaltered if we multiply all the coordinates x_i, y_i, z_i by a constant common factor λ, on condition that we multiply t by $\lambda^{1+\frac{p}{2}}$; the components x'_i, y'_i, z'_i of the velocities are then multiplied by $\lambda^{-\frac{p}{2}}$. Taking $\lambda = 1+\varepsilon$, we arrive at the new infinitesimal transformation

$$A_{10}f = \sum x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i} + z_i \frac{\partial f}{\partial z_i} - \frac{p}{2}\left(x'_i \frac{\partial f}{\partial x'_i} + y'_i \frac{\partial f}{\partial y'_i} + z'_i \frac{\partial f}{\partial z'_i}\right) + \left(1+\frac{p}{2}\right) t \frac{\partial f}{\partial t}.$$

We note that, according to the definition itself of U, we have

$$A_0U = A_1U = \cdots = A_9U = 0, \qquad A_{10}U = -pU.$$

94. Recall the fundamental integral invariant of second degree

$$\omega' = \sum m_i[\delta x'_i \delta x_i] + m_i[\delta y'_i \delta y_i] + m_i[\delta z'_i \delta z_i] - \left[\sum m_i(x'_i \delta x'_i + y'_i \delta y'_i + z'_i \delta z'_i)\delta t\right] + [\delta U \delta t].$$

Denote by ω_i the linear form $\omega'(A_i, \delta)$. There are eleven invariant linear forms $\omega_0, \omega_1, \ldots, \omega_1 0$. It is easy to see *a priori*, according to formula (5), that the first ten are exact differentials, because ω' does not change by any one of the first ten infinitesimal transformations. As for ω_{10}, formula (5) gives

$$(\omega_{10})' = A_{10}(\omega');$$

now ω' has a homogeneity degree (in the sense defined earlier) equal to $1 - \dfrac{p}{2}$; so we have

$$(\omega_{10})' = \left(1 - \frac{p}{2}\right)\omega',$$

and ω_{10} will only be an exact differential if $p = 2$, that is, *if the attraction is proportional to the cube of the distance*.

The calculation of the eleven forms ω_i is not difficult and gives

$$\begin{aligned}
\omega_0 &= \sum m_i(x_i'\delta x_i' + y_i'\delta y_i' + z_i'\delta z_i') - \delta U = \delta H,\\
\omega_1 &= -\sum m_i\,\delta x_i' = \delta H_1,\\
\omega_2 &= -\sum m_i\,\delta y_i' = \delta H_2,\\
\omega_3 &= -\sum m_i\,\delta z_i' = \delta H_3,\\
\omega_4 &= \sum m_i(z_i\,\delta y_i' - y_i\,\delta z_i' + y_i'\,\delta z_i - z_i'\,\delta y_i) = \delta H_4,\\
\omega_5 &= \sum m_i(x_i\,\delta z_i' - z_i\,\delta x_i' + z_i'\,\delta x_i - x_i'\,\delta z_i) = \delta H_5,\\
\omega_6 &= \sum m_i(y_i\,\delta x_i' - x_i\,\delta y_i' + x_i'\,\delta y_i - y_i'\,\delta x_i) = \delta H_6,\\
\omega_7 &= \sum m_i(\delta x_i - t\,\delta x_i' - x_i'\,\delta t) = \delta H_7,\\
\omega_8 &= \sum m_i(\delta y_i - t\,\delta y_i' - y_i'\,\delta t) = \delta H_8,\\
\omega_9 &= \sum m_i(\delta z_i - t\,\delta z_i' - z_i'\,\delta t) = \delta H_9,\\
\omega_{10} &= -\sum m_i(x_i\delta x_i' + y_i\delta y_i' + z_i\delta z_i' + \frac{p}{2}\,x_i'\,\delta x_i + \frac{p}{2}\,y_i'\,\delta y_i + \frac{p}{2}\,z_i'\,\delta z_i)\\
&\qquad + \left(1+\frac{p}{2}\right) t\,\delta H + pH\,\delta t;
\end{aligned}$$

we have put

$$\begin{aligned}
H &= \frac{1}{2}\sum m_i(x_i'^2 + y_i'^2 + z_i'^2) - U,\\
H_1 &= -\sum m_i x_i',\\
H_2 &= -\sum m_i y_i',\\
H_3 &= -\sum m_i z_i',\\
H_4 &= -\sum m_i(y_i z_i' - z_i y_i'),\\
H_5 &= -\sum m_i(z_i x_i' - x_i z_i'),\\
H_6 &= -\sum m_i(x_i y_i' - y_i x_i'),\\
H_7 &= \sum m_i(x_i - t x_i'),\\
H_8 &= \sum m_i(y_i - t y_i'),\\
H_9 &= \sum m_i(z_i - t z_i').
\end{aligned}$$

We verify easily that the bilinear covariant of ω_{10} is equal to $\left(1-\frac{p}{2}\right)\omega'$. If $p = 2$, we have a new first integral

$$\sum m_i(x_i x_i' + y_i y_i' + z_i z_i') - 2Ht = C,$$

which gives, by integrating,

$$\sum m_i(x_i^2 + y_i^2 + z_i^2) = 2Ht^2 + 2Ct + C' :$$

This is *Jacobi's integral.*

The first integrals $H_1, H_2, H_3, H_7, H_8, H_9$ are those which yield the theorem on the centre of gravity; the first integrals H_4, H_5, H_6 are those which yield the law of areas.

95. In the preceding Paragraph,[1] we obtained *directly* only the *differentials* of the first integrals H_i and not the integrals themselves. They are given to us by applying to each of the invariant forms ω_i the operation that corresponds to the infinitesimal transformation $A_i f$. We will thus obtain *invariant* functions, that is, first integrals

$$\alpha_{ij} = \omega_i(A_j) = \omega'(A_i, A_j) = -\alpha_{ji}$$

which we will now write as a two-dimensional table[2] which will be clearly skew-symmetric. The calculations present no difficulties: the quantity α_{ij} is found at the intersection of line i with column j. The letter M denotes the sum of the masses of the n bodies.

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	$-pH$
1	0	0	0	0	0	H	$-H_2$	$-M$	0	0	$-\frac{p}{2}H_1$
2	0	0	0	0	$-H_2$	0	H_1	0	$-M$	0	$-\frac{p}{2}H_2$
3	0	0	0	0	H_2	$-H_1$	0	0	0	$-M$	$-\frac{p}{2}H_3$
4	0	0	H_3	$-H_2$	0	H_6	$-H_5$	0	H_9	$-H_8$	$\left(1-\frac{p}{2}\right))H_4$
5	0	$-H_3$	0	H_1	$-H_6$	0	H_4	$-H_9$	0	$-H_7$	$\left(1-\frac{p}{2}\right))H_5$
6	0	H_2	$-H_1$	0	H_5	$-H_4$	0	H_8	$-H_7$	0	$\left(1-\frac{p}{2}\right))H_6$
7	0	M	0	0	0	H_9	$-H_8$	0	0	0	H_7
8	0	0	M	0	$-H_9$	0	H_7	0	0	0	H_8
9	0	0	0	M	H_8	$-H_7$	0	0	0	0	H_9
10	pH	$\frac{p}{2}H_1$	$\frac{p}{2}H_2$	$\frac{p}{2}H_3$	$\left(\frac{p}{2}-1\right)H_4$	$\left(\frac{p}{2}-1\right)H_5$	$\left(\frac{p}{2}-1\right)H_6$	$-H_7$	$-H_8$	$-H_9$	0

Note that the determinant of the elements of the preceding table is zero, because it is a skew-symmetric determinant of odd order. There are thus eleven coefficients λ_i, not all zero, such that the expression $\sum \lambda_i \omega_i$ becomes zero when we apply to it the operation related to any one of the transformations $A_i f$. We see easily that λ_{10} is zero. Calculation gives for the expression $\sum \lambda_i \omega_i$, which is defined only up to a factor,

[1] Fr. *numero*.

[2] Fr. *un tableau double entre*

$$\frac{\delta K}{K} + \frac{2-p}{p}\,\frac{\delta H}{H},$$

where

$$K = (MH_4 + H_2H_9 - H_3H_8)^2 + (MH_5 + H_3H_7 - H_1H_9)^2 + (MH_6 + H_1H_8 - H_2H_7)^2.$$

In the case of celestial mechanics, $p = 1$; the expression $\dfrac{\delta K}{K} + \dfrac{\delta H}{H}$ is the logarithmic differential of HK. This quantity HK is therefore invariant for all transformations $A_i f$. So it is easy to interpret by choosing the coordinate axes suitably. Taking the centre of gravity as origin, which is allowed since it moves with uniform rectilinear motion, we see that $H_1, H_2, H_3, H_7, H_8, H_9$ are zero. The invariant quantity is thus, up to a constant factor, $H(H_4^2 + H_5^2 + H_6^2)$, that is *the product of the square of the angular momentum of the system in its motion about the centre of gravity, with the total energy in this same motion*. In fact, this quantity is clearly independent of the choice of axes and of the choice of units.

V. — *Application to the kinematics of rigid bodies.*

96. Consider the motion of a solid body referred to three fixed rectangular axes. We know that at each instant it is defined by a system of vectors with general resultant (p, q, r) and moment with respect to the origin (ξ, η, ζ). Suppose that these six quantities are given functions of time. The differential equations of motion of a point of the solid body are

$$\frac{dx}{dt} = \xi + qz - ry = X,$$
$$\frac{dy}{dt} = \eta + rx - pz = Y,$$
$$\frac{dz}{dt} = \zeta + py - qz = Z.$$

These equations admit an obvious invariant integral. In fact, if at time t we consider two infinitely close points

$$(x, y, z), \qquad (x + \delta x, y + \delta y, z + \delta z)$$

of the solid body, the distance between these two points does not change with time. We thus obtain a differential form

$$\delta x^2 + \delta y^2 + \delta z^2$$

which is invariant, if we consider only points at the same instant, and which becomes invariant in an absolute way if we complete it by respectively replacing

$$\delta x, \quad \delta y, \quad \delta z$$

by

$$\delta x - X\,\delta t, \quad \delta y - Y\,\delta t, \quad \delta z - Z\,\delta t.$$

Let

$$F = (\delta x - X\,\delta t)^2 + (\delta y - Y\,\delta t)^2 + (\delta z - Z\,\delta t)^2$$

be this invariant form, to which corresponds the invariant bilinear form

$$F(\delta,\delta') = (\delta x - X\,\delta t)(\delta' x - X\,\delta' t) + (\delta y - Y\,\delta t)(\delta' y - Y\,\delta' t) + (\delta z - Z\,\delta t)(\delta' z - Z\,\delta' t).$$

This bilinear form is not alternating, but *symmetric*; nevertheless, the arguments of n° **88** remain valid.

Suppose that $\xi, \eta, \zeta, p, q, r$ are *constants*; the differential equations of motion then admit the infinitesimal transformation

$$Af = \frac{\partial f}{\partial t};$$

consequently, from the form F we can deduce another invariant form

$$\frac{1}{2}\,\frac{\partial F}{\partial(\delta t)} = -X(\delta x - X\ \delta t) - Y(\delta y - Y\ \delta t) - Z(\delta z - Z\ \delta t).$$

The same procedure can be repeated here and this time gives a first integral

$$\frac{1}{2}\,\frac{\partial^2 F}{\partial(\delta t)^2} = X^2 + Y^2 + Z^2.$$

This first integral is clear geometrically. The motion of the solid body is helicoidal, and the preceding integral is equal to the square of the velocity of the point considered, a velocity that indeed remans equal to itself throughout the motion.

VI. — *Differential equations that admit an infinitesimal transformation.*

97. In the previous examples we assumed that an integral invariant was know. Suppose now that we know only an invariant *equation*, for example the equation

$$\omega(\delta) \equiv a_1\ \delta x_1 + a_2\ \delta x_2 + \cdots + a_n\ \delta x_n = 0.$$

To say that this equation is invariant is to say that we can write in such a way that it contains only first integrals $y_1, \ldots, y_{n-1}$ of the given differential equations and their differentials; said differently, we have

$$\omega(\delta) \equiv \rho[b_1(y)\ \delta y_1 + b_2(y)\ \delta y_2 + \cdots + b_{n-1}(y)\ \delta y_{n-1}],$$

where the b_i depend only on the $y_1, \ldots, y_{n-1}$, and ρ is an arbitrary function. Replace the indeterminate differentiation symbol δ by the symbol of the infinitesimal transformation Af. We have immediately

$$\frac{\omega(\delta)}{\omega(A)} = \frac{b_1(y)\,\delta y_1 + b_2(y)\,\delta y_2 + \cdots + b_{n-1}(y)\,\delta y_{n-1}}{b_1(y) A y_1 + b_2(y) A y_2 + \cdots + b_{n-1}(y) A y_{n-1}},$$

and the right hand side is clearly an *invariant linear form*.

Knowledge of an infinitesimal transformation Af which a given system of differential equations admits and knowledge of an invariant Pfaffian equation $\omega(\delta) = 0$ for this system leads to knowledge of a linear integral invariant $\int \frac{\omega(\delta)}{\omega(A)}$.

For example, suppose that we are dealing with an ordinary differential equation

$$\frac{dx}{X} = \frac{dy}{Y};$$

it is invariant for itself; consequently if it admits an infinitesimal transformation

$$Af = \xi\,\frac{\partial f}{\partial x} + \eta\,\frac{\partial f}{\partial y},$$

it admits by that very fact the invariant linear form

$$\frac{X\,\delta y - Y\,\delta x}{X\eta - Y\xi};$$

since here there is only one first integral, this form is necessarily an exact differential. In other words, we know an integrating factor of the equation. This is a classic result.

Most of the differential equations that we know how to integrate can be related to the preceding remark. This is the case for the equations

$$\frac{dy}{dx} = f(x), \qquad \frac{dy}{dx} = f(y), \qquad \frac{dy}{dx} = f\left(\frac{y}{x}\right);$$

for example, the last of these equations does not change if we multiply x and y by the same factor $1+\varepsilon$; it thus admits the infinitesimal transformation

$$Af = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y};$$

consequently the expression

$$\frac{dy - f\left(\frac{y}{x}\right)dx}{y - f\left(\frac{y}{x}\right)x}$$

is an exact differential. This property becomes clear if we put

$$y = ux,$$

because then the expression becomes

$$\frac{du}{u - f(u)} + \frac{dx}{x}.$$

The integration of this exact differential leads to the same calculation as the classic method.

98. Finally, even if we know nothing *a priori* about a given system of differential equations, knowledge of an infinitesimal transformation admitted by this system allows us to obtain an invariant Pfaffian system. In fact, let us find all Pfaffian equations $\omega = 0$ which are consequences of the given differential equations and such that $\omega(A)$ is zero. If we put

$$\omega(\delta) = \lambda_1\,\delta x_1 + \lambda_2\,\delta x_2 + \cdots + \lambda_n\,\delta x_n,$$

the coefficients λ_i are given by the two conditions

$$\begin{aligned} \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n &= 0, \\ \lambda_1 \xi_1 + \lambda_2 \xi_2 + \cdots + \lambda_n \xi_n &= 0. \end{aligned}$$

The set of equations we seek thus form a Pfaffian system obtained by setting to zero all the determinants of three rows and three columns of the matrix

$$\left\|\begin{matrix} \delta x_1 & \delta x_2 & \cdots & \delta x_n \\ X_1 & X_2 & \cdots & X_n \\ \xi_1 & \xi_2 & \cdots & \xi_n \end{matrix}\right\|.$$

This system has a meaning independent of the choice of variables. Now if we take as variables $n-1$ first integral $y_1,\ldots,y_{n-1}$ and an n^{th} variable, x_n for example, the equations will reduce to

$$\frac{\delta y_1}{\eta_1} = \frac{\delta y_2}{\eta_2} = \cdots = \frac{\delta y_{n-1}}{\eta_{n-1}} \qquad (\eta_i = Ay_i).$$

The Pfaffian system considered is thus invariant, and clearly it is completely integrable, because it reduces to a system of ordinary differential equations in $y_1,\ldots,y_{n-1}$.

For example, if the equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

admits the infinitesimal transformation

$$Af = \xi\,\frac{\partial f}{\partial x} + \eta\,\frac{\partial f}{\partial y} + \zeta\,\frac{\partial f}{\partial z},$$

the total differential equation

$$\begin{vmatrix} dx & dy & dz \\ X & Y & Z \\ \xi & \eta & \zeta \end{vmatrix} = 0$$

is completely integrable. By integrating it, we obtain a first integral of the given equations. Finally, by equating this first integral to a constant, we will get to an ordinary differential equation which admits a known infinitesimal transformation, which will integrate by a quadrature.

VII. — *Expressing that a given system of differential equations admits a given infinitesimal transformation.*

99. We have not yet pointed out the analytic conditions that expressing that a given system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \tag{2}$$

admits a given infinitesimal transformation

$$Af = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n} \tag{3}$$

Put

$$Xf = X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + \cdots + X_n \frac{\partial f}{\partial x_n} \tag{5}$$

Basically, this is about expressing the fact that if f is a first integral, that is, if it satisfies the equation

$$Xf = 0,$$

then Af is also a first integral; differently stated, it is about expressing that the equation

$$X(Af) = 0$$

is a consequence of the equation

$$Xf = 0.$$

We can replace the first equation, *which contains partial derivatives of second order of f*, by the equation

$$X(Af) - A(Xf) = 0$$

which, as an easy calculation shows, is linear and homogeneous with respect to $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$. *The condition we seek is thus*, very simply, *the existence of an identity of the form*

$$X(Af) - A(Xf) = \rho \, Xf, \tag{7}$$

where ρ is an appropriately chosen coefficient.

This condition is clearly satisfied if we take the infinitesimal transformation whose symbol is Xf; this transformation moves each point M of the space along the integral curve passing through this point; it thus leaves invariant each integral curve. If we focus on the effect produced on the integral curves, considered as indivisible entities, this special infinitesimal transformation plays the same role as the *identity* transformation. We can easily verify that the applications of infinitesimal transformations made in this Chapter vanish in this special case. The same comment applies to the infinitesimal transformation λXf, where λ is an arbitrarily given factor.

VIII. — *Equations for variations.*

100. The concept of equations for variations is due to H. Poincare; we can relate it to the concept of infinitesimal transformations.

Consider a system of differential equations which we shall write as

$$\frac{dx_1}{dt} = X_1, \ ..., \ \frac{dx_n}{dt} = X_n \tag{8}$$

where the right hand sides are given functions of $x_1, \ldots, x_n, t$. Let

$$x_1 = f_1(t), \quad x_2 = f_2(t), \quad ..., \quad x_n = f_n(t) \tag{9}$$

be a particular solution of this system. Take an infinitely close solution

$$x_1 = f_1(t) + \varepsilon\xi_1, \quad x_2 = f_2(t) + \varepsilon\xi_2, \ \ldots, \ x_n = f_n(t) + \varepsilon\xi_n,$$

where ε is an infinitely small constant and the ξ are unknown functions of t. Neglecting infinitely small quantities of second order, we obtain, for defining these unknown functions, the equations

$$\frac{d\xi_i}{dt} = \frac{\partial X_i}{\partial x_1}\,\xi_1 + \frac{\partial X_i}{\partial x_2}\,\xi_2 + \cdots + \frac{\partial X_i}{\partial x_n}\,\xi_n \qquad (i = 1, 2, \ldots, n); \tag{10}$$

these are the equations for variations *with respect to the particular solution considered*.

It could happen that we know a particular solution of the variational equations, *independently of the particular solution of the given equations from which was used to form the equations for variations*. The quantities $\xi_1, \ldots, \xi_n$ are then in reality determinate functions of $x_1, \ldots, t$ that satisfy the partial differential equations

$$\frac{\partial \xi_i}{\partial t} + X_1 \frac{\partial \xi_i}{\partial x_1} + \cdots + X_n \frac{\partial \xi_i}{\partial x_n} = \xi_1 \frac{\partial X_i}{\partial x_1} + \xi_2 \frac{\partial X_i}{\partial x_2} + \cdots + \xi_n \frac{\partial X_i}{\partial x_n}. \tag{11}$$

In this case, the given equations clearly admit the infinitesimal transformation

$$Af = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n};$$

in fact, this transformation has equations

$$\begin{aligned} x_1' &= x_1 + \varepsilon\xi_1, \\ &\ \vdots \\ x_n' &= x_n + \varepsilon\xi_n \\ t' &= t; \end{aligned}$$

the transformed curve of the integral curve (9) has as equations

$$x_i + \varepsilon\xi_i = f_i(t)$$

or

$$x_i = f_i(t) - \varepsilon\xi_i;$$

this is still an integral curve because $(-xi_1, ..., -\xi_n)$ is a solution of the equations for variations.

More generally, *to any solution* (ξ_i) *of equations (11) there corresponds an infinity of infinitesimal transformations that leave the given system (8) invariant,* namely the transformations

$$Bf = \xi_1 \frac{\partial f}{\partial x_1} + \cdots + \xi_n \frac{\partial f}{\partial x_n} + \lambda \left(\frac{\partial f}{\partial t} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} \right) \tag{12}$$

with an arbitrary function λ.

Conversely, suppose that we know an infinitesimal transformation that leaves the given system invariant: it can always be put into the form (12). The integral curve (9) is changed by this transformation into the curve

$$x_i + \varepsilon\xi_i + \varepsilon\lambda X_i = f_i(t + \varepsilon\lambda)$$

or

$$x_i = f_i(t) - \varepsilon\xi_i + \varepsilon\lambda[f_i'(t) - X_i] = f_i(t) - \varepsilon\xi_i;$$

consequently the equations for variations (11) admit the solution $(\xi_1, \ldots, \xi_n)$.

Moreover, all these properties follow from the fact that equations (11) only translate analytically relation (7) when, in Af, the coefficient of $\dfrac{\partial f}{\partial t}$ is zero.

Chapter X

Completely Integrable Pfaffian systems.

I. — *The theorem of Frobenius.*

101. A system of h Pfaffian equations

$$\left.\begin{array}{l} \omega_1 \equiv a_{11}\,dx_1 + a_{12}\,dx_2 + \cdots + a_{1n}\,dx_n = 0, \\ \vdots \qquad \vdots \\ \omega^h \equiv a_{h1}\,dx_1 + a_{h2}\,dx_2 + \cdots + a_{hn}\,dx_n = 0, \end{array}\right\} \tag{1}$$

is completely integrable when it can be put into the form

$$dy_1 = dy_2 = \cdots = dy_h = 0. \tag{2}$$

If so, each of the forms ω_i is linear in $dy_1, ..., dy_h$ and consequently its derivative $\omega_i{}'$ vanishes by taking (2) into account, that is, from (1).

For a Pfaffian system to be completely integrable it is necessary that the derivatives of its left hand sides all vanish by taking into account the equations of the system.

To prove the converse, note first that the property that we have just stated does not depend on the choice of variables nor does it depend on the choice of the r left hand sides: in other words, if we write the equations in the form

$$\begin{array}{l} \varpi_1 \equiv \alpha_{11}\,\omega_1 + \alpha_{12}\,\omega_2 + \cdots + \alpha_{1h}\,\omega_h = 0, \\ \vdots \qquad \vdots \\ \varpi_h \equiv \alpha_{h1}\,\omega_1 + \alpha_{h2}\,\omega_2 + \cdots + \alpha_{hh}\,\omega_h = 0, \end{array}$$

the derivatives $\varpi_1{}', \varpi_2{}', \ldots, \varpi_h{}'$ also vanish by taking into account of the equations of the system: in fact, we have

$$\varpi_i{}' \equiv \alpha_{i1}\,\omega_1{}' + \alpha_{i2}\,\omega_2{}' + \cdots + \alpha_{ih}\,\omega_h{}' + [d\alpha_{i1}\,\omega_1] + [d\alpha_{i2}\,\omega_2] + \cdots + [d\alpha_{ih}\,\omega_h]$$

and each of the terms on the right hand side vanish by hypothesis under the conditions indicated.

That said, suppose that the converse has been proved for $n-1$ variables, and let us prove it for n variables. Since the $\omega_i{}'$ vanish by taking equations (1) into account, they will vanish for still stronger reason if moreover we put $dx_n = 0$; consequently, *if we regard x_n as a fixed parameter,* system (1) can be reduced to the form

$$\begin{aligned} dy_1 &= 0,\\ &\ldots\ldots\ldots\\ dy_h &= 0, \end{aligned}$$

where $y_1, \ldots, y_h$ are h independent functions of $x_1, \ldots, x_{n-1}$, but may contain also the parameter x_n. Now, if we no longer regard x_n as a constant, the system is clearly reducible to the form

$$\left.\begin{aligned} \varpi_1 &\equiv dy_1 + b_1\,dx_n = 0,\\ &\vdots \qquad \vdots\\ \varpi_h &\equiv dy_h + b_h\,dx_n = 0, \end{aligned}\right\} \tag{3}$$

where $b_1, \ldots, b_h$ are functions of $y_1, \ldots, y_h$ and, for example, of $x_{h+1}, \ldots, x_n$. Moreover, we have

$$\varpi_1{}' = [db_1\,dx_n], \quad \ldots, \quad \varpi_h{}' = [db_h\,dx_n];$$

by taking equations (3) into account, these formulae reduce to

$$\varpi_i{}' = \frac{\partial b_i}{\partial x_{h+1}}\,[dx_{h+1}\,dx_n] + \cdots + \frac{\partial b_i}{\partial x_{n-1}}\,[dx_{n-1}\,dx_n].$$

The hypothesis thus leads to the consequence that the coefficients b_i depend only on $y_1, \ldots, y_h, x_n$. But then equations (3) form a system of ordinary differential equations, which consequently can be reduced to the form

$$dz_1 = 0, \;\; \ldots, \;\; dz_h = 0,$$

where the letters $z_1, \ldots, z_h$ denote h independent first integrals.

102. The preceding theorem, due to Frobenius, allows us (n° 64) to express the necessary and sufficient conditions for complete integrability of the given system by the relations

$$[\omega_1 \ldots \omega_h\,\omega_1{}'] = 0, \;\; \ldots, \;\; [\omega_1 \ldots \omega_h\,\omega_h{}'] = 0.$$

Take the example of a Pfaffian equation in three variables

$$\omega \equiv P\,dx + Q\,dy + R\,dz = 0;$$

the condition for complete integrability is

$$\begin{aligned}
[\omega\,\omega'] &\equiv \left[(P\,dx + Q\,dy + R\,dz)\left(\overline{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\,dydz} + \overline{\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\,dzdx} + \overline{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dxdy}\right)\right] \\
&\equiv \left\{P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right\}[dxdydz] = 0.
\end{aligned}$$

II. — *Forming the characteristic system of a Pfaffian system.*

103. We can give the above reasoning a different form by looking in a general way for the characteristic Pfaffian system of any given system (1).

For system (1) to be invariant for the differential equations

$$\frac{dx_1}{X_1} = \cdots = \frac{dx_n}{X_n}, \tag{4}$$

it is necessary and sufficient that the equations of (1) can be expressed by means of first integrals of (4) and of their differentials; thus first it is necessary that $\omega_1, \ldots, \omega_h$ vanish by taking (4) into account and it is then necessary that the forms

$$[\omega_1\,\omega_2 \ldots \omega_h\,\omega_1{}'], \qquad \ldots, \qquad [\omega_1\,\omega_2 \ldots \omega_h\,\omega_h{}']$$

can be expressed by means of the differentials of first integrals of (4); in other words *it is necessary that the associate system of the forms*

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_h,\ [\omega_1\,\omega_2 \ldots \omega_h\,\omega_1{}'],\ \ldots,\ [\omega_1\,\omega_2 \ldots \omega_h\,\omega_h{}']$$

be a consequence of equations (4).

Conversely if this condition is met and if $y_1, \ldots, y_{n-1}$ are first integrals of (4), we can reduce the equations to the form

$$\begin{aligned}
\varpi_1 &\equiv dy_1 + b_{1,h+1}dy_{h+1} + \cdots + b_{1,n-1}dy_{n-1} = 0, \\
&\vdots \qquad \vdots \\
\varpi_h &\equiv dy_h + b_{h,h+1}dy_{h+1} + \cdots + b_{h,n-1}dy_{n-1} = 0;
\end{aligned}$$

since the form $[\omega_1\omega_2...\omega_h\omega_i{}']$ does not involve dx_n, the derivatives $\dfrac{\partial b_{i,h+1}}{\partial x_n},\dots,\dfrac{\partial b_{i,n-1}}{\partial x_n}$ are all zero; consequently equations (1) can be written so as to involve only the first integrals of system (4) and their differentials. Thus system (1) is indeed invariant for system (4).

It follows from this that *the characteristic system of* (1) *is the associated system of the forms*

$$\omega_1,\ \omega_2,\ \dots,\ \omega_h;\ [\omega_1\omega_2...\omega_h\omega_1{}'],\ \dots,\ [\omega_1\omega_2...\omega_h\omega_h{}'].$$

In particular for the system to be completely integrable it is necessary and sufficient that this system be identical to (1), that is, that the forms

$$[\omega_1\omega_2...\omega_h\omega_1{}'],\ \dots,\ [\omega_1\omega_2...\omega_h\omega_h{}']$$

be identically zero.

104. We can also obtain the equations of the characteristic system of (1) in the form

$$\begin{cases} \omega_1=0,\ \omega_2=0,\ \dots,\ \omega_h=0,\\ \left\| \begin{matrix} \dfrac{\partial\omega_i{}'}{\partial(dx_1)} & \dfrac{\partial\omega_i{}'}{\partial(dx_2)} & \cdots & \dfrac{\partial\omega_i{}'}{\partial(dx_n)}\\ a_{11} & a_{12} & & a_{1n}\\ \vdots & & & \vdots\\ a_{h1} & a_{h2} & \cdots & a_{hn} \end{matrix} \right\| \end{cases}$$

In particular the characteristic system of one Pfaffian equation

$$\omega \equiv a_1\,dx_1+a_2\,dx_2+\cdots+a_n\,dx_n=0$$

is given by the equations

$$\begin{cases} a_1\,dx_1+a_2\,dx_2+\cdots+a_n\,dx_n=0,\\ \dfrac{a_{12}\,dx_2+a_{13}\,dx_3+\cdots+a_{1n}\,dx_n}{a_1}=\dfrac{a_{21}\,dx_1+a_{23}\,dx_3+\cdots+a_{2n}\,dx_n}{a_2}=\cdots\\ \qquad =\dfrac{a_{n1}\,dx_1+\cdots+a_{n,n-1}\,dx_{n-1}}{a_n}, \end{cases}$$

where

$$a_{ij}=\frac{\partial a_j}{\partial x_i}-\frac{\partial a_i}{\partial x_j}.$$

We will return later to this system (Chapter XIV).

III. — *Integration of a completely integrable Pfaffian system.*

105. Back to a completely integrable Pfaffian system, which we will write as

$$\left.\begin{aligned} dz_1 &= a_{11}\,dx_1 + \cdots + a_{1q}\,dx_q,\\ &\vdots \qquad\qquad \vdots\\ dz_h &= a_{h1}\,dx_1 + \cdots + a_{hq}\,dx_q. \end{aligned}\right\} \tag{5}$$

The integration of this system comes down to that of a system of ordinary differential equations in h unknown functions $z_1, \ldots, z_h$ *of a single independent variable* x_1.

In fact, we know that the system admits one and only one solution corresponding to given initial values (x_i^0, z_j^0). To get the values of the unknown functions $z_1, \ldots, z_h$ corresponding to a system of given numerical values $x_1^1, \ldots, x_q^1$ of the independent variables, go from the point (x_i^0) to the point (x_i^1) on the integral manifold and monitor the variation of the z_j; whatever the sequence of the intermediate values of the independent variables, the result will always be the same. Put for example

$$x_2 - x_2^0 = m_2(x_1 - x_1^0), \ \ \ldots, \ \ x_q - x_q^0 = m_q(x_1 - x_1^0),$$

where we denote by $m_2, \ldots, m_q$ the $q-1$ quantities

$$m_i = \frac{x_i^1 - x_i^0}{x_1^1 - x_1^0} \qquad (i = 2, \ldots, q).$$

We will have

$$\left.\begin{aligned} dz_1 &= (a_{11} + a_{12}m_2 + \cdots + a_{1q}m_q)dx_1,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ dz_h &= (a_{h1} + a_{h2}m_2 + \cdots + a_{hq}m_q)dx_1. \end{aligned}\right\} \tag{6}$$

We need only integrate this system of ordinary differential equations and determine that solution which, for $x_1 = x_1^0$, corresponds to values $z_1^0, \ldots, z_h^0$ of the unknown functions. This solution determined, we will get $z_1^1, \ldots, z_h^1$ by replacing the parametric quantities $m_2, \ldots, m_q$ by their values indicated above.

In fact, we will suppress indices greater than 1: we will simply replace m_i by $\dfrac{x_i - x_i^0}{x_1 - x_1^0}$ in the solution obtained, and thus obtain the expressions for the unknown functions $z_1, \ldots, z_h$.

Note that *knowledge of a first integral of the system of ordinary differential equations* (6) *in* $q-1$ *parameters* m_i *does not necessarily imply knowledge of a first integral of the Pfaffian system* (5).

IV. — *Complete systems.*

106. Return now to the completely integrable system (1), a system of h independent first integrals of which we will denote by $y_1, \ldots, y_h$. Choose arbitrarily $n-h$ linear differential forms

$$\omega_{h+1}, \ \ldots, \ \omega_n$$

independent of each other and independent of the forms $\omega_1, \ldots, \omega_h$. Any linear form in $dx_1, \ldots, dx_n$ can be expressed, in one and only one way, as a linear function of $\omega_1, \ldots, \omega_n$. Take then an *indeterminate* function f and consider its total differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n;$$

we can express it linearly by means of $\omega_1, \ldots, \omega_n$, where the coefficients are obviously linear and homogeneous in $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}$. Let

$$df = X_1 f \,.\, \omega_1 + X_2 f \,.\, \omega_2 + \cdots + X_n f \,.\, \omega_n. \tag{7}$$

The n expressions $X_i f$ are linearly independent in $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}$.

That said, any first integral of the completely integrable system (1) is characterised by the property that its differential, considered as a linear form in $dx_1, \ldots, dx_n$, vanishes subject only to the condition that equations (1) are satisfied; that is, by the property of annulling

$$X_{h+1} f, \ \ldots, \ X_n f.$$

The system of $n-h$ equations in independent linear partial derivatives

$$X_{h+1} f = 0, \ \ldots, \ X_n f = 0 \tag{8}$$

thus admits h independent solutions $y_1, \ldots, y_h$.

Conversely, suppose that system (8) admits h independent solutions (it obviously cannot admit more)

$$y_1, \ \ldots, \ y_h.$$

Identity (7) gives us

$$dy_i = X_1 y_i \,.\, \omega_1 + X_2 y_i \,.\, \omega_2 + \cdots + X_h y_i \,.\, \omega_h \qquad (i = 1, 2, \ldots, h). \tag{9}$$

Since the y_i are independent functions, the right hand sides of equations (**??**) are linearly independent combinations of $\omega_1, \omega_2, \ldots, \omega_h$. Consequently *system* (1) is equivalent to (2), and hence *completely integrable*.

Let us agree to say that equations (8) form a *complete system* if they admit the maximum number h of independent solutions. We see that *to any completely integrable Pfaffian system corresponds a complete system and conversely*. The correspondence is such that if the equations of the Pfaffian system are

$$\omega_1 = \omega_2 = \cdots = \omega_h = 0,$$

those of the complete system are

$$X_{h+1} f = X_{h+2} f = \cdots = X_n f = 0.$$

107. It is easy to find the conditions for a given system of linear partial differential equations of first order to be complete.

Start from identity (7) and find its exterior derivative. We easily get[1]

$$\sum_{k=1}^{k=n} X_k f \,.\, \omega_k{}' + \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} X_i (X_j f) [\omega_i \, \omega_j] = 0. \tag{9}$$

The n covariants $\omega_k{}'$ can be expressed as exterior quadratic forms of $\omega_1, \ldots, \omega_n$; let

$$\omega_k{}' = \sum_{(i\,j)}^{1,\ldots,n} c_{ijk} [\omega_j \, \omega_k]. \tag{10}$$

Equating the set of terms in $[\omega_i \, \omega_j]$ to zero in identity (9), we find

[1] TRANSLATOR'S NOTE. — The equation that follows is labelled (9) in the original; but there has already been an equation (9) in n$^{\text{p}}$ **106**. Nevertheless, I have kept the original numbering in this section to avoid confusion.

$$X_i(X_j f) - X_j(X_i f) + \sum_{k=1}^{k=n} c_{ijk} X_k f = 0. \tag{11}$$

Note the duality of formulae (10) and (11).

Suppose then that system (1) is completely integrable. This means, according to the Frobenius theorem, that $\omega_1', ..., \omega_h'$ vanish with $\omega_1, ..., \omega_h$; in other words that we have

$$c_{h+i,h+j,k} = 0 \qquad (i, j = 1, ..., n-h; k = 1, ..., h).$$

Consequently, according to (11), the combinations

$$X_{h+i}(X_{h+j} f) - X_{h+j}(X_{h+i} f)$$

depend linearly only on $X_{h+1} f, ..., X_n f$. The converse is obvious.

Agree to denote the combination $X(Yf) - Y(Xf)$ by (XY). We see that *the necessary and sufficient condition for a system to be complete is that the parentheses of all the left hand sides, taken two at a time, be linear combinations of the left hand sides.*

Chapter XI

The theory of the last multiplier

I. — *Definition and properties.*

108. Consider a system of differential equations

$$\frac{dx_1}{dt} = X_1, \qquad \frac{dx_2}{dt} = X_2, \qquad \ldots, \qquad \frac{dx_n}{dt} = X_n \tag{1}$$

that admit an integral invariant of maximum degree n

$$\Omega = M[(\delta x_1 - X_1 \delta t)(\delta x_2 - X_2 \delta t)\ldots(\delta x_n - X_n \delta t)].$$

As we have seen (n°**80**), the condition for this to be the case is that the exterior derivative Ω' be zero, which gives by an easy calculation

$$\frac{\partial M}{\partial t} + \frac{\partial (MX_1)}{\partial x_1} + \frac{\partial (MX_2)}{\partial x_2} + \cdots + \frac{\partial (MX_n)}{\partial x_n} = 0. \tag{2}$$

The coefficient M is known as *Jacobi's multiplier*.

Condition (2) states, as we know, that the form Ω can be expressed by means of n independent first integrals

$$y_1,\ y_2,\ \ldots,\ y_n$$

of system (1) and their differentials; in other words that we have an identity

$$M\ [(\delta x_1 - X_1 \delta t)\ldots(\delta x_n - X_n \delta t)] = H(y_1, \ldots, y_n)\ [\delta y_1\ \delta y_2 \ldots \delta y_n]. \tag{3}$$

We can now rediscover the classical theorems relating to the Jacobi multiplier.

THEOREM I. — *The quotient of two multipliers M and M' is a first integral.* In fact, the two identities (3) for the two multipliers M and M' give

$$\frac{M}{M'} = \frac{H(y_1,...,y_n)}{H'(y_1,...,y_n)}.$$

THEOREM II. — *If we know p independent first integrals of equations (1), we can determine a multiplier of the system of $n-p$ differential equations to which the integration of the given system reduces.*

Assume that we know the p independent first integrals $y_1, y_2, ..., y_p$, and suppose, as is always allowed, that these are independent functions of p variables $x_1, ..., x_p$, that is,

$$\frac{D(y_1,...,y_p)}{D(x_1,...,x_p)} \neq 0.$$

Equations (1) can then be written

$$\frac{dy_1}{dt} = 0, \qquad ..., \qquad \frac{dy_p}{dt} = 0 \tag{4}$$

$$\frac{dx_{p+1}}{dt} = X_{p+1}, \qquad ..., \qquad \frac{dx_n}{dt} = X_n, \tag{5}$$

and, equating $y_1, ..., y_p$ to arbitrary constants $C_1, ..., C_p$, the integration of system (1) reduces to that of system (5), in the right hand sides of which we have assumed that the $x_1, ..., x_p$ have been replaced by their values as functions of $x_{p+1}, ..., x_n, t, C_1, ..., C_p$.

That said, the form Ω, which is invariant for equations (4) and (5), can obviously be written

$$\Omega = N[\delta y_1 ..., \ \delta y_p \, (\delta x_{p+1} - X_{p+1}\delta t) ... (\delta x_n - X_n \delta t)];$$

to get the value of the coefficient N, it is sufficient to identify this expression with the original expression; for example, by equating the terms in

$$[\delta x_1 \, \delta x_2, \, ..., \, \delta x_n],$$

we get

$$M = N \frac{D(y_1,...,y_p)}{D(x_1,...,x_p)}.$$

With the quantity N determined in this way, we have the identity

$$N[\delta y_1 ..., \ \delta y_p \, (\delta x_{p+1} - X_{p+1}\delta t) ... (\delta x_n - X_n \delta t)] = H\,[\delta y_1 ... \delta y_p \, \delta y_{p+1} ..., \delta y_n],$$

that is

$$[\delta y_1 \ldots,\ \delta y_p \{N(\delta x_{p+1} - X_{p+1}\delta t)\ldots(\delta x_n - X_n\delta t) - H\,\delta y_{p+1}\ldots\delta y_n\}] = 0.$$

This identity states that (n°**64**), *if we take into account the linear relations*

$$\delta y_1 = 0, \quad \delta y_2 = 0, \quad \ldots, \quad \delta y_p = 0,$$

we have

$$N[(\delta x_{p+1} - X_{p+1}\delta t)\ldots(\delta x_n - X_n\delta t)] = H(y_1 \ldots y_n)\,[\delta y_{p+1} \ldots, \delta y_n]. \tag{6}$$

The left hand side of this equality is thus an invariant form for the system of differential equations (5); in other words *system* (5) *admits the multiplier*

$$N = \frac{M}{\dfrac{D(y_1, \ldots, y_p)}{D(x_1, \ldots, x_p)}}.$$

THEOREM III. — *If we know* $n-1$ *independent first integrals of equations* (1), *the integration of the equations is completed with a quadrature.*

It is sufficient to apply Theorem II to the case $p = n-1$: we then see that the linear differential form

$$\frac{M}{\dfrac{D(y_1, \ldots, y_{n-1})}{D(x_1, \ldots, x_{n-1})}}(\delta x_n - X_n\,\delta t)$$

is an exact differential, when we assume that the variables are related by the relations

$$y_1 = C_1, \quad y_2 = C_2, \quad \ldots, \quad y_{n-1} = C_{n-1}.$$

The general solution of equations (1) is thus obtained by equating to a constant C_n the integral of the total differential

$$\int \frac{M}{\dfrac{D(y_1, \ldots, y_{n-1})}{D(x_1, \ldots, x_{n-1})}}(dx_n - X_n\,dt).$$

II. — *Generalisations.*

109. The theorem of the last multiplier can be generalised to the much more general case where we know an invariant form Ω *of any degree* $r < n$. Suppose that we know $n-1$ independent first integrals $y_1, y_2, \dots, y_{n-1}$. Choose in every possible way $n-r$ of these integrals

$$y_{\alpha_1},\ y_{\alpha_2},\ \dots, y_{\alpha_{n-r}}$$

and consider the forms, clearly invariant,

$$[y_{\alpha_1}\, y_{\alpha_2} \cdots y_{\alpha_{n-r}}\, \Omega];$$

all these forms are of degree n. *If they are not all zero*, we are brought back to the case studies previously; *there is a multiplier*, we can even have several, and in some cases Theorem I can give, by division of two of these multipliers, the last first integral.

The exceptional case is the one where all the forms written previously are zero. Now, imagine Ω expressed by means of $\delta y_1, \dots, \delta y_{n-1}$ and of the differential of an n^{th} (unknown) first integral δy_n. The hypothesis made amounts to saying that Ω *does not contain* δy_n, because if, for example, Ω were to contain a non zero term such as

$$A\,[\delta y_1 \dots \delta y_{r-1}\, \delta y_n],$$

the exterior product of Ω with $\delta y_{r+1}\, \delta y_{r+2} \dots \delta y_{n-1}$ would not be zero.

That being so, Ω would thus be an exterior form in $\delta y_1 \dots \delta y_{n-1}$, an exterior form whose coefficients we could calculate. Each of these coefficients would be a first integral. If at least one of these coefficients is independent of $y_1 \dots y_{n-1}$, we complete the integration by equating it to an arbitrary constant. The only dubious case is that where all the coefficients are functions of $y_1 \dots y_{n-1}$; now it is obvious that in this case knowledge of the invariant form Ω can not be of any help for completing the integration. We note simply that in this case the given equations *are not* the characteristic system of Ω.

We can now state the following general theorem:

Knowledge an invariant differential form Ω whose characteristic system is the given system of differential equations (1) allows us, in the least favourable case, to complete the integration of this system by a quadrature when we already know $n-1$ independent first integrals.

110. Another generalisation of Jacobi's theory of the last multiplier concerns completely integrable Pfaff systems. Let

$$\omega_1 = 0, \qquad \omega_2 = 0, \qquad \dots, \qquad \omega_r = 0$$

be a completely integrable system for which we know an invariant form of maximum degree r

$$\Omega = M[\omega_1 \ \omega_2 ... \omega_r].$$

Knowing $r-1$ first integrals $y_1, ..., y_{r-1}$ of the system allows us to complete the integration by a quadrature. In fact, by equating $y_1, ..., y_{r-1}$ to arbitrary constants, the given system reduces to a single equation, for example $\omega_r = 0$, and we get a formula such as

$$\Omega = N[\delta y_1 ... \delta y_{r-1} \omega_r],$$

where the coefficient N is deduced from M by an easy identification. It follows that $N\omega_r$ is an invariant form for the single equation $\omega_r = 0$ which remains to be integrated, in other words that $N\omega_r$ is an exact differential. The integration is thus completed by a quadrature.

Finally the completely general theorem which summarises all the cases considered is the following:

Knowledge of a differential form Ω allows us, in the least favourable case, to complete the integration of the characteristic system of this form by means of a quadrature when we already know $r-1$ independent first integrals, where r is the class of the form.

III. — *Case where the independent variable is not distinguished.*

111. If the given system of differential equations is put into the form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n},$$

any invariant integral of degree $n-1$ is of the form

$$\Omega = MX_1[dx_2\, dx_3 \ldots dx_{n-1}] - MX_2[dx_1\, dx_3 \ldots dx_{n-1}] + \cdots - (-1)^n MX_n[dx_1\, dx_2 \ldots dx_{n-1}],$$

and the condition $\Omega' = 0$ becomes

$$\frac{\partial (MX_1)}{\partial x_1} + \frac{\partial (MX_2)}{\partial x_2} + \cdots \frac{\partial (MX_n)}{\partial x_n} = 0.$$

Apart from this difference in writing, the theory is identical to the one above.

IV. — *Case where the given equations admit an infinitesimal transformation.*

112. Consider the general case of a completely integrable system

$$\omega_1 = 0, \quad \omega_2 = 0, \quad \dots, \quad \omega_r = 0 \tag{7}$$

which admits an invariant form of degree r, which we can always assume reduced to the form

$$\Omega = [\omega_1\, \omega_2 \dots \omega_r].$$

Suppose that this system admits a known infinitesimal transformation Af, and form the quantities

$$\omega_1(A), \quad \omega_2(A), \quad \dots, \quad \omega_r(A),$$

which we will assume are not all zero. We can always suppose the equations of the system written in such a way as to have

$$\omega_1(A) = 1, \quad \omega_2(A) = 0, \quad \dots, \quad \omega_r(A) = 0, \tag{8}$$

where Ω keeps the same form.

As we saw above (n° **88**), knowing the infinitesimal transformation Af allows us to deduce another invariant form invariant form $\Omega(A,\delta)$ from the form $\Omega(\delta)$ which, with the assumptions made here, reduces to

$$\Omega(A,\delta) = [\omega_2 \dots \omega_r].$$

We will denote this new invariant form by the letter Π. We have

$$\Omega = [\omega_1\, \Pi].$$

The *associated* (non-characteristic) system of Π is

$$\omega_2 = 0, \quad \omega_3 = 0, \quad \dots, \quad \omega_r = 0; \tag{9}$$

it is completely integrable. This follows from an earlier theorem (n° **98**), but this also follows from the fact that, since Ω is expressible by means of r first integrals $y_1, \dots, y_r$ of the given system and their differentials, the associated system of $\Omega(A,\delta)$ will also only contain $y_1, \dots, y_r$ and their differentials; and their differentials; it will therefore be a system of ordinary differential equations, and thus completely integrable.

Now form the exterior derivative Π' of the form Π; this is a new invariant form of degree r, we thus have

$$\Pi' = m\Omega = m[\omega_1\, \Pi].$$

The coefficient m is a first integral. But there is a discussion to be had.

1° $m = 0$. — Π' is zero, and the associated system (9) of Π is its characteristic system. We thus know a multiplier of system (9); consequently when we know $r-2$ independent first integrals of this system, the integration will end in a quadrature. A second quadrature will then complete the integration of the given system (7); this quadrature is obviously $\int \omega_1$.

It is clear that Π and Ω are reducible to

$$\Pi = [\delta y_2\, \delta y_3 \ldots \delta y_r], \qquad \Omega = [\delta y_1\, \delta y_2 \ldots \delta y_r];$$

the transformation Af, applied to first integrals of the given system, reduce to

$$Af = \frac{\partial f}{\partial y_1}.$$

A second quadrature will then complete the integration of the given system (7); this quadrature is obviously

There is an infinity of ways of choosing the first integrals so that the data remain the same, that is, so that Ω and Af do not change; we can perform any transformation on the $y_2, y_3, \ldots, y_r$ with functional determinant 1, and add an arbitrary function of $y_2, \ldots, y_r$ to y_1. This explains the nature of the simplifications that arise in the integration.

2° *m is a non-zero constant*. — In this case, suppose that we have integrated system (9), and let $y_2, y_3, \ldots, y_r$ be a system of $r-1$ independent integrals. We will have

$$\Pi = H[\delta y_2\, \delta y_3 \ldots \delta y_r],$$

the coefficient H being independent of $\delta y_2, \ldots, \delta y_r$ without which Π' would be zero, but being a first integral of the given system. We thus have an r^{th} integral of the given system by simple differentiations.

By writing y_1 in place of H, we have

$$\Omega = \frac{1}{m}[\delta y_1\, \delta y_2 \ldots \delta y_r], \qquad \Pi = y_i\, [\delta y_2 \ldots \delta y_r], \qquad Af = m y_1 \frac{\partial f}{\partial y_1}.$$

The most general transformation in $y_1, \ldots, y_r$ which preserves the data is obtained by performing an *arbitrary* transformation on $y_2, \ldots, y_r$ and putting

$$\bar{y}_1 = \frac{y_1}{\dfrac{D(\bar{y}_2,\dots,\bar{y}_r)}{D(y_2,\dots,y_r)}}\,.$$

This explains why the integration of the system (9) cannot be simplified and also why, once this integration has been carried out, that of the given system (7) is deduced from it.

3° *The coefficient m is not constant, but $A(m)$ is zero..* — The function m is a first integral of system (9). The integration of this system reduces to that of a system of differential equations with $r-2$ unknown functions; the integration of the given system can be deduced as in the previous case.

The form Π is reducible to

$$\Pi = y_i\,[\delta m\,\delta y_3\dots\delta y_r]$$

and we have

$$\Omega = \frac{1}{m}\,[\delta y_1\,\delta m\,\delta y_3\dots\delta y_r], \qquad Af = my_1\,\frac{\partial f}{\partial y_1}\,.$$

The transformations which preserve the data are

$$\left.\begin{array}{l} \overline{m} = m, \qquad \bar{y}_3 = f_3(m,y_3,\dots,y_r), \qquad \dots, \qquad \bar{y}_r = f_r(m,y_3,\dots,y_r), \\ \bar{y}_1 = \dfrac{y_1}{\dfrac{D(f_3,\dots,f_r)}{D(y_3,\dots,y_r)}}\,. \end{array}\right\}$$

They explain the simplifications that arise in the integration.

4° *The coefficient m is not constant, and $Am = m_1 \neq 0$..* — Take right away the general case

$$Am = m_1, \qquad Am_1 = m_2, \qquad \dots, \qquad Am_{i-1} = m_i,$$

assuming that $m, m_1, \dots, m_{i-1}$ are i independent first integrals of the given system, and that m_i is a function of $m,\dots,m_{i-1}$.

The given system thus admits i known independent first integrals and *its integration reduces to that of a system of differential equations with $r-i$ unknown functions for which we know a multiplier*.

Look for the reduced forms of Ω and Af. We can always put

$$\Omega = H[\delta m\,\delta m_1\dots\delta m_{i-1}\,\delta y_{i+1}\dots\delta y_r],$$

where $y_{i+1},...,y_r$ are $r-i$ firs integrals of system (9) and H is a function of $m_1,...,m_{i-1},y_{i+1},...,y_r$. Obviously we have

$$Af = m_1\frac{\partial f}{\partial m} + m_2\frac{\partial f}{\partial m_1} + \cdots + m_i\frac{\partial f}{\partial m_{i-1}}.$$

Let us say that the exterior derivative of $\Pi = \Omega(A,\delta)$ is equal to $m\Omega$, or which comes down to the same thing,

$$A(\Omega) = m\Omega.$$

We have

$$A(\Omega) = \left(A(H) + \frac{\partial m_i}{\partial m_{i-1}}H\right)[\delta m\,\delta m_1\ldots\delta m_{i-1}\,\delta y_{i+1}\ldots\delta y_r];$$

we must thus have

$$A(H) + \frac{\partial m_i}{\partial m_{i-1}}H = mH.$$

Let $h(\delta m,\delta m_1,...,\delta m_{i-1})$ be a *particular* solution of this partial differential equation; the latter can be written as

$$A\left(\frac{H}{h}\right) = 0;$$

in other words, $\dfrac{H}{h}$ is an integral of equations (9). We can thus choose $y_{i+1},...,y_r$ such as to reduce this function to unity. We will then have

$$\begin{aligned}\Omega &= h(m,...,m_{i-1})[\delta m\,\delta m_1\ldots\delta m_{i-1}\,\delta y_{i+1}\ldots\delta y_r],\\ Af &= m_1\frac{\partial f}{\partial m} + m_2\frac{\partial f}{\partial m_1} + \cdots + m_i\frac{\partial f}{\partial m_{i-1}}.\end{aligned}$$

The transformations which preserve the data are clearly

$$\begin{aligned}&\overline{m} = m, \quad \overline{m}_1 = m_1, \quad ..., \quad \overline{m}_{i-1} = m_{i-1},\\ &\overline{y}_{i+1} = f_{i+1}(\mu_1,...,\mu_{i-1},y_{i+1},...,y_r) \quad ..., \quad \overline{y}_r = f_r(\mu_1,...,\mu_{i-1},y_{i+1},...,y_r)\end{aligned}$$

with

$$\frac{D(\overline{y}_{i+1},...,\overline{y}_r)}{D(y_{i+1},...,y_r)} = 1.$$

(We have denoted $i-1$ independent functions of $m, m_1, ...m_{i-1}$ satisfying $A\mu = o$ by $\mu_1, ..., \mu_{i-1}$).

The nature of the preceding transformations explains the simplifications that arise in the integration.

It can also happen that $i = r$; in this case no integration is required, since in which case no integration is necessary, since we get r independent first integrals by differentiation.

V. — *Applications.*

113. The theory of the last multiplier applies to all the previously indicated examples where an invariant form of degree equal to the number of unknown functions is involved. Recall these examples:

1º The equations of motion of the molecules of a continuous medium, when we know the density ρ and the components u, v, w of the velocity as a function of x, y, z, t, are:

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w.$$

Since the integral invariant is

$$\Omega = \rho\left[(\delta x - u\,\delta t)(\delta y - v\,\delta t)(\delta z - w\,\delta t)\right],$$

the multiplier is ρ. If we know two independent first integrals, the integration ends in a quadrature.

If the motion is *permanent*, the invariant form

$$\Pi = \Omega(A, \delta) = -\rho u[\delta y\,\delta z] - \rho v[\delta z\,\delta x] - \rho w[\delta x\,\delta y]$$

has zero derivative. The equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w},$$

which determine the geometric trajectories, admit a multiplier ρ; consequently, if we know a first integral, determination of the trajectories requires only one quadrature, and a final quadrature gives t.

2º The equations which determine the vortex lines of a given vector field (X, Y, Z) are the characteristic equations of the form

$$[\delta X\,\delta x]+[\delta Y\,\delta y]+[\delta Z\,\delta z]$$
$$=\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)[\delta y\,\delta z]+\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)[\delta y\,\delta z]+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)[\delta y\,\delta z];$$

these equations

$$\frac{dx}{\dfrac{\partial Z}{\partial y}-\dfrac{\partial Y}{\partial z}}=\frac{dy}{\dfrac{\partial X}{\partial z}-\dfrac{\partial Z}{\partial x}}=\frac{dz}{\dfrac{\partial Y}{\partial x}-\dfrac{\partial X}{\partial y}}$$

thus admit a known multiplier, which is unity.

3° The dynamical equations, in their canonical form

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i},$$

admit the multiplier 1: this follows from a direct calculation; this also follows from the fact that the existence of the invariant form

$$\Omega=\sum_{i=1}^{i=n}\left[\left(\delta q_i-\frac{\partial H}{\partial p_i}\,\delta t\right)\left(\delta p_i+\frac{\partial H}{\partial q_i}\,\delta t\right)\right]$$

leads to that of the invariant form Ω^n:

$$\frac{1}{n!}\,\Omega^n=\prod_{i=1}^{i=n}\left[\left(\delta q_i-\frac{\partial H}{\partial p_i}\,\delta t\right)\left(\delta p_i+\frac{\partial H}{\partial q_i}\,\delta t\right)\right].$$

114. But the theory of the last multiplier does not apply only to material systems for which Hamilton's canonical equations are valid, but to any system *with ideal holonomic constraints with given forces that depend only on the position of the system*.

For such a system we have Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i}=Q(q_1,\ldots,q_n,t).$$

If the Q_i are zero, the introduction of Hamilton's canonical variables will lead to the equations

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}.$$

It follows that the complete equations of motion can be put into the form

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}+Q_i.$$

They admit the multiplier 1, in other words the invariant form

$$\Omega = \left[\left(\delta q_1 - \frac{\partial H}{\partial p_1}\,\delta t \right) \dots \left(\delta q_n - \frac{\partial H}{\partial p_n}\,\delta t \right) \left(\delta p_1 + \overline{\frac{\partial H}{\partial q_1} - Q_1\,\delta t} \right) \dots \left(\delta p_n + \overline{\frac{\partial H}{\partial q_n} - Q_n\,\delta t} \right) \right]$$

which, with Lagrange's variables, is written

$$\Omega = \prod_{i=1}^{i=n} \left[(\delta q_i - q_i'\,\delta t) \left(\delta\,\frac{\partial T}{\partial q_i'} - \overline{\frac{\partial T}{\partial q_i} + Q_i\ \delta t} \right) \right].$$

If the constrains are independent of time, as well as the given forces, the equations of motion admit the infinitesimal transformation

$$Af = \frac{\partial f}{\partial t}$$

and, consequently, the invariant form $\Pi = \Omega(A,\delta)$, whose derivative is zero. According to the general theory, the integration of the equations of motion reduces to that of the equations of the (geometrical) trajectories

$$\frac{dq_i}{q_i'} = \frac{d\,\dfrac{\partial T}{\partial q_i'}}{\dfrac{\partial T}{\partial q_i} + Q_i} \quad \text{or} \quad \frac{dq_i}{\dfrac{\partial H}{\partial p_i}} = \frac{dp_i}{-\dfrac{\partial H}{\partial q_i} + Q_i},$$

to which the theory of the last multiplier applies, and to a quadrature that gives the time: in fact, we have clearly, for example,

$$\Omega = \left[\left(\delta t - \frac{\delta q_1}{q_1'} \right) \Pi \right]$$

since $\Omega(A,\delta)$ is equal to Π.

115. As an example of *time-dependent forces*, but with a known infinitesimal transformation, consider the simple case of a moving point on a fixed straight line and attracted by a fixed point on the straight line according to a force proportional to the distance, where the factor of proportionality is a known function of time. The motion is given by the differential equation of the second order

$$\frac{d^2x}{dt^2} = -k(t)x,$$

or by the system

$$\frac{dx}{dt} = x', \qquad \frac{dx'}{dt} + k(t)x = 0. \tag{10}$$

The second-order equation does not change if we change x to λx, where λ is an arbitrary constant factor; consequently the system equivalent to it admits the infinitesimal transformation whose effect is to change respectively

$$x, \quad x', \quad t$$

into

$$(1+\varepsilon)x, \quad (1+\varepsilon)x', \quad t;$$

the symbol of this transformation is

$$Af = x\frac{\partial f}{\partial x} + x'\frac{\partial f}{\partial x'}.$$

System (10) admits the invariant form

$$\Omega = [(\delta x - x'\,\delta t)(\delta x' + kx\,\delta t)]$$

corresponding to the multiplier 1. Here the derived form $\Omega(A,\delta)$ is

$$\varpi = x(\delta x' + kx\,\delta t) - x'(\delta x - x'\,\delta t) = x\,\delta x' - x'\,\delta x + (x'^2 + kx^2)\delta t:$$

it is an invariant form. Its exterior derivative is

$$\varpi' = 2[\delta x\,\delta x'] + 2x'[\delta x'\,\delta t] + 2kx[\delta x\,\delta t] = 2\Omega.$$

Since the coefficient of Ω on the right hand side is constant, we know (n° **112**) that it is sufficient to integrate the *completely integrable* equation $\varpi = 0$ to deduce from it by differentiations the general solution of the given system: the form ϖ is in fact reducible to $y_1\,\delta y_2$. This form ϖ is written, by changing δ to d,

$$\varpi = x^2\left[d\,\frac{x'}{x} + \left(\frac{x'^2}{x^2} + k\right)dt\right].$$

We are thus led, by putting

$$\frac{x'}{x} = u,$$

to the Riccati equation

$$\frac{du}{dt} + u^2 + k = 0.$$

Suppose this equation integrated; we have a first integral of the form

$$y_2 = \frac{\alpha(t)u + \beta(t)}{\gamma(t)u + \delta(t)} = \frac{\alpha(t)x' + \beta(t)x}{\gamma(t)x' + \delta(t)x}.$$

By identifying ϖ with $y_1 dy_2$, we find, by taking for example the terms in dx',

$$x = y_1 x \frac{\alpha\delta - \beta\gamma}{(\gamma x' + \delta x)^2},$$

hence

$$y_1 = \frac{(\gamma x' + \delta x)^2}{\alpha\delta - \beta\gamma}.$$

If we assume, which is always allowed, that the determinant $\alpha\delta - \beta\gamma$ is equal to 1 (or even simply constant), the general solution of the system is provided by the equations

$$\alpha x' + \beta x = C_1,$$
$$\gamma x' + \delta x = C_2,$$

and we have

$$x = C_2 \alpha(t) - C_1 \gamma(t).$$

In other words, the coefficients $\alpha(t)$ and $\gamma(t)$ that appear in the general integral of the Riccati equation form a system of fundamental solutions of the given second order equation.

There is still another way to present things. Suppose that we know the general solution u of the Riccati equation expressed by means of t and the constant of integration y_2. The identity

$$y_1\, dy_2 = x^2[du + (u^2 + k)dt]$$

gives

$$y_1 = x^2 \frac{\partial u}{\partial y_2},$$

from which we deduce x as a function of y_1, y_2 and t. Since we have

$$u = \frac{\delta y_2 - \beta}{-\gamma y_2 + \alpha}$$

we get

$$x^2 = y_1(\alpha - \gamma y_2)^2,$$

from which

$$x = C_1\alpha + C_2\gamma.$$

116. Note. — The theory of Jacobi's last multiplier can be applied to problems of Mechanics other than those indicated above. Take for example the motion of a material point subject to a force that is a function only of its position in space, but where the system of reference rotates uniformly about Oz. The equations of motion are of the form

$$\begin{aligned}
\frac{d^2x}{dt^2} + 2\alpha\frac{dy}{dt} - X &= 0,\\
\frac{d^2y}{dt^2} - 2\alpha\frac{dx}{dt} - Y &= 0,\\
\frac{d^2z}{dt^2} \qquad\qquad - Z &= 0,
\end{aligned}$$

where X, Y, Z are given functions of x, y, z, t. By writing them as

$$\begin{aligned}
\frac{dx}{dt} &= x', & \frac{dx'}{dt} &= -2\alpha\frac{dy'}{dt} + X,\\
\frac{dy}{dt} &= y', & \frac{dx'}{dt} &= 2\alpha\frac{dx'}{dt} + Y,\\
\frac{dz}{dt} &= z', & \frac{dz'}{dt} &= \qquad\qquad Z,
\end{aligned}$$

we obtain a system that clearly admits the multiplier 1.

117. The last application we will consider is provided by the integral invariant of Hydrodynamics

$$\Omega = \xi[\delta y\,\delta z] + \eta[\delta z\,\delta x] + \zeta[\delta x\,\delta y] + (\eta w - \zeta v)[\delta x\,\delta t] + (\zeta u - \xi w)[\delta y\,\delta t] + (\xi v - \eta u)[\delta z\,\delta t].$$

The characteristic system of this invariant is formed by the two Pfaffian equations

$$\frac{dx - u\,dt}{\xi} = \frac{dy - v\,dt}{\eta} = \frac{dz - w\,dt}{\zeta}.$$

The integral manifolds are the two-dimensional manifolds, in the universe (x, y, z, t), generated for example by a vortex line in its various successive positions.

Integration of this system reduces to integrating a system of two differential equations with two unknown functions for which we know a multiplier. The search for the trajectories of the molecules (fluid trajectories) also requires the integration of an ordinary differential equation, *which can be arbitrary*.

If the motion is permanent, the characteristic manifolds are given by two quadratures, namely

$$\int \begin{vmatrix} dx & dy & dz \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix} = C,$$

and then, taking the preceding equation into account, assumed solved with respect to z,

$$t + \int \frac{\eta\, dx - \xi\, dy}{\xi v - \eta u} = C'.$$

Chapter XII

Equations which admit a relative linear integral invariant

I. — *General method of integration.*

118. Consider a Pfaffian form ω and the characteristic system of the relative *integral* invariant $\int \omega$. This is the associated system of the form ω.

First, suppose ω has $2n+1$ variables; if the form ω is of even rank (n° **59**), its characteristic system will consist in general of $2n$ equations. Consequently, *there exists in general one and only one system of differential equations that admits a relative integral invariant* $\int \omega$*, where* ω *is an arbitrary Pfaff form with* $2n+1$ *variables*. This is the case for the integral invariant of Dynamics.

In general, let $2n$ be the rank (or the class) of the form ω'. It is easy to indicate a method of integration for the characteristic equations of ω'.

In fact, let y_1 be a first integral of these equations (it is obtained by an operation of order $2n$). If we relate the variables by the relation $y_1 = C_1$, that is, the differentials by the relation $dy_1 = 0$, the rank of ω' decreases, and since it always remains even, it reduces to $2n-2$. Let y_2 be a first integral of the new characteristic system; by supposing

$$y_i = C_1, \qquad y_2 = C_2,$$

the rank of ω' reduces to $2n-4$, and so on. Thus, by successive operations of orders

$$2n,\ 2n-2,\ \ldots,\ 4,\ 2,$$

we will be able to find n first integrals

$$y_1, y_2, \ldots, y_n$$

such that, if we suppose that the variables are related by the relations

$$y_1 = C_1, \;\; y_2 = C_2, \;\; \ldots, \;\; y_n = C_n,$$

the rank of ω' becomes zero. At this point, since ω' is identically zero, the form ω is an exact differential; thus a quadrature puts it into the form

$$\omega = dS.$$

The function S depends on n constants $C_1, C_2, \ldots, C_n$. If we no longer assume that the variables are related by the n relationships shown, we have obviously

$$\omega = dS + z_1\,dy_1 + z_2\,dy_2 + \cdots + z_n\,dy_n$$

and consequently

$$\omega' = [dz_1\,dy_1] + [dz_2\,dy_2] + \cdots + [dz_n\,dy_n].$$

Since ω' is of rank $2n$, the $2n$ differentials dy_i and dz_i are linearly independent; thus *the* $2n$ *functions* y_i *and* z_i *form a system of first integrals independent of the given equations*, the integration of which is thus complete.

Finally, the integration required $n+1$ *operations of order*

$$2n, \;\; 2n-2, \;\; \ldots, \;\; 4, \;\; 2, \;\; 0$$

followed by differentiations.

NOTE I. — The quantity S here serves only as an intermediary; it is not in general a first integral of the characteristic equations of the invariant $\int \omega$.

NOTE II. — We see from the result obtained that *any exterior quadratic form with zero exterior derivative can be put into the form*

$$[dz_1\,dy_1] + [dz_2\,dy_2] + \cdots + [dz_n\,dy_n].$$

119. It is important to note the indefiniteness of the choice of functions y_i and z_i that enter the canonical form. The equality

$$[dz'_1\,dy'_1] + [dz'_2\,dy'_2] + \cdots + [dz'_n\,dy'_n] = [dz_1\,dy_1] + [dz_2\,dy_2] + \cdots + [dz_n\,dy_n]$$

leads to the property that the difference

$$z'_1\,dy'_1 + z'_2\,dy'_2 + \cdots + z'_n\,dy'_n - (z_1\,dy_1 + z_2\,dy_2 + \cdots + z_n\,dy_n)$$

is an exact differential dV. Suppose, which is the general case, that $y'_1, \dots, y'_n$ are independent functions of $z_1, \dots, z_n$; then there is no relationship between the y_i and the y'_i. By expressing V as a function of the y_i and the y'_i, we deduce

$$z'_i = \frac{\partial V}{\partial y'_i}, \qquad z_i = -\frac{\partial V}{\partial y_i}.$$

These equations, which involve an arbitrary function of $2n$ arguments, allows the y' and the z' to be expressed in terms of the y and the z; in fact, the last n give $y'_1, \dots, y'_n$ and the first n then give $z'_1, \dots, z'_n$. This assumes that we do not have

$$\frac{D\left(\dfrac{\partial V}{\partial y_1}, \dfrac{\partial V}{\partial y_2}, \dots, \dfrac{\partial V}{\partial y_n}\right)}{D(y'_1, y'_2, \dots, y'_n)} = 0.$$

Under the same condition, we can solve for the y as a function of the y' and z' by means of the n first equations and then obtain the z by means of the last n equations.

We would treat similarly the case where one or more relationships exist between the y and the y'.

The set of transformations thus defined on the variables y and z, *that is, on the integral curves of the given equations*, defines an infinite group which plays the same role in this theory as the group of transformations with functional determinant equal to 1 in the theory of the Jacobi multiplier.

120. Return to the integration of the characteristic equations of ω'. Suppose that, by some process, we have come to know $N > n$ independent first integrals $y_1, y_2, \dots, y_N$ such that, by equating them to arbitrary constants, the rank of ω' becomes zero, that is, ω becomes an exact differential. A quadrature followed by differentiations then gives

$$\omega = dS + z_1\, dy_1 + z_2\, dy_2 + \cdots + z_N\, dy_N$$

It is easy to see that $z_1, z_2, \dots, z_N$ *are first integrals*.

In fact, suppose that among the functions y_i and z_i there are $N + r$ independent functions; we can then express the functions z_i as functions of the y_i and of r among them, which we will call $t_1, t_2, \dots, t_r$. That said, we have

$$\omega' = [dz_1\, dy_1] + [dz_2\, dy_2] + \cdots + [dz_N\, dy_N].$$

The characteristic system of ω' includes by hypothesis the equations

$$dy_1 = 0, \qquad dy_2 = 0, \qquad \dots, \qquad dy_N = 0.$$

It includes also the equation

$$\frac{\partial\omega'}{\partial[dy_i]} \equiv -dz_i + \frac{\partial z_1}{\partial y_i}dy_1 + \frac{\partial z_2}{\partial y_i}dy_2 + \cdots + \frac{\partial z_N}{\partial y_i}dy_N = 0,$$

and hence the equation

$$dz_i = 0.$$

We thus see that the z_1 are first integrals, and on the other hand that $N+r$ must be equal to $2n$.

Finally, *knowledge of N first integrals that make ω an exact differential when we equate them to arbitrary constants, allows us to complete the integration by a quadrature and differentiations*.

121. In practice, it may happen that we are not looking for all the solutions of the given differential equations, but only those for which the N first integrals $y_1,\ldots,y_N$ have given *numerical* values. We can then proceed as follows. Since the form ω' is zero when we set $dy_1,\ldots,dy_N$ to zero, it can be put into the form

$$\omega' = [dy_1\,\varpi_1] + [dy_2\,\varpi_2] + \cdots + [dy_N\,\varpi_N]$$

in an infinite number of ways, where the ϖ_i are conveniently chosen linear forms. Among these N forms ϖ_i, there are among them $2n-N$ independent ones and independent of the dy_i; suppose that this is so for $\varpi_1,\ldots,\varpi_{2n-N}$. The characteristic system of ω' is obviously formed by the equations

$$dy_1 = dy_2 = \cdots = dy_N = 0,$$
$$\varpi_1 = \varpi_2 = \cdots = \varpi_{2n-N} = 0.$$

Expressing the fact that the exterior derivative of ω' is zero: we get

$$[dy_1\,\varpi_1'] + [dy_2\,\varpi_2'] + \cdots + [dy_N\,\varpi_N'] = 0,$$

from which, in particular, by exterior multiplication with $[dy_2\,dy_3\ldots dy_N]$,

$$[dy_1\,dy_2\ldots dy_N\,\varpi_1] = 0.$$

The form ϖ_1 (and also the forms $\varpi_2,\ldots,\varpi_{2n-N}$) are thus exact differentials when we give the y_i fixed numerical values. Consequently, *the solutions we seek are obtained by $2n-N$ independent quadratures*

$$\int\varpi_1 = \gamma_1, \quad \ldots, \quad \int\varpi_{2n-N} = \gamma_{2n-N}.$$

There is no reason to be surprised that we encounter here $2n-N$ quadratures, while the search for the general solution required only one quadrature. In fact, to perform the $2n-N$ quadratures indicated above is to perform the single quadrature

$$\int \lambda_1 \varpi_1 + \lambda_2 \varpi_2 + \cdots + \lambda_{2n-N} \varpi_{2n-N} = C$$

with $2n - N$ arbitrary *parameters* $\lambda_1, \ldots, \lambda_{2n-N}$.

The preceding process of integration uses only the invariant form ω' and does not involve the form ω. *What plays an essential role therefore is knowledge of the absolute integral invariant of second degree* $\int \omega'$*, and the property that the form* ω' *is an exact derivative*. The form ω (or the forms ω) whose derivative is ω' plays only a secondary role.

II. — *Poisson brackets and the Jacobi identity.*

122. Let $2n$ be the rank of the exterior derivative ω', and let f and g be two first integrals of its characteristic system. The two differential forms

$$[\omega'^{n-1}\, df\, dg] \quad \text{and} \quad [\omega'^{n}]$$

are invariants of maximum degree $2n$; they thus differ only by a factor, and this factor is a first integral. We put

$$\frac{1}{(n-1)!}[\omega'^{n-1}\, df\, dg] = \frac{1}{n!}(fg)[\omega'^{n}]$$

or

$$(fg)[\omega'^{n}] = n[\omega'^{n-1}\, df\, dg].$$

The quantity (fg) thus defined is called the *Poisson bracket*: it is an alternating bilinear form of the partial derivatives of f and g.

The bracket of two first integrals is again a first integral.

This theorem, due to Poisson in the particular case of the canonical equations, had its importance demonstrated by Jacobi.

Before moving on to the applications of this theorem, we make a few remarks.

The condition $(fg) = 0$ expresses the fact that the rank of ω' is equal to an $2n - 4$: in this case we say that the integrals f and g are *in involution*.

If this condition is not met, the defining formula for (fg) expresses the fact that the form

$$\omega' - \frac{[df\, dg]}{(fg)}$$

is of rank $2n-2$; in fact, the n^{th} power of this form is

$$[\omega'^n] - \frac{n}{(fg)}[\omega'^{n-1} df\, dg] = 0.$$

We note furthermore that if we have reduced ω' to its normal form

$$\omega' = [\omega\,\omega_2] + [\omega_3\,\omega_4] + \cdots + [\omega_{2n-1}\,\omega_{2n}],$$

and if we put

$$\begin{aligned} df &= f_1\omega_1 + f_2\omega_2 + \cdots + f_{2n}\omega_{2n}, \\ dg &= g_1\omega_1 + g_2\omega_2 + \cdots + g_{2n}\omega_{2n}, \end{aligned}$$

we have

$$(fg) = f_1g_2 - f_2g_1 + f_3g_4 - f_4g_3 + \cdots + f_{2n-1}g_{2n} - f_{2n}g_{2n-1}.$$

We note finally, according to what we saw earlier (n^0 **118**), that we can always assume that the ω_i are exact differentials. An easy calculation then gives the following identity, due to Jacobi,

$$((fg)h) + ((gh)f) + ((hf)g) = 0,$$

which applies to any three first integrals f, g, h.

But we can verify this also without assuming anything about the linear forms $\omega_1, \omega_2, ..., \omega_{2n}$. It relies on the identity

$$\frac{1}{(n-1)!}[\omega'^{n-1}((fg)dh + (gh)df + (hf)dg)] = \frac{1}{(n-2)!}[\omega'^{n-2} df\, dg\, dh] \qquad (1)$$

which is none other than the identity (8) proved in n^0 **68**. Finding its exterior derivative and noting that the exterior derivative of the right hand side is zero, we get

$$[\omega'^{n-1} d(fg)\, dh] + [\omega'^{n-1} d(gh)\, df] + [\omega'^{n-1} d(hf)\, dg] = 0,$$

which is none other than Jacobi's identity.

123. The integration method indicated at the beginning of the Chapter can be stated using the Poisson brackets. Let

$$Xf = 0$$

be the equation which expresses the fact that f is a first integral. We first look for a particular solution y_1 of this equation; we then look for a particular solution y_2 of the system

$$Xf = 0, \qquad (y_1 f) = 0,$$

then a particular solution y_3 of the system

$$Xf = 0, \qquad (y_1 f) = 0, \qquad (y_2 f) = 0$$

and so on up to a particular solution y_n of the system

$$Xf = 0, \qquad (y_1 f) = 0, \qquad (y_2 f) = 0, \qquad ..., \qquad (y_{n-1} f) = 0.$$

In the case of the canonical equations of dynamics

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

corresponding to the invariant form

$$\omega' = \sum_{i=1}^{i=n} [\delta p_i \, \delta q_i] - [\delta H \, \delta t],$$

the partial differential equation of the first integrals of the given equations is

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{i=n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0.$$

As regards the Poisson bracket (fg) of two first integrals, it is defined by the equality

$$n\,[\omega'^{\,n-1} \, \delta f \, \delta g] = (fg)[\omega'^{\,n}];$$

equate on the two sides the terms in

$$[\delta p_1 \, \delta q_1 \, \delta p_2 \, \delta q_2 \ldots \delta p_n \, \delta q_n];$$

we get

$$(fg) = \sum_{i=1}^{i=n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right).$$

The partial differential equation of the first integrals can thus be written, by extending the definition of the bracket (fg) to any two functions functions of q_i, p_i and t,

$$\frac{\partial f}{\partial t} - (Hf) = 0.$$

III. — *Use of known first integrals.*

124. We now return to the problem of integrating the characteristic equations of the differential form ω', assuming that we know a certain (arbitrary) number of first integrals $y_1, y_2, ..., y_p$. Equating these integrals to arbitrary constants $C_1, C_2, ..., C_p$, the rank of the form ω' is reduced by a certain even number $2p' \leq 2p$ of units. Then we need only integrate the characteristic equations of this new form, or rather, to look for $n - p'$ first integrals in involution: we are thus back to the problem of n° **120**.

The preceding method does not in general make full use of the known integrals. In fact, according to the Poisson-Jacobi theorem, the brackets of the p given integrals, taken two at a time, are themselves first integrals of the equations to be integrated. So, form the brackets $(y_i y_j)$; if they provide new integrals, then form the brackets of these integrals with each other and with the given integrals, and so on until this operation gives nothing new. This amounts to saying that, *by prior differentiations, we can always assume that the brackets $(y_i y_j)$ are functions of the first integrals $y_1, y_2, ..., y_p$.*

Now, to find out by how many units the rank of ω' is reduced when we assume that the variables are related by the relations

$$y_1 = C_1, \quad y_2 = C_2, \quad ..., \quad y_p = C_p,$$

we need only apply the theorem of n° **69** to the exterior quadratic form ω'' constructed with the variables $\delta x_1, ..., \delta x_{2n+1}$ related by the relations

$$\delta y_1 = 0, \quad \delta y_2 = 0, \quad ..., \quad \delta y_1 = 0.$$

The coefficients a_{ij} of n° **69** are here the brackets $(y_i y_j)$ and the quadratic form $\overline{\Phi}$ is here

$$\overline{\Phi} = \sum (y_i y_j)[\xi_i \xi_j];$$

the number of units by which the rank of ω' is reduced is equal to the maximum number $2p$ minus the rank of the form $\overline{\Phi}$.

125. We can see in the following way that all possible advantage has been taken of the given first integrals.

Perform a linear substitution (with coefficients that are functions of $y_1, ..., y_p$) on the p variables $\xi_1, ..., \xi_p$ so as to $\overline{\Phi}$ back to its normal form

$$\overline{\Phi} = [\xi_1 \, \xi_2] + \cdots + [\xi_{2q-1} \, \xi_{2q}] \quad (2q \leq p).$$

This amounts to replacing the linear forms $\delta y_1, ..., \delta y_p$ by new differential forms

$$\varpi_1, ..., \varpi_p,$$

linear in $\delta y_1, ..., \delta y_p$ with coefficients that are functions of $y_1, ..., y_p$ and such that we have identically

$$\xi_1 \delta y_1 + \xi_2 \delta y_2 + \cdots + \xi_p \delta y_p = \overline{\xi}_1 \varpi_1 + \overline{\xi}_2 \varpi_2 + \cdots + \overline{\xi}_p \varpi_p.$$

The exterior quadratic form ω' will then take the form

$$\omega' = [\varpi_1\, \varpi_2] + \cdots + [\varpi_{2q-1}\, \varpi_{2q}] + [\varpi_{2q+1}\, \omega_1] + \cdots + [\varpi_p\, \omega_{p-2q}] \\ + [\omega_{p-2q+1}\, \omega_{p-2q+2}] + \cdots + [\omega_{2n-p-1}\, \omega_{2n-p}],$$

by introducing $2n-p$ new linear forms $\omega_1, ..., \omega_{2n-p}$.

Denote by Π the form

$$\Pi = [\varpi_1\, \varpi_2] + \cdots + [\varpi_{2q-1}\, \varpi_{2q}]$$

and formulate the fact that the exterior derivative of ω' is zero. If we neglect all the terms which contain one of the linear forms

$$\varpi_{2q+1}, ..., \varpi_p; \qquad \omega_{p-2q+1}, ..., \omega_{2n-p},$$

we obtain

$$\Pi' + [\varpi_{2q+1}\, \omega_1] + \cdots + [\varpi_p'\, \omega_{p-2q}] = 0. \tag{2}$$

Since the form Π is constructed only with the functions y_i and their differentials, it is the same for Π'; consequently, no reduction of similar terms can be made between the different parts of the left hand side of (2). In particular, it follows that each of the forms

$$\varpi_{2q+1}', ..., \varpi_p',$$

is zero (assuming that the forms $\varpi_{2q+1}', ..., \varpi_p'$ are zero); consequently *the Pfaffian system*

$$\varpi_{2q+1} = \cdots = \varpi_p = 0$$

is completely integrable. Denote a system of first integrals of these equations by

$$\overline{y}_{2q+1}, ..., \overline{y}_p.$$

We also have

$$\Pi' = 0,$$

always by considering the forms $\varpi_{2q+1}', ..., \varpi_p'$ as zeros; in other words, if we assume $\overline{y}_{2q+1}, ..., \overline{y}_p$ to be constants, the form Π is an exact derivative, and consequently (n° **118**) reducible to

$$\Pi = [d\overline{y}_1\, d\overline{y}_2] + \cdots + [d\overline{y}_{2q-1}\, d\overline{y}_{2q}].$$

Finally, we see easily that we can put ω' into the form

$$\begin{aligned}\omega' = [d\bar{y}_1\, d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\, d\bar{y}_{2q}] + [d\bar{y}_{2q+1}\, \overline{\omega}_1] + \cdots \\ +[d\bar{y}_p\, \overline{\omega}_{p-2q}] + [\omega_{p-2q+1}\, \omega_{p-2q+2}] + \cdots\end{aligned} \quad (3)$$

Basically, this comes down to the following theorem:

We can find p functions

$$\bar{y}_1, \bar{y}_2, ..., \bar{y}_p$$

of the given p first integrals that satisfy the conditions

$$(\bar{y}_1\, \bar{y}_2) = \cdots = (\bar{y}_{2q-1}, \bar{y}_{2q}) = 1$$

where all other brackets $(\bar{y}_i, \bar{y}_j)$ *are zero.*

126. Besides the intrinsic interest of this theorem, its form brings into relief the fact stated above that the indicated method of integration has taken full advantage of the given integrals. In fact, the form (3) found for ω' allows us to write

$$\begin{aligned}\omega = dS + \bar{y}_1\, d\bar{y}_2 + \cdots + \bar{y}_{2q-1}\, d\bar{y}_{2q} + w_1\, d\bar{y}_{2q+1}] + \cdots \\ + w_{p-2q}\, d\bar{y}_p + v_1\, du_1 + \cdots + v_{n-p+q}\, du_{n-p+q'}], \\ \omega' = [d\bar{y}_1\, d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\, d\bar{y}_{2q}] + [dw_1\, d\bar{y}_{2q+1}] + \cdots \\ + [dw_{p-2q}\, d\bar{y}_p] + [dv_1\, du_1] + \cdots + [dv_{n-p+q}\, du_{n-p+q}].\end{aligned}$$

The group of the most general transformations on the integral curves which preserves the data, that is, which leave invariant $\omega', y_1, ..., y_p$ is defined by the following equations, where the accented letters indicate the transformed variables, and where V denotes an arbitrary function of the arguments $u_i, u'_i, \bar{y}_{2q+1}, ..., \bar{y}_p$:

$$\begin{aligned}\overline{y'}_i = \bar{y}_i \qquad (i = 1, 2, ..., p), \\ v'_1 = \frac{\partial V}{\partial u'_1}, ..., v'_{n-p+q} = \frac{\partial V}{\partial u'_{n-p+q}}, \\ v_1 = -\frac{\partial V}{\partial u_1}, ..., v_{n-p+q} = -\frac{\partial V}{\partial u_{n-p+q}}, \\ w'_1 = w_1 + \frac{\partial V}{\partial \bar{y}_{2q+1}}, ..., w'_{p-2q} = w_{p-2q} + \frac{\partial V}{\partial \bar{y}_{p-2q}}.\end{aligned}$$

Any unambiguous procedure which, starting from ω' and p first integrals $y_1, ..., y_p$, would allow us to deduce another first integral by operations *that have a meaning independent of the choice of variables*, necessarily leads to a first integral invariant by the most general group of transformations preserving $\omega', y_1, ..., y_p$; and the only functions invariant by this group are obviously arbitrary functions of $y_1, ..., y_p$.

IV. — *Generalisation of the theorem of Poisson-Jacobi.*

127. The Poisson-Jacobi theorem is immediately generalised if, instead of two first integrals, we know two invariant linear forms ϖ_1 and ϖ_2: the quantity α defined by the equality

$$n[\omega'^{n-1}\,\varpi_1\,\varpi_2] = \alpha[\omega'^n] \tag{4}$$

is obviously a first integral; it reduces to $(y_1 y_2)$ if ϖ_1 and ϖ_2 are the differentials of two first integrals y_1, y_2.

Apply this remark to the case where, were the characteristic equations of ω' to admit two infinitesimal transformations $A_1 f$ and $A_2 f$, we would have

$$\varpi_1 = \omega'(A_1, \delta), \qquad \varpi_2 = \omega'(A_2, \delta).$$

To calculate the quantity α in this case, apply to the two sides of equality (4) the operation which takes us from an invariant form $\Omega(\delta)$ to the form $\Omega(A_1, \delta)$. We get

$$n(n-1)[\omega'^{n-2}\,\varpi_1\,\varpi_1\,\varpi_2] - n[\omega'^{n-1}\,\varpi_1]\varpi_2(A_1) = n\alpha[\omega'^{n-1}\,\varpi_1],$$

from which, since the form $[\omega'^{n-1}\,\varpi_1]$ is certainly not zero,

$$\alpha = -\varpi_2(A_1) = \omega'(A_1, A_2) = \varpi_1(A_2).$$

The generalised Poisson-Jacobi theorem, applied to two invariant forms $\omega'(A_1, \delta)$ and $\omega'(A_2, \delta)$, thus leads to the first integral $\omega'(A_1, A_2)$ provided by the twice-repeated application to ω' of the operation corresponding to the infinitesimal transformations $A_1 f$ and $A_2 f$.

Chapter XIII

Equations admitting an absolute linear integral invariant

I. — *General method of integration.*

128. Let ω be a linear differential form; its bilinear covariant ω' has even rank, say 2n. Two cases can arise, according as the equation $\omega = 0$ is not or is not part of the characteristic system of ω'. We will first consider the first case.

I. We can obviously put

$$\omega' = [\omega_1\, \omega_2] + \cdots + [\omega_{2n-1}\, \omega_{2n}],$$

where the $2n+1$ forms $\omega, \omega_1, \ldots, \omega_{2n}$ are independent. In this case the characteristic equations of ω are (n° **78**)

$$\omega = \omega_1 = \omega_2 = \cdots = \omega_{2n} = 0.$$

We can easily indicate a reduced form for ω. In fact, operations of order

$$2n,\ 2n-2,\ \ldots,\ 2$$

successively yield n first integrals

$$y_1,\ y_2,\ \ldots,\ y_n$$

of the characteristic equations of ω', reducing its rank to zero when we equate them to arbitrary constants. A quadrature then puts ω in the form

$$\omega = du + z_1\, dy_1 + z_2\, dy_2 + \cdots + z_n\, dy_n.$$

Such is the reduced form we sought, which is obtained by operations of orders

$$2n,\ 2n-2,\ \ldots,\ 2,\ 0$$

and which, once obtained, gives the general solution of the characteristic equations of ω.

We see that in this case the integration of the characteristic equations of ω' and that of the characteristic equations of ω are two equivalent problems, and the fact that $\int \omega$ is an *absolute* integral invariant has no more importance for the integration than if $\int \omega$ were a *relative* integral invariant. This is true at least if, for the integration of the characteristic equations of ω', we follow the method indicated in n° **118**; it would no longer be the same if we were to apply the method of n° **121**.

II. In the second case we can put

$$\omega' = [\omega\,\omega_1] + [\omega_2\,\omega_3] + \cdots + [\omega_{2n-2}\,\omega_{2n-1}]$$

with $2n$ linearly independent forms $\omega, \omega_1, \ldots \omega_{2n-1}$. The equations

$$\omega = 0, \qquad \omega_2 = \omega_3 = \cdots = \omega_{2n-1} = 0,$$

which are obtained by writing, in addition to equation $\omega = 0$, the equations of the associated system of the form ω', *where we assume that the differentials are related by the relation* $\omega = 0$, have intrinsic significance. This is the associated system of the two forms ω and $[\omega\,\omega']$ and consequently (n° **103**) *it is the characteristic system of the Pfaffian equation* $\omega = 0$. We will call it the system (Σ), denoting by (S) the characteristic system of ω, which also contains the equation $\omega_1 = 0$.

By an operation of order $2n-1$, we can obtain a first integral y_1 of system (Σ). By equating it to an arbitrary constant, the system (Σ) of the new form ω, that is, the characteristic system of the new equation $\omega = 0$, has the number of its equations reduced by two units; thus, by operations of order

$$2n-3,\ \ldots, 3,$$

we can find new integrals

$$y_2,\ \ldots, y_{n-1}$$

such that, by equating them to new arbitrary constants, the new system (Σ) corresponding to ω contains only one equation, which will obviously be $\omega = 0$. This means that this equation is completely integrable and a new operation of order 1 gives a new integral y_n which allows us to write

$$\omega = z_1\,dy_1 + z_2\,dy_2 + \cdots + z_n\,dy_n.$$

In this way we arrive at the *reduced form* of ω, which effectively involves the minimum number $2n$ of variables, where $2n$ is the number of equations in the characteristic system (S) of ω, that is, the class of ω.

We easily find the most general transformation performed on the characteristic variables y_i and z_i, which preserves the form ω; given the equality

$$z'_1\, dy'_1 + z'_2\, dy'_2 + \cdots + z'_n\, dy'_n = z_1\, dy_1 + z_2\, dy_2 + \cdots + z_n\, dy_n,$$

remaining in the most general case,

$$V(y'_1, ..., y'_n, y_1, ..., y_n) = 0,$$
$$\frac{z'_1}{\dfrac{\partial V}{\partial y'_1}} = \frac{z'_2}{\dfrac{\partial V}{\partial y'_2}} = \cdots = \frac{z'_n}{\dfrac{\partial V}{\partial y'_n}} = \frac{-z_1}{\dfrac{\partial V}{\partial y_1}} = \cdots = \frac{-z_n}{\dfrac{\partial V}{\partial y_n}}.$$

These formulae show clearly that the $y_1, ..., y_n, \dfrac{z_2}{z_1}, ..., \dfrac{z_n}{z_1}$ are transformed among themselves; this is the smallest number of variables in terms of which we can write the equation $\omega = 0$; they are the first integrals of the system (Σ) characteristic of this equation.

Were there to exist p independent relations ,

$$V_1 = 0, \quad V_2 = 0, \quad ..., \quad V_p = 0$$

between the y_i and the y'_i, the formulae that define the transformation would be

$$z'_i = \lambda_1 \frac{\partial V_1}{\partial y'_i} + \lambda_2 \frac{\partial V_2}{\partial y'_i} + \cdots + \lambda_p \frac{\partial V_p}{\partial y'_i},$$
$$-z_i = \lambda_1 \frac{\partial V_1}{\partial y_i} + \lambda_2 \frac{\partial V_2}{\partial y_i} + \cdots + \lambda_p \frac{\partial V_p}{\partial y_i},$$

with p auxiliary unknowns $\lambda_1, ..., \lambda_p$.

129. From the point of view of the integration, note the difference between the two cases, where the characteristic system of ω is odd $(2n+1)$ or even $(2n)$: in the first case, the integration requires operations of orders

$$2n,\ 2n-2,\ ...,\ 2,\ 0;$$

in the second case, it requires operations of orders

$$2n-1,\ 2n-3,\ ...,\ 1.$$

Note also that the two cases in practice distinguish from one from the other in the following way. Let $2n$ be the rank of ω', that is, let n be the largest exponent such that the form $[\omega'^n]$ is not zero; *in the first case in the first case* $[\omega\, \omega'^n]$ *is not zero; in the second case* $[\omega\, \omega'^n]$ *is zero.*

II. — *Generalisation of the Poisson-Jacobi formulae.*

130. I. *Suppose that the form ω is of the first type.* — Let f be any first integral of its characteristic system; the form $[\omega'^n\, df]$ is invariant and of maximum order $2n+1$. We can therefore put

$$[\omega'^n\, df] = \{f\}[\omega\, \omega'^n],$$

where $\{f\}$ is a finite quantity linear with respect to the first-order partial derivatives of the function f. *This quantity $\{f\}$ is either a constant or a first integral of the characteristic equations of ω.*

Now let f and g be two first integrals of the characteristic equations of ω. We can define a quantity (fg) by the relation

$$n[\omega\, \omega'^{n-1}\, df\, dg] = (fg)[\omega\, \omega'^n];$$

this quantity (fg) is also a constant or a first integral.

If the form ω has been reduced:

$$\omega = du + z_1\, dy_1 + \cdots + z_n\, dy_n,$$

we have

$$\begin{aligned} \{f\} &= \frac{\partial f}{\partial u}, \\ (fg) &= \sum_{i=1}^{i=n} \frac{\partial f}{\partial z_i}\left(\frac{\partial g}{\partial y_i} - z_i \frac{\partial g}{\partial u}\right) - \frac{\partial g}{\partial z_i}\left(\frac{\partial f}{\partial y_i} - z_i \frac{\partial f}{\partial u}\right). \end{aligned}$$

From this we deduce easily the important identities

$$\begin{aligned} &\{(fg)\}(\{f\}g) + (f\{g\}), \\ &((fg)h) + ((gh)f) + ((hf)g) = (fg)\{h\} + (gh)\{f\} + (hf)\{g\}. \end{aligned}$$

131. To prove these identities directly, note that the form $df - \{f\}\omega$ is a linear combination of the $2n$ independent linear forms by means of which ω' can be expressed, since we have

$$[\omega'^{\,n}\, (df - \{f\}\omega)] = 0.$$

We deduce immediately and identity of the form

$$n[\omega'^{\,n-1}\, (df - \{f\}\omega)\, (dg - \{g\}\omega)] = \lambda\, [\omega'^n]$$

and exterior multiplication by ω gives $\lambda = (fg)$. We thus have

$$n[\omega'^{\,n-1}\, df\, dg] - n\{f\}[\omega\, \omega'^{\,n-1}\, dg] + n\{g\}[\omega\, \omega'^{\,n-1}\, df] = (fg)[\omega'^{\,n}]. \qquad (1)$$

Identity (8) of n° **68**, applied to the three linear forms $df - \{f\}\omega$, $dg - \{g\}\omega$, $dh - \{h\}\omega$ then gives

$$(fg)[\omega'^{\,n-1}\,(dh - \{h\}\omega)] + (gh)[\omega'^{\,n-1}\,(df - \{f\}\omega)] + (hf)[\omega'^{\,n-1}\,(dg - \{g\}\omega)]$$
$$= (n-1)[\omega'^{\,n-2}\,(df - \{f\}\omega)\,(dg - \{g\}\omega)\,(dh - \{h\}\omega)],$$

from which we deduce, by multiplying by ω,

$$[\omega\,\omega'^{\,n-1}\;((fg)dh + (gh)df + (hf)dg)] = (n-1)[\omega\,\omega'^{\,n-2}\,df\,dg\,dh]. \tag{2}$$

That said, exterior derivation of identity (1) gives

$$n[\omega\,\omega'^{\,n-1}\,d\{f\}\,dg] + n[\omega\,\omega'^{\,n-1}\,df\,d\{g\}] = [\omega'^{\,n}\,d(fg)],$$

that is, the first identity to prove:

$$(\{f\}g) + (f\{g\}) = \{(fg)\}.$$

Exterior derivation of identity (2) consequently gives

$$[\omega'^{\,n}\;((fg)dh + (gh)df + (hf)dg)] - [\omega\,\omega'^{\,n-1}\,d(fg)\,dh] - [\omega\,\omega'^{\,n-1}\,d(gh)\,df]$$
$$-[\omega\,\omega'^{\,n-1}\,d(hf)\,dg] = (n-1)[\omega'^{\,n-1}\,df\,dg\,dh];$$

but, on the other hand, exterior multiplication of (1) by dh gives

$$n[\omega'^{\,n-1}\,df\,dg\,dh] - n\{f\}[\omega\,\omega'^{\,n-1}\,dg\,dh] + n\{g\}[\omega\,\omega'^{\,n-1}\,df\,dh] = (fg)[\omega'^{\,n}\,dh];$$

we deduce from this last formula

$$n[\omega'^{\,n-1}\,df\,dg\,dh] = [\{f\}(gh) + \{g\}(hf) + \{h\}(fg)][\omega\,\omega'^{\,n}]$$

and from the preceding one the identity to be proved

$$(fg)\{h\} + (gh)\{f\} + (hf)\{g\} = ((fg)h) + ((gh)f) + ((hf)g).$$

132. *Suppose now that the form ω is of the second type.* — Given two first integrals f and g of the characteristic equations of ω, the quantities of w, we will define similarly the quantities $\{f\}$ and (fg) by the formulae

$$n[\omega\,\omega'^{\,n-1}\,df] = \{f\}[\omega'^{\,n}],$$
$$n[\omega'^{\,n-1}\,df\,dg] = (fg)[\omega'^{\,n}].$$

If ω is the reduced form

$$\omega = z_1\,dy_1 + z_2\,dy_2 + \cdots + z_n\,dy_n,$$

we have

$$\{f\} = -\left(z_1\frac{\partial f}{\partial z_1} + z_2\frac{\partial f}{\partial z_2} + \cdots + z_n\frac{\partial f}{\partial z_n}\right),$$
$$(fg) = \sum\left(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial f}{\partial z_i}\right).$$

We then verify easily the formulae

$$\{(fg)\} = (fg) + (\{f\}g) + (f\{g\}); \tag{3}$$
$$((fg)h) + ((gh)f) + ((hf)g) = 0, \tag{4}$$

the second of which is none other than the Jacobi identity, since f,g,h are first integrals of the characteristic equations of ω'.

To prove the first identity directly, apply identity (8) of n° **68** to the three linear forms ω, df, dg; the relations

$$n[\omega'^{\,n-1}\,\omega\, df] = \{f\}[\omega'^{\,n}],$$
$$n[\omega'^{\,n-1}\, df\, dg] = (fg)[\omega'^{\,n}],$$
$$n[\omega'^{\,n-1}\, dg\, \omega] = -\{g\}[\omega'^{\,n}],$$

lead to the identity

$$[\omega'^{\,n-1}\,(\{f\}dg + (fg)\omega - \{g\}df)] = (n-1)[\omega'^{\,n-2}\,\omega\, df\, dg]$$

which, taking its exterior derivative, gives

$$[\omega'^{\,n-1}\, d\{f\}\, dg] + [\omega'^{\,n-1}\, df\, d\{g\}] + (fg)[\omega'^{\,n}] - [\omega\,\omega'^{\,n-1}\, d(fg)] = (n-1)[\omega'^{\,n-1}\, df\, dg];$$

by replacing each term by its value and simplifying, we get the identity to be proved.

III. — *Use of known first integrals.*

133. *Suppose that the form w is of the first type* and that we know p independent first integrals $y_1, \ldots, y_p$ of its characteristic equations. We will form the quantities $\{y_i\}, (y_i y_j)$; if these introduce new integrals, we will add them to those that are given and we will repeat the operation until it introduces no new integrals. We can thus assume that this first result has been obtained, that is, that the quantities $\{y_i\} = a_i, (y_i y_j) = a_{ij}$ are functions of $y_1, \ldots, y_p$.

If we now introduce auxiliary variables $\xi_1, \ldots, \xi_p$, we obtain two forms, one linear

$$\varphi = a_1\xi_1 + a_2\xi_2 + \cdots + a_p\xi_p,$$

the other exterior quadratic

$$\Phi = \sum a_{ij}[\xi_i\,\xi_j];$$

the first provides the value of the quantity $\{f\}$ when f is an arbitrary function of $y_1, ..., y_p$ admitting as partial derivatives $\xi_1, ..., \xi_p$; the second, or rather the corresponding alternating bilinear form

$$\sum a_{ij}[\xi_i\,\eta_j];$$

provides the value of the parenthesis (fg).

That said, we will reduce the two preceding forms by a suitable linear substitution on the variables ξ_i.

Three cases are possible; with the form Φ reduced to

$$\Phi = [\xi_1'\,\xi_2'] + \cdots + [\xi_{2q-1}'\,\xi_{2q}']$$

we can have

$a)$ $\qquad \varphi = 0,$

$b)$ $\qquad \varphi = \xi_1',$

$c)$ $\qquad \varphi = \xi_{2q+1}'.$

A linear substitution, with coefficients that are functions of the y_i, performed on the δy_i, will give p diffewrential forms $\varpi_1, ..., \varpi_p$ that satisfy the identity

$$\xi_1'\varpi_1 + \xi_2'\varpi_2 + \cdots + \xi_p'\varpi_p = \xi_1\delta y_1 + \xi_2\delta y_2 + \cdots + \xi_p\delta y_p.$$

That said, *in case* a), all the forms

$$[\omega'^{\,n}\,\varpi_i], \qquad [\omega\,\omega'^{\,n-1}\,\varpi_i\,\varpi_j]$$

are zero, except

$$n[\omega\,\omega'^{\,n-1}\,\varpi_1\,\varpi_2] = \cdots = n\omega'^{\,n-1}[\varpi_{2q-1}\,\varpi_{2q}] = [\omega\,\omega'^{\,n}].$$

We deduce easily that

$$\omega' = [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + [\varpi_{2q+1}\,\omega_1] + \cdots$$
$$+ [\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \cdots$$

By equating the y_i to arbitrary constants, the form ω remains of the first type and the rank of ω' is decreased by $2p-2q$ units. The case is identical to the one that was studied in the previous Chapter, where the given first integrals are integrals of the characteristic system of ω'.

In case b), we have

$$[\omega'^{\,n}\,(\varpi_1-\omega)] = [\omega'^{\,n}\,\varpi_2] = \cdots = [\omega'^{\,n}\,\varpi_p] = 0,$$

and ω' is reducible to the form

$$\omega' = [(\varpi_1-\omega)\,\varpi_2] + [\varpi_3\,\varpi_4] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + \\ +[\varpi_{2q+1}\,\omega_1] + \cdots + [\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \cdots$$

By equating the y_i to arbitrary constants, the form ω again remains of the first type and the rank of ω' is decreased by $2p-2q$ units.

In case c), we have

$$[\omega'^{\,n}\,\varpi_1] = \cdots = [\omega'^{\,n}\,\varpi_{2q}] = [\omega'^{\,n}\,(\varpi_{2q+1}-\omega)] = \cdots = [\omega'^{\,n}\,\varpi_p] = 0;$$

the form ω' is reducible to

$$\omega' = [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + [(\varpi_{2q+1}-\omega)\,\omega_1] + \cdots \\ +[\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \cdots$$

By equating the y_i to arbitrary constants, the form ω becomes of second type, the rank of ω' is decreased by $2p-2q-2$ units; the characteristic system of the new equation $\omega=0$ is formed from $2n-2p+2q+1$ equations. In this case the integration requires operations of orders

$$2n-2p+2q+1, \quad \ldots, \quad 3, \quad 1,$$

while in cases a) and b) they require operations of orders

$$2n-2p+2q, \quad \ldots, \quad 2, \quad 0$$

In summary, the form ω remains of first type if the exterior product $[\varphi\,\Phi]$ is zero, and becomes of second type otherwise.

134. *Suppose now that the form ω is of second type.* — We here again have the forms

$$\varphi = a_1\xi_1 + \cdots + a_p\xi_p,$$
$$\Phi = \sum a_{ij}[\xi_i\,\xi_j].$$

Since the coefficients a_{ij} are given by the equations

$$n[\omega'^{\,n-1}\,dy_i\,dy_j] = a_{ij}[\omega'^{\,n}],$$

the rank of ω' is reduced by $2p-2q$ units when we equate the integrals y_i to arbitrary constants, if $2q$ is the rank of the form Φ.

If Φ is reduced to its normal form

$$\Phi = [\xi_1'\,\xi_2'] + \cdots + [\xi_{2q-1}'\,\xi_{2q}'],$$

we can assume that, at the same time, we have for φ one of the following three forms:

$a)$ $$\varphi = 0,$$

$b)$ $$\varphi = \xi_1',$$

$c)$ $$\varphi = \xi_{2q+1}'.$$

Case a), according to identity (3) requires that all the brackets $(y_i y_j)$ be zero, that is, that the form Φ be identically zero. We thus have $q=0$. In this case, we obviously have

$$\omega' = [\omega\,\omega_1] + [\varpi_1\,\omega_2] + \cdots + [\varpi_p\,\omega_{p+1}] + [\omega_{p+2}\,\omega_{p+3}] + \cdots$$

The form ω remains of the second type, with the number n reduced by p units.

In case b), we have

$$n[\omega'^{\,n-1}\,\omega\,\varpi_1] = n[\omega'^{\,n-1}\,\varpi_1\,\varpi_2] = \cdots = n[\omega'^{\,n-1}\,\varpi_{2q-1}\,\varpi_{2q}] = [\omega'^{\,n}],$$

and ω' is reducible to the form

$$\begin{aligned}\omega' = {} & [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + [(\omega+\varpi_2)\,\omega_1] \\ & + [\varpi_{2q+1}\,\omega_2] + \cdots + [\varpi_p\,\omega_{p-2q+1}] + [\omega_{p-2q+2}\,\omega_{p-2q+3}] + \cdots\end{aligned}$$

By equating the y_i to arbitrary constants, the form ω remains of the second type, the rank of ω' is decreased by $2p-2q$ units.

In case c), ω' is reducible to the form

$$\begin{aligned}\omega' = {} & [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + [\omega\,\varpi_{2q+1}] + [\varpi_{2q+2}\,\omega_1] + \cdots \\ & + [\varpi_p\,\omega_{p-2q-1}] + [\omega_{p-2q}\,\omega_{p-2q+1}] + \cdots\end{aligned}$$

By equating the y_i to arbitrary constants, the form ω becomes of first type, the rank of ω' is decreased by $2p-2q$ units.

In summary, the form ω' remains of second type if the product $[\varphi\,\Phi]$ is zero, and becomes of first type otherwise.

135. In summary, we have obtained four essentially distinct reduced problems, ignoring the two cases a), one of which was treated in the previous chapter and the other corresponds to knowing p first integrals in involution of the characteristic system of the equation $\omega = 0$.

We can take a closer look at the four reduced problems and ask whether all possible advantage has been taken of the known first integrals. The method for answering this question is the same as that which has been used in the previous Chapter; it relies on the reduction of ω to a canonical form involving p suitably chosen functions of $y_1, \ldots, y_p$ and of other independent primefirst integrals. Once this canonical form has been obtained, we can deduce the equations of the largest group of transformations which, when performed on the integral curves, preserve the data.

We will rapidly point out the canonical forms of ω and ω' in each of the four cases, the calculations to get there being done in the same way as in the as in the previous Chapter (n$^{\text{o}}$ **125**).

1$^{\text{o}}$ *The form ω is of the first type and the forms φ and Φ are reducible to*

$$\varphi = \xi_1', \qquad \Phi = [\xi_1'\,\xi_2'] + \cdots + [\xi_{2q-1}'\,\xi_{2q}'].$$

We have in this case

$$\omega' = [(\varpi_1 - \omega)\,\varpi_2] + [\varpi_3\,\varpi_4] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] \\ + [\varpi_{2q+1}\,\omega_1] + \cdots + [\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \cdots$$

Putting

$$\Pi = [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}],$$

the exterior derivative of ω' gives, neglecting terms in

$$\varpi_{2q+1},\ \ldots, \varpi_p,\ \omega_{p-2q+1},\ \ldots,\ \omega_{2n-p},$$

the identity

$$\Pi' - [\Pi\,\varpi_2] + [\omega\,\varpi_2'] + [\varpi_{2q+1}'\,\omega_1] + \cdots + [\varpi_p'\,\omega_{p-2q}] = 0.$$

We can then put

$$\varpi_2 = \frac{d\bar{y}_2}{\bar{y}_2}, \quad \varpi_{2q+1} = d\bar{y}_{2q+1}, \quad ..., \quad \varpi_p = d\bar{y}_p\,;$$

since the exterior derivative of the form $\frac{1}{y_2}\Pi$ is zero, we can consequently put

$$\Pi = \bar{y}_2\left([d\bar{y}_1\,d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\,d\bar{y}_{2q}]\right).$$

We have finally

$$\omega' = \left[\frac{d\bar{y}_2}{\bar{y}_2}\,\omega\right] + \bar{y}_2[d\bar{y}_1\,d\bar{y}_2] + \cdots + \bar{y}_2[d\bar{y}_{2q-1}\,d\bar{y}_{2q}] + [d\bar{y}_{2q+1}\,\overline{\varpi}_1] + \cdots + [d\bar{y}_p\,\overline{\varpi}_{p-2q}] + \cdots$$

The result can be put into a more intuitive form, by putting

$$\overline{\varpi} = \frac{1}{\bar{y}_2}\,\omega - \bar{y}_1\,d\bar{y}_2 - \cdots - \bar{y}_{2q-1}\,d\bar{y}_{2q}.$$

In fact, we get

$$\overline{\varpi}' = [d\bar{y}_{2q+1}\,\overline{\varpi}_1] + \cdots + [d\bar{y}_p\,\overline{\varpi}_{p-2q}] + [\overline{\varpi}_{p-2q+1}\,\overline{\varpi}_{p-2q+2}] + \cdots + [\overline{\varpi}_{2n-p-1}\,\overline{\varpi}_{2n-p}].$$

In this form, it is clear that we have drawn all possible advantage from the known integrals. We have also obtained the canonical relations

$$\{\bar{y}_1\} = 1, \qquad \{\bar{y}_i\} = 0 \qquad (i = 2, ..., p);$$
$$(\bar{y}_1\,\bar{y}_2) = (\bar{y}_3\,\bar{y}_4) = \cdots = (\bar{y}_{2q-1}\,\bar{y}_{2q}) = \bar{y}_2\,,$$

where all other brackets are zero.

2° *The form ω is of the first type and the forms φ and Φ are reducible to*

$$\varphi = \xi'_{2q+1}, \qquad \Phi = [\xi'_1\,\xi'_2] + \cdots + [\xi'_{2q-1}\,\xi'_{2q}].$$

We have in this case

$$\omega' = [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + [(\varpi_{2q+1} - \omega)\,\omega_1] + \cdots + [\varpi_p\,\omega_{p-2q}] + [\omega_{p-2q+1}\,\omega_{p-2q+2}] + \cdots$$

The exterior derivative of the right hand side easily shows that we can put

$$\begin{aligned} \overline{\varpi}_{2q+2} &= d\overline{y}_{2q+2}, \quad \dots, \quad \overline{\varpi}_p = d\overline{y}_p, \\ \overline{\varpi}_{2q+1} &= d\overline{y}_{2q+1} + \overline{y}_1\, d\overline{y}_2 + \dots + \overline{y}_{2q-1}\, d\overline{y}_{2q}, \\ \Pi &= [d\overline{y}_1\, d\overline{y}_2] + [d\overline{y}_3\, d\overline{y}_4] + \dots + [d\overline{y}_{2q-1}\, d\overline{y}_{2q}]. \end{aligned}$$

Putting

$$\overline{\varpi} = -\omega + d\overline{y}_{2q+1} + \overline{y}_1\, d\overline{y}_2 + \dots + \overline{y}_{2q-1}\, d\overline{y}_{2q}\,,$$

we get

$$\overline{\varpi}' = [\overline{\varpi}\,\overline{\varpi}_1] + [d\overline{y}_{2q+2}\,\overline{\varpi}_2] + \dots + [d\overline{y}_p\,\overline{\omega}_{p-2q}] + [\overline{\varpi}_{p-2q+1}\,\overline{\omega}_{p-2q+2}] + \cdots,$$

a formula which clearly shows the fact that all possible advantage has been drawn from the given integrals.

Moreover, we have obtained the canonical relations

$$\begin{aligned} \{\overline{y}_{2q+1}\} &= 1, \\ (\overline{y}_1\,\overline{y}_2) &= \dots = (\overline{y}_{2q-1}\,\overline{y}_{2q}) = 1, \\ (\overline{y}_1\,\overline{y}_{2q+1}) &= -\overline{y}_1, \quad (\overline{y}_3\,\overline{y}_{2q+1}) = -\overline{y}_3, \quad \dots, \quad (\overline{y}_{2q-1}\,\overline{y}_{2q+1}) = -\overline{y}_{2q-1}, \end{aligned}$$

where all the other quantities $\{\overline{y}_i\}, (\overline{y}_i\,\overline{y}_j)$ are zero.

3° *The form* ω *is of the second type and the forms* φ *and* Φ *are reducible to*

$$\varphi = \xi_1', \qquad \Phi = [\xi_1'\,\xi_2'] + \dots + [\xi_{2q-1}'\,\xi_{2q}'].$$

We have in this case

$$\omega' = [\overline{\omega}_1\,\overline{\omega}_2] + \dots + [\overline{\omega}_{2q-1}\,\overline{\omega}_{2q}] + [(\omega + \overline{\omega}_2)\,\omega_1] + [\overline{\omega}_{2q+1}\,\omega_2] + \dots + [\overline{\omega}_p\,\omega_{p-2q+1}] + \cdots$$

Again put

$$\Pi = [\overline{\omega}_1\,\overline{\omega}_2] + \dots + [\overline{\omega}_{2q-1}\,\overline{\omega}_{2q}]$$

and find the exterior derivative of ω' omitting terms in

$$\omega + \overline{\omega}_2, \quad \overline{\omega}_{2q+1},\ \dots, \overline{\omega}_p, \quad \omega_{p-2q+2},\ \dots$$

We get

$$\Pi' + [\Pi\,\omega_1] + [\overline{\omega}_2'\,\omega_1] + [\overline{\omega}_{2q+1}'\,\omega_2] + \dots + [\overline{\omega}_p'\,\omega_{p-2q+1}] = 0.$$

This identity allows us to put

$$\varpi_{2q+1} = d\bar{y}_{2q+1}, \quad ..., \quad \varpi_p = d\bar{y}_p\,;$$

we then see that if we regars $\bar{y}_{2q+1}, ..., \bar{y}_p$ as constants, ϖ_2' is equal to $-\Pi$ of rank $2q$, where the equation $\varpi_2 = 0$ is part of the associated system of ϖ_2'. We can thus assume

$$\varpi_2 = -(\bar{y}_1\, d\bar{y}_2 + \cdots + \bar{y}_{2q-1}\, d\bar{y}_{2q}).$$

Finally by putting

$$\overline{\omega} = \omega + \varpi_2 = \omega - \bar{y}_1\, d\bar{y}_2 - \cdots - \bar{y}_{2q-1}\, d\bar{y}_{2q},$$

we get

$$\overline{\omega}' = [\omega\,\overline{\varpi}_1] + [d\bar{y}_{2q+1}\,\overline{\varpi}_2] + \cdots + [d\bar{y}_p\,\overline{\varpi}_{p-2q+1}] + \cdots$$

We see that we have drawn all possible advantage from the known integrals, and we arrive moreover at the canonical relations

$$\begin{gathered} \{\bar{y}_1\} = -\bar{y}_1, \quad \{\bar{y}_3\} = -\bar{y}_3, \quad ..., \quad \{\bar{y}_{2q-1}\} = -\bar{y}_{2q-1}, \\ (\bar{y}_1\,\bar{y}_2) = (\bar{y}_3\,\bar{y}_4) = ... = (\bar{y}_{2q-1}\,\bar{y}_{2q}) = 1, \end{gathered}$$

where all other quantities $\{\bar{y}_i\}$, $(\bar{y}_i\,\bar{y}_j)$ are zero.

4° *The form* ω *is of the second type and the forms* φ *and* Φ *are reducible to*

$$\varphi = \xi_{2q+1}', \qquad \Phi = [\xi_1'\,\xi_2'] + \cdots + [\xi_{2q-1}'\,\xi_{2q}'].$$

We have in this case

$$\omega' = [\varpi_1\,\varpi_2] + \cdots + [\varpi_{2q-1}\,\varpi_{2q}] + [\omega\,\varpi_{2q+1}] + [\varpi_{2q+2}\,\omega_1] + \cdots + [\varpi_p\,\omega_{p-2q-1}] + \cdots$$

If we keep the same meaning as above for the letter Π, by omitting the terms in

$$\varpi_{2q+2},\ ..., \varpi_p, \quad \omega_{p-2q},\ ...,\ \omega_{2n-1-p},$$

we get the identity

$$\Pi' + [\Pi\,\varpi_{2q+1}] - [\omega\,\varpi_{2q+1}'] + [\varpi_{2q+2}'\,\omega_1] + \cdots + [\varpi_p'\,\omega_{p-2q-1}] = 0.$$

The exterior derivatives $\varpi_{2q+1}', \varpi_{2q+2}', ..., \varpi_p'$ are zero with $\varpi_{2q+1}, ..., \varpi_p$; we can thus put

$$\varpi_{2q+1} = \frac{d\bar{y}_{2q+1}}{\bar{y}_{2q+1}}, \quad \varpi_{2q+2} = d\bar{y}_{2q+2} \quad ..., \quad \varpi_p = d\bar{y}_p\,.$$

The exterior derivative of the form $\bar{y}_{2p+1}\,\Pi$ is then zero when we regard $\bar{y}_{2q+2},\ldots,\bar{y}_p$ as constants. We can thus put

$$\bar{y}_{2q+1}\,\Pi = [d\bar{y}_1\,d\bar{y}_2] + \cdots + [d\bar{y}_{2q-1}\,d\bar{y}_{2q}].$$

Finally, by putting

$$\overline{\omega} = \bar{y}_{2q+1}\,\omega - \bar{y}_1\,d\bar{y}_2 - \cdots - \bar{y}_{2q-1}\,d\bar{y}_{2q},$$

we get

$$\overline{\omega}' = [d\bar{y}_{2q+2}\,\overline{\omega}_1] + \cdots + [d\bar{y}_p\,\overline{\omega}_{p-2q-1}] + \cdots$$

We see clearly that we have drawn all possible advantage from the known integrals. Moreover, we have obtained the canonical relations

$$\begin{gathered}\{\bar{y}_{2q+1}\}\\ (\bar{y}_1\,\bar{y}_2) = (\bar{y}_3\,\bar{y}_4) = \ldots = (\bar{y}_{2q-1}\,\bar{y}_{2q}) = \bar{y}_{2q+1},\end{gathered}$$

where all other quantities $\{\bar{y}_i\}$, $(\bar{y}_i\,\bar{y}_j)$ are zero.

Chapter XIV

Differential equations that admit an invariant Pfaffian equation

I. — *General method of integration.*

136. We have already encountered (n^o **104**) the characteristic system of a Pfaffian equation

$$\omega \equiv a_1\,dx_1 + a_2\,dx_2 + \cdots + a_r\,dx_r = 0; \tag{1}$$

it is formed by the equations

$$\omega = 0, \qquad \frac{\dfrac{\partial \omega'}{\partial dx_1}}{a_1} = \frac{\dfrac{\partial \omega'}{\partial dx_2}}{a_2} = \cdots = \frac{\dfrac{\partial \omega'}{\partial dx_r}}{a_r}, \tag{2}$$

the last $r-1$ of which provide the associated system of the exterior quadratic form ω', when we assume that the variables are related by the relation $\omega = 0$.

This characteristic system was also encountered in the previous Chapter (n^o **128**) in connection with a Pfaffian expression ω of the second type.

The number of independent equations in the characteristic system (2) is always odd; in fact, taking into account the relation $\omega = 0$, we can put ω' into the form

$$\omega' = [\omega_1\,\omega_2] + \cdots + [\omega_{2n-1}\,\omega_{2n}]$$

where we denote by $\omega_1, \omega_2, ..., \omega_{2n}$ independent linear differential forms that are independent of each other and independent of ω. The characteristic system of equation (1) is then defined by the equations

$$\omega = \omega_1 = \omega_2 = \cdots = \omega_{2n} = 0.$$

As we can see, the integer n is the largest integer such that the form $[\omega\,\omega'\,n]$ is not is not zero. The class of the equation $\omega = 0$ is equal to the degree of this form.

137. It is easy to find a canonical form for equation (1). In fact, let y_1 be any first integral of the characteristic system (2); if we equal y_1 to an arbitrary constant C_1 and dy_1 to zero, the rank of the characteristic system of the new equation (1) is reduced by at least one unit, and *since this rank is odd*, it is reduced by at least two units. Let y_2 be a first integral of the new characteristic system. Putting

$$y_1 = C_1, \quad y_2 = C_2; \qquad dy_1 = 0, \quad dy_2 = 0,$$

the rank of the characteristic system of the given equation is reduced by at least four units, and so on. Finally, after at most $n+1$ operations, the equation $\omega = 0$ will be satisfied identically; in other words, this equation is of the form

$$z_1\,dy_1 + z_2\,dy_2 + \cdots + z_q\,dy_q + dy_{q+1} = 0 \qquad (q \leq n).$$

Moreover, the integer q is equal to n, otherwise equation (1) could be written by means of less than $2n+1$ variables.

Thus *if the characteristic system of equation* (1) *is of rank* $2n+1$*, this equation is reducible to the form*

$$dy_{q+1} + z_1\,dy_1 + z_2\,dy_2 + \cdots + z_n\,dy_n = 0, \qquad (q \leq n).$$

and the quantities

$$y_1, \ldots, y_{n+1}; \qquad z_1, \ldots, z_n$$

form a system of independent first integrals of its characteristic equations.

By this method, we see that *the reduction of equation (1) to its canonical form, and consequently the integration of its characteristic system, requires* $n+1$ *successive operations of orders*

$$2n+1, \quad 2n-1, \quad \ldots, \quad 3,, \quad 1,$$

and differentiations.

138. Note, as in Chapter XII (n° **120**), that knowing $N \geq n+1$ first integrals

$$y_1, y_2, \ldots, y_N$$

such that equation (1) is identically satisfied by equating these integrals to arbitrary constants, allows the integration of the characteristic equations to be completed by differentiations. In fact, equation (1) can be put, in one way and only one way, into the form

$$dy_n + z_1\,dy_1 + z_2\,dy_2 + \cdots + z_{N-1}\,dy_{N-1} = 0,$$

and we show that the coefficients $z_1, \dots, z_{N-1}$ are again first integrals of the characteristic of the characteristic equations.

More generally, we can propose to see to what the integration of the characteristic system reduces when we know a certain number r of independent first integrals of this system.

139. *First integrals in involution.* — We say that two first integrals f and g of the characteristic system of equation (1) are *in involution* if we have

$$[\omega\, \omega'^{\,n-1}\, df\, dg] = 0; \tag{3}$$

This definition is clearly independent of the choice of variables and also independent of the arbitrary factor by which we can multiply the left hand side of equation (1).

The property of two first integrals of being in involution leads to the important consequence that the *rank of the characteristic system is reduced by four units when the variables are assumed to be related by the two relations*

$$f = C, \qquad g = C',$$

where C and C' are two arbitrary constants. In fact, condition (3) expresses the fact that, if we assume that $df = dg = 0$ and $\omega = 0$, the rank of ω' is less than $2n-2$ and therefore equal to $2n-4$.

II. — *Using known integrals*

140. *The case where we know p independent first integrals $y_1, \dots, y_p$ pairwise in involution.* — In this case it follows from the developments in Chapter VI (n° **67**) that the rank of ω', when the differentials are assumed to be related by the relations

$$\omega = 0, \qquad dy_1 = 0, \qquad \dots, \qquad dy_p = 0,$$

is reduced to $2n-2p$. The characteristic system of equation (1), where we assume the variables to be related by the relations

$$y_1 = C_1, \qquad y_2 = C_2, \qquad \dots, \qquad y_p = C_p,$$

is thus of rank $2n-2p+1$, and its integration requires operations of order

$$2n-2p+1, \quad 2n-2p-1, \quad \dots, \quad 3, \quad 1$$

followed by differentiations.

The case just examined is one where the rank of the characteristic system is reduced at the outset by the maximum number of $2p$ units.

141. *Case where the given first integrals are not all in pairwise involution. —* Here the reduction of the rank of the characteristic system, when we equate the given first integrals to arbitrary constants, does not reach its upper limit $2p$. However, *we can determine a linear absolute integral invariant for the characteristic equations*, which in some cases can produce a much greater reduction in the problem of integration than in the first, apparently more favourable, case.

In fact, suppose that y_1 and y_2 are two first integrals of the characteristic equations that are not in involution; we will have

$$n[\omega\,\omega'^{\,n-1}\,dy_1\,dy_2] = A[\omega\,\omega'^{\,n}],$$

where the coefficient A is not zero. There are infinitely many (unknown) functions m such that

$$\varpi = m\omega$$

is an invariant form, that it, can be expressed by means of first integrals of the characteristic equations and their differentials. For such a form we have

$$\varpi' = m\omega' + [dm\,\omega]$$

and consequently

$$[\varpi\,\varpi'^{\,n-1}\,dy_1\,dy_2] = m^n[\omega\,\omega'^{\,n-1}\,dy_1\,dy_2], = [\varpi\,\varpi'^{\,n}] = m^{n+1}[\omega\,\omega'^{\,n}].$$

Comparing, we thus have

$$n[\varpi\,\varpi'^{\,n-1}\,dy_1\,dy_2] = \frac{A}{m}[\varpi\,\varpi'^{\,n}].$$

The two forms in square brackets are obviously invariant; consequently $\frac{A}{m}$ is a first integral; thus

$$\frac{A}{m}\,\varpi = A\omega$$

is an invariant form. This is the result we wanted to achieve:

If we know two first integrals y_1, y_2 such that the function A defined by the equality

$$n[\omega\,\omega'^{\,n-1}\,dy_1\,dy_2] = A[\omega\,\omega'^{\,n}].$$

is not zero, the linear form $A\omega$ is an absolute invariant form.

Note furthermore that the variables, whose number is the smallest in terms of which the form $A\omega$ can be expressed, are obviously the $2n+1$ first integrals of the given characteristic equations; *the characteristic system of the form $A\omega$ is therefore the same as that of equation (1)* and consequently of odd rank. *The form $A\omega$ is of the first type.*

142. We can related the previous theorem to a method of integration which can be widely generalised and which involves integrating the characteristic equations of the *form $u\,\omega$*, where u is an *auxiliary* variable. In fact, it is clear that to any solution to these equations there will correspond a solution of the *equation* $\omega = 0$ of the characteristic equations, namely that which is obtained by eliminating the auxiliary variable u between the relations that define the solution.

The form $u\,\omega$ is clearly of the second type, and the general method of integration of its characteristic equations explained in n° **128** is identical to that recalled in n° **137** for the characteristic equations of the equation $\omega = 0$. There is therefore no advantage, *if we know nothing a priori about the integrals*, in substituting consideration of the form $u\,\omega$ for that of the equation $\omega = 0$. But the advantage becomes obvious if we know *a priori* some first integrals of the characteristic equations, because we can apply to the integration of characteristic equations of the form $u\,\omega$ the method of use explained in n° **134**. In particular, if we know two first integrals y_1 and y_2 of the characteristic equations of the equation $\omega = 0$, we have

$$\{y_1\} = 0, \qquad \{y_2\} = 0,$$

and calculation of the parenthesis $(y_1 y_2)$, defined by (n° **132**)

$$(n+1)[(u\,\omega)'^{\,n}\, dy_1\, dy_2] = (y_1 y_2)(u\,\omega)'^{\,n+1},$$

gives, by expanding and equating the terms that contain du,

$$(y_1 y_2) = \frac{A}{u},$$

where A is the quantity defined in the preceding paragraph. We could continue the application of the general method by retaining the auxiliary variable u, forming the quantity $\{\frac{A}{u}\}$, the brackets of this quantity with y_1 and y_2, and so on. We note also that, since $\{\frac{A}{u}\}$ is a first integral of the characteristic equations of the form $u\,\omega$, the form $A\,\omega$ is itself invariant.

III. — *Application to first order partial differential equations.*

143. The problem of integrating the characteristic equations of a Pfaffian equation finds immediate application in the theory of first-order partial differential equations of the first order. In fact, integrating an equation

$$F\left(z,x_1,x_2,...,x_n;\frac{\partial z}{\partial x_1},\frac{\partial z}{\partial x_2},...,\frac{\partial z}{\partial x_n}\right)=0$$

or, using a classical notation,

$$F\left(z,x_1,x_2,...,x_n;p_1,p_2,...,p_n\right)=0, \tag{4}$$

is to determine $(n+1)$ functions $z,p_1,p_2,...,p_n$ of $x_1,x_2,...,x_n$ that satisfy equation (4) and the Pfaffian equation

$$\omega \equiv dx - p_1\,dx_1 - p_2\,dx_2 - \cdots - p_n\,dx_n = 0. \tag{5}$$

Now, suppose that we have solved equation (4) for one of the $2n+1$ arguments z,x_i,p_i as a function of the $2n$ others, Pfaff's equation (5) now contains only $2n$ variables and its characteristic system is necessarily of odd rank $2n-1$. Consequently, taking equation (4) into account, Pfaffian equation (5) is reducible to the canonical form

$$dZ - P_1\,dX_1 - P_2\,dX_2 - \cdots - P_{n-1}\,dX_{n-1} = 0, \tag{6}$$

where $Z,X_1,...,X_{n-1},P_1,...,P_{n-1}$ are $2n-1$ independent functions: these are the first integrals of the characteristic equations of equation (5).

That said, suppose that we know how to reduce equation (5) to the canonical form (6). Since the integration of equation (4) basically comes down to determining $n+1$ independent relationships between $z,x_1,...,x_n,p_1,...,p_n$ (including the relationship given in (4)) such that equation (5) follows, to arrive at this result it is sufficient to establish n independent relations between $Z,X_1,...,P_{n-1}$ which would result in equation (6). Now, in general, this is possible by taking

$$\begin{aligned} Z &= f(X_1,...,X_{n-1}),\\ P_1 &= \frac{\partial f}{\partial X_1},...,P_{n-1}=\frac{\partial f}{\partial X_{n-1}}, \end{aligned}$$

where f denotes an arbitrary function of its arguments. More generally, we will establish between $X,X_1,...,X_{n-1}$ any number $h \le n$ independent relations

$$\Phi(Z,X_1,...,X_{n-1})=0, \qquad \Phi_2=0, \qquad ..., \qquad \Phi_h=0,$$

will add the relations obtained by eliminating the homogeneous parameters $\lambda_1,...,\lambda_h$ between the $n-1$ equations

$$\begin{aligned} \lambda_1\left(\frac{\partial\Phi_1}{\partial X_1}+P_1\frac{\partial\Phi_1}{\partial Z}\right)+\lambda_2\left(\frac{\partial\Phi_2}{\partial X_1}+P_1\frac{\partial\Phi_2}{\partial Z}\right)+\cdots+\lambda_h\left(\frac{\partial\Phi_h}{\partial X_1}+P_1\frac{\partial\Phi_h}{\partial Z}\right) &= 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdots\\ \lambda_1\left(\frac{\partial\Phi_1}{\partial X_{n-1}}+P_{n-1}\frac{\partial\Phi_1}{\partial Z}\right)+\lambda_2\left(\frac{\partial\Phi_2}{\partial X_{n-1}}+P_{n-1}\frac{\partial\Phi_2}{\partial Z}\right)+\cdots+\lambda_h\left(\frac{\partial\Phi_h}{\partial X_{n-1}}+P_{n-1}\frac{\partial\Phi_h}{\partial Z}\right) &= 0. \end{aligned}$$

144. The equations

$$\begin{aligned} &X_1 = a_1, \quad \ldots, \quad X_{n-1} = a_{n-1},\\ &P_1 = b_1, \quad \ldots, \quad P_{n-1} = b_{n-1},\\ &Z = c \end{aligned}$$

define one-dimensional multiplicities, which are the characteristic multiplicities of the Pfaffian equation (5) (where we assume the variables are related by relation (4)). *These are known as the characteristics of the partial differential equation* (4). We see immediately that any integral surface is generated by characteristics.

It is easy to form the differential equations of the characteristics; in fact, these are the equations of the associated system of ω', where the differentials of the variables are assumed to be related by the relation $\omega = 0$, and *also by the relation* $dF = 0$. We thus obtain them (n° **104**) by adding to equation (5) the equations

$$\left\| \begin{matrix} \dfrac{\partial \omega'}{\partial(dz)} & \dfrac{\partial \omega'}{\partial(dx_1)} & \dfrac{\partial \omega'}{\partial(dx_2)} & \cdots & \dfrac{\partial \omega'}{\partial(dp_1)} & \cdots & \dfrac{\partial \omega'}{\partial(dp_n)} \\ 1 & -p_1 & -p_2 & \cdots & 0 & \cdots & 0 \\ \dfrac{\partial F}{\partial(dz)} & \dfrac{\partial F}{\partial(dx_1)} & \dfrac{\partial F}{\partial(dx_2)} & \cdots & \dfrac{\partial F}{\partial(dp_1)} & \cdots & \dfrac{\partial F}{\partial(dp_n)} \end{matrix} \right\| = 0,$$

which can be written as

$$\frac{dx_1}{\dfrac{\partial F}{\partial p_1}} = \cdots = \frac{dx_n}{\dfrac{\partial F}{\partial p_n}} = \frac{-dp_1}{\dfrac{\partial F}{\partial x_1} + p_1 \dfrac{\partial F}{\partial z}} = \cdots = \frac{-dp_n}{\dfrac{\partial F}{\partial x_n} + p_n \dfrac{\partial F}{\partial z}}. \tag{7}$$

We rediscover the classical equations.

IV. — *Cauchy's method*

145. The method just described basically comes down to integrating the characteristic equations and to reducing equation (5) to its canonical form (6), this reduction resulting from the rest of the integration, if the latter is managed properly (n° **137**). It is easy to see that, whatever the method used to integrate the characteristic equations, the reduction of equation (5) to its normal form is always possible, once the integration of the characteristic equations has been carried out. In fact, it is necessary to determine the first integrals which, for a given numerical value x_n^0 of x_n, reduce respectively to

$$z, \; x_1, \; \ldots, \; x_{n-1}, \; p_1, \; \ldots, \; p_n.$$

If we denote these first integrals by

$$Z,\ X_1,\ \ldots,\ X_{n-1},\ P_1,\ \ldots,\ P_n,$$

necessarily related by the rleation

$$F(Z, X_1, \ldots, X_{n-1}, x_n^0; P_1, \ldots, P_n) = 0,$$

relation (5) which, as we know, can be expresses by means of the first integrals, will clearly reduce to

$$dZ - P - 1\, dX_1 - \cdots - P_{n-1}\, dX_{n-1} = 0.$$

This is the principle of *Cauchy's method*.

V. — *Lagrange's method*

146. Lagrange's method of the *complete integral* is also easily related to the preceding point of view. The equation

$$V(z, x_1, \ldots, x_n; a_1, \ldots, a_n) = 0 \tag{8}$$

defines a complete integral if it defines a function of z that satisfies equation (4) whatever the n arbitrary *constants* $a_1, \ldots, a_n$ may be. Moreover, equation (4) is the only equation satisfied by all the functions z defined by (8); because eliminating $a_1, \ldots, a_n$ between equation (8) and the equations

$$\left.\begin{array}{c} \dfrac{\partial V}{\partial x_1} - p_1 \dfrac{\partial V}{\partial z} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial V}{\partial x_n} - p_n \dfrac{\partial V}{\partial z} = 0, \end{array}\right\} \tag{9}$$

deduced from it, in general lead (and this is what we assume) to only one relation which is naturally equation (4).

Since equation (4) is the result of eliminating $a_1, \ldots, a_n$ from the $(n+1)$ equations (8) and (9), to integrate equation (4) amounts to satisfying Pfaffian equation (5), assumed to have $3n+1$ variables z, x_i, p_i, a_i related by the $(n+1)$ relations (8) and (9). Now, taking these relations into account, we have

$$0 = \frac{\partial V}{\partial z} dz + \frac{\partial V}{\partial x_1} dx_1 + \cdots + \frac{\partial V}{\partial a_1} da_1 + \cdots$$
$$= \frac{\partial V}{\partial z}(dz - p_1\, dx_1 - \cdots - p_n\, dx_n) + \frac{\partial V}{\partial a_1} da_1 + \cdots + \frac{\partial V}{\partial a_n} da_n.$$

The Pfaffian equation (5) is thus equivalent to the equation

$$\frac{\partial V}{\partial a_1} da_1 + \frac{\partial V}{\partial a_2} da_2 + \cdots + \frac{\partial V}{\partial a_n} da_n = 0;$$

but the latter is reduced by itself to its normal form by putting

$$X_1 = a_1, \quad \ldots, \quad X_{n-1} = a_{n-1}, \quad Z = a_n,$$

$$P_1 = -\frac{\dfrac{\partial V}{\partial a_1}}{\dfrac{\partial V}{\partial a_n}}, \quad \ldots, \quad P_{n-1} = -\frac{\dfrac{\partial V}{\partial a_{n-1}}}{\dfrac{\partial V}{\partial a_n}}.$$

We see that the characteristics are defined by the equations

$$V = 0,$$
$$\frac{\partial V}{\partial a_1} + b_1 \frac{\partial V}{\partial a_n} = 0, \quad \ldots, \quad \frac{\partial V}{\partial a_{n-1}} + b_{n-1} \frac{\partial V}{\partial a_n} = 0;$$

this is a classical result.

147. We now apply the theorem of n$^{\text{o}}$ **141** to the special case of an equation in two independent variables

$$F(x,y,z,p,q) = 0. \tag{10}$$

The knowledge of two independent first integrals u and v of the characteristic equations leads, when they are not in involution, to the determination of a linear integral invariant for the characteristic equations. This integral invariant is $A\,\omega$, where A is defined by the equality

$$[\omega\, du\, dv] = A[\omega\, \omega'],$$

or rather, since we assume here that the variables are related by relation (10),

$$[dF\, \omega\, du\, dv] = A[dF\, \omega\, \omega'],$$

In particular, take the terms in $[dx\, dz\, dp\, dq]$ on the two sides, we find

$$A = \frac{1}{\dfrac{\partial F}{\partial q}} \begin{vmatrix} \dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial p} & \dfrac{\partial F}{\partial q} \\ \dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial p} & \dfrac{\partial u}{\partial q} \\ \dfrac{\partial v}{\partial x} + p\dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial p} & \dfrac{\partial v}{\partial q} \end{vmatrix}.$$

If therefore the determinant on the right hand side is not zero, the expression $A(dz - p\,dx - q\,dy)$ is an invariant form for the equations of the characteristics.

VI. — *First order partial differential equations that admit an infinitesimal transformation.*

148. If the first order partial differential equation

$$F(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0 \tag{4}$$

admits an infinitesimal transformation Af involving the variables $z, x_1, \ldots, p_n$, this means that any system of $n+1$ relations between these $2n+1$ variables which defines an integral multiplicity is changed by the transformation into another system of $n+1$ relations again defining an integral multiplicity. Therefore, taking into account equation (4), the Pfaffian equation

$$\omega \equiv dz - p_1 x_1 - \cdots - p_n\,dx_n = 0 \tag{5}$$

admits the infinitesimal transformation Af. It follows immediately (n$^{\text{p}}$ **97**) that the linear form

$$\frac{\omega(\delta)}{\omega(A)} = \frac{dz - p_1 x_1 - p_2\,dx_2 - \cdots - p_n\,dx_n}{A(z) - p_1 A(x_1) - p_2 A(x_2) - \cdots - p_n A(x_n)}$$

is an invariant for the system of differential equations of the characteristics.

Knowledge of an infinitesimal transformation thus leads to knowledge of a linear integral invariant for the equations of the characteristics and consequently integration of the given equation, which was a problem of the second type requiring operations of orders

$$2n+1, \quad 2n-1, \quad \ldots, \quad 3, \quad 1,$$

is reduced to a problem of the first type requiring operations of orders

$$2n, \quad 2n-2, \quad \ldots, \quad 2, \quad 0.$$

149. A classic example is that where the given equation (1) does not depend explicitly on z: it is then obvious that from any solution of the equation we deduce another by adding an arbitrary constant to z; in other words, the given equation admits the infinitesimal transformation

$$Af = \frac{\partial f}{\partial z}.$$

The absolute integral invariant which admits the equations of the characteristics is then

$$\int \omega_\delta = \int dz - p_1\, dx_1 - \cdots - p_n\, dx_n.$$

The method of integration of equations of this kind follows from the theory of Chapter XII. The characteristic equations of ω' are here

$$\frac{dx_1}{\dfrac{\partial F}{\partial p_1}} = \cdots = \frac{dx_n}{\dfrac{\partial F}{\partial p_n}} = \frac{-dp_1}{\dfrac{\partial F}{\partial x_1}} = \cdots = \frac{-dp_n}{\dfrac{\partial F}{\partial x_n}};$$

once we have determined $n-1$ first integrals in pairwise involution, the integration of the characteristic equations of ω reduces to a quadrature, the expression ω becoming an exact differential when we equate the $n-1$ first integrals to arbitrary constants.

VII. — *Jacobi's first method*

150. Jacobi's first method for integrating first-order partial differential equations is related to the preceding considerations. Jacobi reduces equation (4), assumed arbitrary, to an equation in which the unknown function no longer appears, namely

$$F\left(z, x_1, \ldots, x_n, -\frac{\dfrac{\partial V}{\partial x_1}}{\dfrac{\partial V}{\partial z}}, \ldots, -\frac{\dfrac{\partial V}{\partial x_n}}{\dfrac{\partial V}{\partial z}}\right) 0.$$

To abbreviate, put

$$\frac{\partial V}{\partial z} = u,$$

the characteristic equations to be integrated are those of the absolute invariant form

$$\delta V - u(\delta z - p_1\, \delta x_1 - \cdots - p_n\, \delta x_n),$$

whose $2n+3$ variables are related by the relation (4); they admit the *relative* integral invariant

$$\int u(\delta z - p_1\,\delta x_1 - \cdots - p_n\,\delta x_n), \tag{11}$$

and it is the characteristic equations of this integral invariant that we integrate first using the methods of Chapter XII.

Jacobi's method is similar to that described in n$^{\text{o}}$ **142**, with the difference that the latter uses integral (11) as an *absolute* integral invariant, whereas Jacobi's method uses it as a *relative integral invariant*. Moreover, Jacobi's method leads to operations of orders

$$2n+2, \quad 2n, \quad \ldots, \quad 2, \quad 0,$$

instead of

$$2n+1, \quad 2n-1, \quad \ldots, \quad 1.$$

Its advantage is that it allows one to use knowledge of first integrals given by applying the Poisson-Jacobi theorem. But, this advantage is retained by the method of n$^{\text{o}}$ **142**, which takes full advantage of given first integrals.

VIII. — *Reducing certain differential equations to a first order partial differential equation.*

151. We can now adopt a point of view inverse to that of the previous paragraphs.

Consider first a Pfaffian equation in an even number $2s$ of variables, but suppose that only $s+1$ of the coefficients are different from zero:

$$\omega \equiv a_1\,dx_1 + a_2\,dx_2 + \cdots + a_{s+1}\,dx_{s+1} = 0.$$

The characteristic equations of this Pfaffian equation are obviously the same as those of the first order partial differential equation with s independent variables $x_1, x_2, \ldots, x_n$ obtained by putting

$$x_{s+1} = z, \qquad a_1 + p_1 a_{s+1} = 0, \qquad \ldots, \qquad a_s + p_s a_{s+1} = 0,$$

and by eliminating $x_{s+1}, x_{s+2}, \ldots, x_{2s}$ between these $s+1$ equations. Of course, we must assume that elimination is possible and gives a single relation.

152. Secondly, consider a system of differential equations that admit a relative linear integral invariant $\int \omega$, where the form ω has $2s+1$ variables, and where $[\omega'^{s}]$ is different from zero. The differential equations considered are the characteristic equations of ω'. Their integration can be reduced to that of a first-order partial differential equation which does not explicitly contain the unknown function if the coefficients of s of the differentials are zero in ω:

$$\omega \equiv a_1\,dx_1 + a_2\,dx_2 + \cdots + a_{s+1}\,dx_{s+1}.$$

In fact, consider the Pfaffian equation

$$dV - \omega \equiv dV - a_1\,dx_1 - a_2\,dx_2 - \cdots - a_{s+1}\,dx_{s+1} = 0,$$

and put

$$p_1 = a_1, \qquad p_2 = a_2, \qquad \ldots, \qquad p_{s+1} = a_{s+1};$$

the elimination of $x_{s+2},\ldots,x_{2s+1}$ between the $s+1$ equations leads to a relation

$$F(x_1,\ldots,x_{s+1};\ p_1,\ldots,p_{s+1}) = 0, \tag{12}$$

which is none other than the stated partial differential equation. The differential equations of the characteristics of this equation are formed by the equations of the characteristics of ω', to which we add the equation

$$dV - \omega = 0.$$

We see easily that the method of integration, pointed out in Chapter XII, of the characteristic equations of ω' leads to the same operations as the operations as the search for the characteristics of the partial differential equation equation (12).

If the invariant ω is that of the equations of Dynamics :

$$\omega = p_1\,\delta q_1 + \cdots + p_n\,\delta q_n - H\,\delta t,$$

equation (12) is none other than Jacobi's equation

$$\frac{\partial V}{\partial t} + H\left(t, q_1,\ldots,q_n;\ \frac{\partial V}{\partial q_1},\ldots,\frac{\partial V}{\partial q_n}\right) = 0.$$

153. Jacobi's method for integrating the equations of Dynamics is thus rests ultimately on the identity of two problems of integration, that of the characteristic system of a relative linear integral invariant $\int \omega$, and that of the characteristic equations of a first order partial differential equation that admits an infinitesimal transformation (*for example*, not containing explicitly the unknown function). In both cases, the nature of the problem is determined by the existence of an integral invariant $\int \omega$.

This method of reduction to a partial differential equation only succeeds if the form ω in $2s+1$ variables has s zero coefficients, but it should not be thought because of this that, in the absence of this particular, the integration of the characteristic equations of ω' is a problem more complicated than the search for the characteristics of a first order partial differential equation which does not explicitly contain the unknown function or, which comes to the same thing, the integration of a system of *canonical* differential equations. Basically, *the importance of canonical equations lies solely in their property of admitting an integral invariant* $\int \omega$, *and not in their simple form*: it is the existence of the integral invariant that is the fundamental property from which all others derive.

IX. — *Remarks on the nature of the main applications of Jacobi's method.*

154. In fact, most of the fruitful applications that Jacobi's method has had in Dynamics have their origin in the simplifications offered by the search for a complete integral of Jacobi's partial differential equation, obtained as the sum of functions in each of which only a part of the variables $q_1, ..., q_n$ other than t appears. But these simplifications can be made clear without recourse to the theory of first order partial differential equations and of the complete integral.

Let ω be a linear differential form with $2s+1$ variables, which we will denote by

$$x_1,\ ...,\ x_{2s},\ t.$$

Suppose that ω can be decomposed into a sum of p terms

$$\omega = \omega_1 + \omega_2 + \cdots + \omega_p,$$

where the form ω_i is constructed with a certain number $2h_i$ of the variables x and the variable t, in such a way that the variables x which enter into the formation of any two of the forms $\omega_1, ..., \omega_p$ are different. We then have

$$s = h_1 + h_2 + \cdots + h_p.$$

If we suppose that the exterior quadratic form ω' is of rank $2s$, it is necessary that the forms $\omega_1', \omega_2', ..., \omega_p'$ be respectively of rank $2h_1, 2h_2, ..., 2h_p$. Reduction of each of these p forms to its canonical form then leads to the same reduction for ω'. Consequently, *integration of the characteristic equations of ω' is equivalent to integration of the characteristic equations of $\omega_1', \omega_2', ..., \omega_p'$*, and the p corresponding problems can be solved independently of each other.

An even greater simplification would occur if the numbers k_i of the variables x (different for the different forms ω_i) which enter into these forms at the same time as t, were not all even. In this case, the variable t would be a first integral of the characteristic equations of ω': in fact, by giving t an arbitrary constant value, the rank of the quadratic form ω_i' would be reduced

for k_i even, to k_i at most,
for k_i odd to $k-1$ at most;

now $2s$ is equal to the sum of all the k_i; the rank of ω', for constant t, would therefore be less than $2s$, which is what needed to be proved. We see moreover that only two of the numbers k_i can be odd and that the reduction of ω' to its normal form, when t is constant, is provided by the reductions to their normal forms of $\omega_1', \ldots, \omega_p'$, when we also make t constant.

Chapter XV

Differential equations which admit several linear integral invariants

I. — *Case where as many are known integral invariants as there are unknown functions.*

155. In these lectures we will not deal with the general problem of integration of differential equations that admit any number of integral invariants. We confine ourselves to the particularly simple case where a system of n first-order ordinary differential equations with n unknown functions admits n (independent) invariant linear forms

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_n,$$

that is, n linear absolute integral invariants

$$\int \omega_1, \quad \int \omega_2, \quad \ldots, \quad \int \omega_n.$$

In this particular case the given differential equations can be written as

$$\omega_1 = \omega_2 = \cdots = \omega_n = 0. \tag{1}$$

The exterior quadratic forms $\omega_1', \omega_2', \ldots, \omega_n'$ being invariant, they can be expressed in terms of $\omega_1, \omega_2, \ldots, \omega_n$ by formulae such as

$$\omega_s' = \sum_{(i,k)}^{1,\ldots,n} c_{iks}[\omega_i\, \omega_k] \qquad (s = 1, 2, \ldots, n). \tag{2}$$

The coefficients c_{iks} are clearly first integrals of the given differential equations. We will see that this can always be reduced to the case where they are constants.

In fact, suppose that among the integrals c_{iks} there is a certain number r of independent ones, which we will denote by

$$y_1,\ y_2,\ \ldots,\ y_r;$$

the c_{iks} are thus specific functions of these r integrals. Each differential dy_i is in turn an invariant form which can be expressed linearly in terms of $\omega_1,\ldots,\omega_n$:

$$dy_i = b_{i1}\omega_1 + b_{i2}\omega_2 + \cdots + b_{in}\omega_n \qquad (i = 1,2,\ldots,r).$$

The coefficients b_{ik} are in turn first integrals; if among these there are r' independent of each other and independent of y_i, their differential differentials $dy_{r+1},\ldots,dy_{r+r'}$ can also be expressed linearly as a function the ω_i and the coefficients can provide new first integrals, and so on. There will come a time when these operations will come to an end and we will arrive at a certain number $\rho \le n$ of first integrals $y_1,\ldots,y_\rho$ such that the coefficients c_{iks} of formulae (2) and the coefficients b_{ik} of the formulae

$$dy_i = b_{i1}\omega_1 + b_{i2}\omega_2 + \cdots + b_{in}\omega_n \qquad (i = 1,2,\ldots,\rho). \tag{3}$$

will be specific functions of the $y_1, y_2, \ldots, y_\rho$.

That said, suppose for definiteness that the determinant obtained by taking the first ρ columns of the matrix of the b_{ik} is not zero. We can then then substitute for the ρ invariant forms $\omega_1,\ldots,\omega_\rho$ the invariant forms $dy_1,\ldots,dy_\rho$. If we assign arbitrary constant values to $y_1,\ldots,y_\rho$, the system of equations to be integrated will admit $n-\rho$ invariant forms $\omega_{\rho+1},\ldots,\omega_n$ and we will have

$$\omega_s' = \sum_{(ik)}^{\rho+1,\ldots,n} c_{iks}[\omega_i\,\omega_k] \qquad (s = \rho+1,\ldots,n),$$

where the coefficients c_{iks} are now constants.

156. Take the case where in formulae (2) the coefficients c_{iks} are all constants. Firstly, it is easy to see that, conversely, the existence of relations like (2) leads as a consequence to the property of the forms $\omega_1,\ldots,\omega_n$ that they are invariant as regards the differential equations (1): in fact, the characteristic system of the set of forms $\omega_1,\omega_2,\ldots,\omega_n$ is obtained (n° **78**) by adding to equations (1) the equations of the associated systems of $\omega_1', \omega_2', \ldots, \omega_n'$, equations which are all consequences of the equations (1).

If we substitute for the forms $\omega_1,\ldots,\omega_n$ the linear combinations with constant coefficients

$$\left.\begin{aligned} \overline{\omega}_1 &= a_{11}\omega_1 + a_{12}\omega_2 + \cdots a_{1n}\omega_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \overline{\omega}_n &= a_{n1}\omega_1 + a_{n2}\omega_2 + \cdots a_{nn}\omega_n, \end{aligned}\right\} \tag{4}$$

the n new forms are again invariant and we again have the relations

$$\overline{\varpi}'_s = \sum_{(ik)}^{1,\dots,n} \overline{c}_{iks}[\overline{\varpi}_i \overline{\varpi}_k]$$

with new constants $\overline{c}_{iks}$. We will say that the matrix of the $\overline{c}_{iks}$ has the same same *structure* as the matrix of the c_{iks}.

It may be possible to choose the constant coefficients a_{ij} of the substitution (4) in such a way that, in the expression of the first $\nu < n$ derivatives $\overline{\varpi}'_1, \dots, \overline{\varpi}'_\nu$, only $\overline{\varpi}_1, \dots, \overline{\varpi}_\nu$ appear, that is, such that we have

$$\overline{c}_{\nu+i,ks} = \overline{c}_{k,\nu+i,s} = \overline{c}_{\nu+i,\nu+j,s} = 0 \qquad (k,s = 1,2,\dots,\nu;\ i,j = 1,\dots,n-\nu).$$

In this case, the forms $\overline{\varpi}_1, \dots, \overline{\varpi}_\nu$ are invariant for the *completely integrable* system of Pfaffian equations

$$\overline{\varpi}_1 = \overline{\varpi}_2 = \cdots = \overline{\varpi}_\nu = 0.$$

If we know how to integrate this system and we equate its first integrals to arbitrary constants, the given system reduces to a system similar to the first, except that n is replaced by $n-\nu$.

We will say that the matrix of the c_{iks} is *simple* if it is impossible to find a linear substitution with constant coefficients (4) that achieves the preceding reduction. We then see that *the given system of differential equations can be reduced to successive systems for each of which the matrix of the c_{iks} is simple*. Each simple matrix corresponds to a particular integration problem.

157. Leaving aside for the moment this method of reduction, imagine a second system of differential equations

$$\varpi_1 = \varpi_2 = \cdots = \varpi_n = 0 \tag{1'}$$

which admits the n invariant forms ϖ_i with the relations

$$\varpi'_s = \sum_{(ik)}^{1,\dots,n} c_{iks}[\varpi_i \varpi_k]. \tag{2'}$$

where the coefficients c_{iks} have the same numerical values as in formulae (2). Let

$$\begin{array}{llll} y_1, & y_2, & \dots, & y_n, \\ z_1, & z_2, & \dots, & z_n \end{array}$$

be two systems of independent first integrals, the first for equations (1), the second for equations (1'). Then $\omega_1, \omega_2, \dots, \omega_n$ can be expressed by means of y_i and their differentials, in the same way as $\varpi_1, \varpi_2, \dots, \varpi_n$ can be expressed by means of the z_i and their differentials. *It is possible to choose the first integrals z_i such that the ϖ_i are expressed by means of the z_i and the dz_i in the same way as the ω_i are expressed by means of y_i and the dy_i*. This comes down to saying that if this condition is not fulfilled, we can at least find functions

$$f_1(y_1,\ldots,y_n),\ \ldots,\ f_n(y_1,\ldots,y_n)$$

such that by putting

$$z_1 = f_1(y_1,\ldots,y_n),\ \ldots,\ z_n = f_n(y_1,\ldots,y_n)$$

the ϖ_i become respectively equal to the ω_i. For this, it is sufficient to integrate the total differential equations

$$\left.\begin{array}{l} \varpi_1 - \omega_1 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \varpi_n - \omega_n = 0, \end{array}\right\} \tag{5}$$

where the z_i are unknown functions of the independent variables y_i. Pfaffian system (5) is *completely integrable* (n$^{\text{o}}$ **101**) because, if we take into account equations (5), the exterior derivatives

$$\varpi'_s - \omega'_s = \sum c_{iks}[\varpi_i\,\varpi_k] - \sum c_{iks}[\omega_i\,\omega_k]$$

of their left hand sides all cancel. Thus it is possible, and that in an infinite number of ways (depending on n arbitrary constants), to satisfy the stated conditions.

This proves in particular that the integrations of the two systems (1) and (1') are two problems of *essentially* the same nature, in the sense that any method *that uses for the integration only the given properties* $\omega_1,\ldots,\omega_n$ *of being invariant forms*, can be applied in parallel to the systems (1) and (1'), any progress in the integration of (1) having its equivalent in the integration of (1').

II. — *The group that conserves the given invariants.*

158. Return to system (1), and imagine a choice of n independent first integrals

$$y_1,\ y_2,\ \ldots,\ y_n.$$

It is possible to find an infinite number of other systems of n independent first integrals

$$\overline{y}_1,\ \overline{y}_2,\ \ldots,\ \overline{y}_n$$

such that the forms ω_s are expressed by means of $\overline{y}_i$ and their differentials in the same way as with the y_i and their differentials. For this, it is sufficient to integrate the Pfaffian system

$$\left.\begin{array}{l} \overline{\omega}_1 - \omega_1 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \overline{\omega}_n - \omega_n = 0, \end{array}\right\} \tag{6}$$

where $\overline{\varpi}_s$ denotes the same function of the $\bar{y}_i$ and the $d\bar{y}_i$ as ω_s is of the y_i and the dy_i. In this Pfaffian system, regard the arguments $\bar{y}_1,...,\bar{y}_n$ as unknown functions of the independent variables $y_1,...,y_n$. Such a system is completely integrable for the same reason as bas been indicated for the system system (5). There are thus functions

$$\bar{y}_s = f_s(y_1,...,y_n;\, C_1,...,C_n) \qquad (s=1,...,n) \tag{7}$$

which depend on n arbitrary constants and satisfy the conditions stated above.

Equations (7) define an infinite number of transformations performed on the first integrals $y_1,...,y_n$ and *which preserve the data of the problem*, that is, which leave the forms $\omega_1,...,\omega_n$ invariant. These transformations form a group G, because being characterised by the property of conserving $\omega_1,...,\omega_n$, it is very obvious that by performing two transformations of type (7) one after the other, we again obtain a resultant transformation of the same type. This group G is a finite group with n parameters: *it is the largest group which, applied to the first integrals of the given system, preserves the given invariant forms*. As is easily understood, the benefit derived from knowledge of these n invariant forms depends on the nature of this group. Moreover, this is a general fact that applies to all cases where we know *a priori* integral invariants, systems of invariant equations, infinitesimal transformations, etc. The nature of the largest group of transformations which, applied to first integrals of the given differential equations (or, which amounts to the same thing, to their integral curves considered as indivisible entities), preserves the known information, is of paramount importance in the integration of the system.

In the case at hand, we see in particular that it is impossible, based solely on the fact that $\omega_1,...,\omega_n$ are invariant forms, to obtain any first integral without integration[1]; in fact, otherwise the property of the forms $\omega_1,...,\omega_n$ of being invariant would of itself make it possible to *single out* a first integral, say y, which would consequently have to be be equal to any one of the integrals $\bar{y}_i$ defined by the formulae (7); but this is clearly impossible, because equations (6) always admit a solution such that to given numerical values of the $y_1,...,y_n$, correspond *arbitrary* numerical values of the $\bar{y}_1,...,\bar{y}_n$.

159. The constants c_{iks} play an important role with respect to the group G: they are what, in group theory, are called the *structure constants* of this group. The method of reduction described above (n° **156**) is based precisely on the decomposition of G into a normal series of sub-groups. The case where the matrix of the c_{iks} is *simple* corresponds to *simple* groups.

We know that the structure constants of a group are not arbitrary; we can verify this here by expressing the fact that the exterior derivatives of $\omega'_1,...,\omega'_n$ are zero. The exterior derivative of ω'_s, using the expressions (2) for $\omega'_1,...,\omega'_n$, is (n° **73**)

$$\sum_{(ik)}^{1,...,n} c_{iks}\left([\omega'_i\,\omega_k]-[\omega_i\,\omega'_k]\right) = \sum_{(\alpha\beta\gamma)}^{1,...,n}\left(\sum_{i=1}^{i=n} c_{\alpha\beta i}c_{i\gamma s}+c_{\beta\gamma i}c_{i\alpha s}+c_{\gamma\alpha i}c_{i\beta s}\right)[\omega_\alpha\,\omega_\beta\,\omega_\gamma].$$

[1] This means by any sequence of one-to-one operations applied to $\omega_1,...,\omega_n$ and capable of being carried out whatever the nature of the coefficients of these forms.

We thus get the necessary relations

$$\sum_{i=1}^{i=n} c_{\alpha\beta i}c_{i\gamma s} + c_{\beta\gamma i}c_{i\alpha s} + c_{\gamma\alpha i}c_{i\beta s} = 0 \qquad (\alpha, \beta, \gamma, s = 1, 2, ..., n).$$

In the theory of groups, we show that they are sufficient for the existence of a group that admits the c_{iks} as its structure constants.

III. — *Examples.*

160. Suppose all the constants c_{iks} are zero. It is clear then that, since all the forms $\omega_1, ..., \omega_n$ are exact differentials, the integration requires only n independent quadratures. Since the forms $\omega_1, ..., \omega_n$ are reducible to

$$\omega_1 = dy_1, \qquad \omega_2 = dy_2, \qquad ..., \qquad \omega_n = dy_n,$$

the group G has equations

$$y'_s = y_s + C_s, \qquad (s = 1, 2, ..., n).$$

The preceding case always arises if $n = 1$.

We look at all possible cases for $n = 2$. Apart from the case just examined, we can have

$$\omega'_1 = a[\omega_1\, \omega_2],$$
$$\omega'_2 = b[\omega_1\, \omega_2],$$

where the coefficients a and b are not both zero. Suppose for example that $b \neq 0$. By taking $a\omega_2 - b\omega_1$ as a new form $\overline{\omega}_1$, we see immediately that we have

$$\overline{\omega}'_1 = 0,$$
$$\omega'_2 = [\overline{\omega}_1\, \omega_2].$$

A first quadrature gives

$$\overline{\omega}'_1 = dy_1;$$

then equating y_1 to an arbitrary constant, ω_2 becomes an exact differential and a second quadrature completes the integration. Changing the notation a little, we can assume that

$$\omega_1 = \frac{dy_1}{y_1},$$
$$\omega_2 = \frac{dy_2}{y_1}.$$

The group G has equations

$$y_1' = C_1 y_1,$$
$$y_2' = C_1 y_2 + C_2.$$

161. We will not go through a general discussion for $n = 3$. We point out only the most interesting case in which we can reduce formulae (2) to

$$\omega_1' = [\omega_1\, \omega_2],$$
$$\omega_2' = [\omega_1\, \omega_3],$$
$$\omega_3' = [\omega_2\, \omega_3].$$

In this case, *the integration of equations (1) reduces to that of a Riccati equation*.

In fact, consider the Pfaffian equation

$$dt + \omega_1 + t\omega_2 + \frac{1}{2}t^2\omega_3 = 0, \tag{8}$$

where t is regarded as an unknown function of the dependent and independent variables which appear in the given differential equations. *This equation is completely integrable*: in fact, we can easily verify that the exterior derivative of its left hand side is zero if we take into account the equation itself (and if we use the expressions for $\omega_1', \omega_2', \omega_3'$). Consequently, as we know, we can reduce its integration to that of an ordinary differential equation, which is obviously a Riccati equation. Now if we denote by y_1, y_2, y_3 a system of independent first integrals of the given equations (1), the expressions $\omega_1, \omega_2, \omega_3$ can be expressed by means of the three quantities y_1, y_2, y_3 and their differentials: the general solution t of equation (1) is thus a function of y_1, y_2, y_3 (and of an arbitrary constant C). Consequently, if we have integrated the Riccati equation (8) in the classical form

$$t = \frac{\alpha + C\beta}{\gamma + C\delta}$$

the mutual ratios of the four functions $\alpha, \beta, \gamma, \delta$ provide three first integrals of the given equations, and it we can show easily that they are independent.

IV. — *Generalisations.*

162. We will not press this theory any further which, to be developed properly, would require quite extensive knowledge of group theory. We see how the latter is necessarily introduced if we want to push all the way the methods of integration of differential equations which admit given integral invariants. We only point out that the method indicated in n° **142** can be generalised to any system of differential equations that admit invariant forms, invariant Pfaffian equations, etc. It consists in forming, *by the introduction of auxiliary variables*, as many linear integral invariants as the given system of equations has independent first integrals. An example will be sufficient for understanding the spirit of the method.

Suppose that we are to integrate a system of differential equations (S) in four variables

$$\omega_1 = \omega_2 = \omega_3 = 0,$$

and that each of these equations $\omega_1 = 0, \omega_2 = 0, \omega_3 = 0$ is invariant for the given system. We will introduce three new auxiliary variables u_1, u_2, u_3 and we will consider the three forms

$$\overline{\omega}_1 = u_1\omega_1, \qquad \overline{\omega}_2 = u_2\omega_2, \qquad \overline{\omega}_3 = u_3\omega_3.,$$

The integration of the characteristic equations (Σ) of these three forms will lead to that of the differential equations given by the elimination of u_1, u_2, u_3 between the relations which define any solution of (Σ). Form, then, the exterior derivatives $\overline{\omega}'_1, \overline{\omega}'_2, \overline{\omega}'_3$; by supposing that we have

$$\begin{aligned}
\omega'_1 &= a_1[\omega_2\,\omega_3] \qquad (\mathrm{mod}\ \omega_1),\\
\omega'_2 &= a_2[\omega_3\,\omega_1] \qquad (\mathrm{mod}\ \omega_2),\\
\omega'_3 &= a_3[\omega_1\,\omega_2] \qquad (\mathrm{mod}\ \omega_3),
\end{aligned}$$

with coefficients a_1, a_2, a_3 functions of the primitive variables, we will have

$$\begin{aligned}
\overline{\omega}'_1 &= \frac{a_1u_1}{u_2u_3}[\overline{\omega}_2\,\overline{\omega}_3] \qquad (\mathrm{mod}\ \overline{\omega}_1),\\
\overline{\omega}'_2 &= \frac{a_2u_2}{u_3u_1}[\overline{\omega}_3\,\overline{\omega}_1] \qquad (\mathrm{mod}\ \overline{\omega}_2),\\
\overline{\omega}'_3 &= \frac{a_3u_3}{u_1u_2}[\overline{\omega}_1\,\overline{\omega}_2] \qquad (\mathrm{mod}\ \overline{\omega}_3).
\end{aligned}$$

The coefficients

$$\overline{a}_1 = \frac{a_1u_1}{u_2u_3}, \quad \overline{a}_2 = \frac{a_2u_2}{u_3u_1}, \quad \overline{a}_3 = \frac{a_3u_3}{u_1u_2}$$

are thus first integrals of the characteristic system (Σ) of the forms $\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3$. Consequently, the same applies to

$$\sqrt{\bar{a}_2\bar{a}_3} = \frac{\sqrt{a_2a_3}}{u_1},$$

and the form

$$\sqrt{\bar{a}_2\bar{a}_3}\ \bar{\omega}_1 = \sqrt{a_2a_3}\ \omega_1$$

is also invariant. But it does not contain the auxiliary variables u_1, u_2, u_3; it is thus an invariant form for the given equations (S), and the same applies to $\sqrt{a_3a_1}\ \omega_2, \sqrt{a_1a_2}\ \omega_3$. Consequently, *if none of the coefficients a_1, a_2, a_3 is zero, the given differential system admits three invariant linear forms* and it reduces to the problem treated in this Chapter.

Of course, this will not always be the case, but *in all cases we will be able to take full advantage of the information known about the given equations.*

Chapter XVI

Differential equations which admit given infinitesimal transformations

I. — *Reduction of the problem.*

163. We have already considered differential equations which admit infinitesimal transformations, but these equations were assumed to admit an integral invariant or an invariant Pfaffian equation. We now adopt a slightly more general point of view, which will moreover provide an illustration of the theories outlined in the previous Chapter.

Consider a system of n ordinary differential equations (or a completely integrable system of n Pfaffian equations)

$$\omega_1 = \omega_2 = \cdots = \omega_n = 0, \tag{1}$$

and suppose that this system admits a certain number $r \leq n$ of infinitesimal transformations

$$A_1 f, \ A_2 f, \ \ldots, \ A_r f.$$

We investigate what advantage can be gained for the integration from knowing these r infinitesimal transformations. This problem has been solved by S. Lie. We will confine ourselves to essential generalities.

Consider the matrix of quantities $\omega_i(A_k)$ obtained by replacing in the form ω_i the indeterminate differentiation symbol by the symbol of the infinitesimal transformation $A_k f$. Suppose that in this table

$$\left\| \begin{matrix} \omega_1(A_1) & \omega_1(A_2) & \cdots & \omega_1(A_r) \\ \omega_2(A_1) & \omega_2(A_2) & \cdots & \omega_2(A_r) \\ \vdots & & & \vdots \\ \omega_n(A_1) & \omega_n(A_2) & \cdots & \omega_n(A_r) \end{matrix} \right\| \tag{2}$$

the determinant formed by the first r rows and r columns is not zero. We can then substitute for the left hand sides of equation (1) linear combinations of these left hand sides such that the table becomes

$$\left\|\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{matrix}\right\| \tag{3}$$

that is, such that all the $\omega_i(A_k)$ are zero except for

$$\omega_1(A_1) = \omega_2(A_2) = \cdots = \omega_r(A_r) = 1.$$

It is clear that if n is greater than r, the new forms $\omega_1, ..., \omega_n$ are not completely determined: we can still perform an arbitrary linear substitution on

$$\omega_{r+1}, \ \ ..., \ \ \omega_n$$

and we can add to each of the forms $\omega_1, ..., \omega_r$ any linear combination of $\omega_{r+1}, ..., \omega_n$.

If equations (1) had been written in the form

$$dy_1 = dy_2 = \cdots = dy_n = 0,$$

it is clear that, since the quantities $\omega_i(A_j) = A_j y_i$ are first integrals, the new forms $\omega_1, ..., \omega_n$ obtained by reducing the table of $\omega_i(A_j)$ to its canonical form could always be assumed to be formed with the y_j and their differentials. The following two consequences ensue from this, and from what was said above,

1° *Whenever the matrix of the $\omega_i(A_k)$ is reduced to its normal form (3), the Pfaffian system*

$$\omega_{r+1} = \cdots = \omega_n = 0 \tag{4}$$

is an invariant system;

2° *Each of the linear forms $\omega_1 = \cdots = \omega_r$ is an invariant form, up to a linear combination of the left hand sides of the preceding invariant Pfaffian system.*

164. Before going further, note that if system (1) admits the two infinitesimal transformations Af and Bf, it admits the infinitesimal transformation Cf whose symbol is defined by

$$Cf = A(Bf) - B(Af).$$

Assume, which does not restrict generality, that the symbols of the infinitesimal transformations that can be deduced from the r given transformations taken two at a time, are linear combinations of $A_1 f, ..., A_r f$. In other words, suppose that we have

$$A_i(A_k f) - A_k(A_i f) = \sum_{s=1}^{s=r} \gamma_{iks} A_s f \qquad (i,k = 1,2,...,r). \tag{5}$$

With this assumption, we will show that *the Pfaffian system (4) is completely integrable*.

To prove this, it is necessary to return to the definition of the of the bilinear covariant $\omega'(\delta, \delta')$ of a linear form ω, in the case where the two differentiation symbols δ, δ' *do not commute*, a case we have not yet considered. If we put

$$\omega(\delta) = a_1\, \delta x_1 + a_2\, \delta x_2 + \cdots + a_n\, \delta x_n,$$

we have

$$\begin{aligned}\delta\omega(\delta') - \delta'\omega(\delta) = a_1(\delta\delta' x_1 - \delta'\delta x_1) + a_2(\delta\delta' x_2 - \delta'\delta x_2) + \cdots + a_n(\delta\delta' x_n - \delta'\delta x_n) \\ + \sum \left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k} \right) (\delta x_i\, \delta' x_k - \delta x_k\, \delta' x_i),\end{aligned}$$

or also, by agreeing to put

$$\delta'' = \delta\delta' - \delta'\delta,$$

we get

$$\delta\omega(\delta') - \delta'\omega(\delta) = \omega(\delta'') + \omega'(\delta, \delta'). \tag{6}$$

Apply this formula to the case where the symbols δ and δ' are replaced by the symbols $A_i f$ and $A_k f$; it will then be appropriate to replace the symbol δ'' by the symbol

$$A_i(A_k f) - A_k(A_i f) = \sum \gamma_{iks} A_s f.$$

Finally, suppose that we take for ω any one of the forms $\omega_{r+1}, ..., \omega_n$ which, as we have seen, can be assumed to be expressed by means of the $y_1, ..., y_n$ and their differentials. We will get

$$\omega'_{r+\alpha} = \sum c_{\lambda,\mu,r+\alpha} [\omega_\lambda\, \omega_\mu].$$

Now,

$$\omega'_{r+\alpha}(A_i) = \omega'_{r+\alpha}(A_k) = \omega'_{r+\alpha}(A_s) = 0;$$

thus we have the relation

$$\sum c_{\lambda,\mu,r+\alpha} \left[\omega_\lambda(A_i)\,\omega_\mu(A_k) - \omega_\mu(A_i)\,\omega_\lambda(A_k)\right] = 0.$$

It follows from this that the coefficients $c_{\lambda,\mu,r+\alpha}$ are zero as soon as the indices λ, μ are both less than or equal to r, because the preceding relation then obviously reduces to

$$c_{i,k,r+\alpha} = 0 \qquad (i,k = 1,2,...,r).$$

Consequently, since the exterior derivatives $\omega'_{r+1},...,\omega'_n$ are all zero when equations (4) are taken into account, system (4) is indeed completely integrable (n° **101**).

II. — *The case where there are as many infinitesimal transformations as there are unknown functions.*

165. Assume now that system (4), which is an absolutely arbitrary completely integrable Pfaffian system, has been integrated; simply assume even that a solution of this system (4) is known: to this solution there correspond an infinite number of solutions in the given system obtained by integrating the equations

$$\omega_1 = \omega_2 = \cdots = \omega_r = 0. \tag{7}$$

This is a system for which we know r invariant forms $\omega_1,...,\omega_r$. We are brought back to the problem considered in the previous Chapter.

It is easy here to determine *a priori* the coefficients c_{iks} that enter into expressions $\omega'_1,...,\omega'_r$:

$$\omega'_s = \sum_{(ik)}^{1,...,n} c_{iks}[\omega_i\,\omega_k].$$

In fact, apply formula (6), where we replace the symbol δ by the symbol A_α, the symbol δ' by the symbol A_β, and the symbol δ'' by $\sum_{\rho=1}^{\rho=r} \gamma_{\alpha\beta\rho} A_\rho$. Since all the $\omega_i(A_k)$ are equal to 0 or to 1, that is, to constants, formula (6) reduces to

$$0 = \gamma_{\alpha\beta s} + c_{\alpha\beta s}.$$

We thus have

$$c_{iks} = -\gamma_{iks}.$$

166. We now confine ourselves to the case where the coefficients γ_{iks} are constants. We show that in this case *the given infinitesimal transformations $A_1 f, ..., A_r f$ generate an r-parameter group Γ* whose structure constants are the γ_{iks}. We see that system (7) is in the category of those studied in the previous Chapter (n$^{\text{o}}$ **156**), and the group G corresponding to it has the same structure as the group Γ that is admitted by the given differential system (7). *This group G is the largest group which, when applied to the first integrals $y_1, ..., y_r$, preserves the law by which these integrals are exchanged among themselves by the given infinitesimal transformations*. In fact, denote by f an arbitrary function of $y_1, ..., y_r$: it is clear that we can determine, in one and only one way, r Pfaffian expressions $\varpi_1, ..., \varpi_r$ such that we have identically, that is, whatever the differential $dy_1, ..., dy_r$, and also whatever the arguments $\dfrac{\partial f}{\partial y_1}, ..., \dfrac{\partial f}{\partial y_r}$,

$$df \equiv \frac{\partial f}{\partial y_1} dy_1 + \cdots + \frac{\partial f}{\partial y_r} dy_r = \varpi_1 A_1 f + \cdots + \varpi_r A_r f.$$

If we replace in this identity the indeterminate symbol of differentiation d by the symbol $A_k f$, we will get

$$A_k f = \varpi_1(A_k) A_1 f + \cdots + \varpi_r(A_k) A_r f;$$

consequently, all the $\varpi_i(A_k)$ are zero except for

$$\varpi_1(A_1) = \varpi_2(A_2) = \cdots = \varpi_r(A_r) = 1;$$

consequently at last *the forms ϖ_i are identical to the forms ω_i*. Thus perform on the $y_1, ..., y_r$ a transformation of the group G, where these quantities become $\overline{y}_1, ..., \overline{y}_r$; the function f of $y_1, ..., y_r$ becomes a function $\overline{f}$ of $\overline{y}_1, ..., \overline{y}_r$; the symbols $A_1 f, ..., A_r f$ become $\overline{A}_1 \overline{f}, ..., \overline{A}_r \overline{f}$ and we have

$$d\overline{f} = \overline{\omega}_1 \overline{A}_1 \overline{f} + \cdots + \overline{\omega}_r \overline{A}_r \overline{f};$$

but since the $\overline{\omega}_i$ being formed with the $\overline{y}_i$ and their differentials like the ω_i were formed with the y_i and their differentials, the coefficient of $\dfrac{\partial \overline{f}}{\partial \overline{y}_k}$ in $\overline{A}_i \overline{f}$ will be the same function of the $y_1, ..., y_r$ as the coefficient of $\dfrac{\partial f}{\partial y_k}$ in $A_i f$ was a function of $y_1, ..., y_r$. In other words, the given infinitesimal transformations transform the $\overline{y}_i$ in the same way as the y_i.

Here again, we see that the group G is the largest group of transformations which, applied to first integrals, preserve the given information. we see that the group G is the largest group of transformations transformations which, when applied to prime integrals, preserve the given information.

III. — *Application to second-order differential equations.*

167. We have already dealt directly with the case $n = r = 1$. Take some other examples. A second order differential equation of the form

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}\right)$$

is equivalent to the system

$$\begin{aligned} dy - y'\,dx &= 0, \\ dy' - F(y')\,dx &= 0, \end{aligned}$$

which admits the two infinitesimal transformations

$$Af = \frac{\partial f}{\partial x}, \qquad Bf = \frac{\partial f}{\partial y}.$$

To reduce the matrix of quantities $\omega_i(A_k)$ to its normal form, we need to take

$$\begin{aligned} \omega_1 &= dx - \frac{dy'}{F(y')}, \\ \omega_2 &= dy - \frac{y'\,dy'}{F(y')}. \end{aligned}$$

These two invariant forms are exact differentials and we have the general solution we looked for by two independent quadratures

$$x - \int \frac{dy'}{F(y')} = C_1, \qquad y - \int \frac{y'\,dy'}{F(y')} = C_2.$$

Take now a second order differential equation of the form

$$y\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}\right);$$

it admits a translation parallel to the x-axis and a homothety with centre O, which corresponds to two infinitesimal transformations

$$Af = \frac{\partial f}{\partial x}, \qquad Bf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

The given equation is equivalent to the system

$$\begin{aligned} dy - y'\,dx &= 0, \\ y\,dy' - F(y')\,dx &= 0. \end{aligned}$$

To normalise the matrix of the $\omega_i(A_k)$, we must take

$$\begin{aligned}\omega_1 &= dx - \frac{x}{y}\,dy - \frac{y - xy'}{F(y')}\,dy',\\ \omega_2 &= \frac{dy}{y} - \frac{y'}{F(y')}\,dy'.\end{aligned}$$

Since we have here

$$A(Bf) - B(Af) = Af,$$

we will have, as is easily verified,

$$\begin{aligned}\omega_1' &= -[\omega_1\,\omega_2],\\ \omega_2' &= 0.\end{aligned}$$

Consequently, the integration is performed by two quadratures:

$$\begin{aligned}y &= C_1\, e^{\displaystyle\int \frac{y'\,dy'}{F(y')}},\\ x &= C_1 \int \frac{1}{F(y')}\, e^{\displaystyle\int \frac{y'\,dy'}{F(y')}}\, dy' + C_2.\end{aligned}$$

IV. — *Generalisations. — Examples.*

168. It could happen, in the case of system (1) of n Pfaffian equations that admit r infinitesimal transformations

$$A_1 f,\ \ldots,\ A_r f,$$

that the rank of the matrix of the $\omega_i(A_k)$ was smaller than r (this is certainly the case if $r > n$). Let ρ then be the rank of this matrix and suppose, which is allowed, that the determinant formed by the first ρ rows and the first ρ columns is not zero. We will then have, whatever the index s, $r-\rho$ relations of the form

$$\begin{aligned}\omega_s(A_{\rho+1}) &= \lambda_{11}\,\omega_s(A_1) + \cdots + \lambda_{1\rho}\,\omega_s(A_\rho),\\ &\ \ \vdots\\ \omega_s(A_r) &= \lambda_{r-\rho,1}\,\omega_s(A_1) + \cdots + \lambda_{r-\rho,\rho}\,\omega_s(A_\rho).\end{aligned}$$

The coefficients λ_{ij} introduced in these relations are first integrals; in fact, for any linear combination ω of $\omega_1,\dots,\omega_n$, in particular for the differentials $dy_1,\dots,dy_n$ of n independent first integrals, we will have the same relations

$$A_{\rho+1}(y_s) = \lambda_{11}A_1(y_s) + \cdots + \lambda_{1\rho}A_\rho(y_s),$$

which leads to values for the $\lambda_{11},\dots,\lambda_{1\rho}$ that depend only on the $A_i(y_s)$, that is, on the $y_1,\dots,y_n$.

We will not pursue the theory further in this general case; it rests on the same principles as previously.

169. EXAMPLE I. — Consider the differential equation

$$(1+y'^2)^{3/2} = Ry''$$

of curves in the plane which have a given radius of curvature. It is equivalent to the system

$$\begin{aligned}\omega_1 &\equiv dy - y'\,dx = 0,\\ \omega_2 &\equiv R\,dy' - (1+y'^2)^{3/2}\,dx = 0.\end{aligned}$$

This system admits the three infinitesimal transformations corresponding to a translation parallel to Ox, to a translation parallel to Oy and to a rotation around O. These transformations myst be calculated, not only as regards their effects on x and on y, but also on y'. We find without difficulty

$$\begin{aligned}A_1 f &= \frac{\partial f}{\partial x},\\ A_2 f &= \frac{\partial f}{\partial y},\\ A_3 f &= -y\,\frac{\partial f}{\partial x} + x\,\frac{\partial f}{\partial y} + (1+y'^2)\,\frac{\partial f}{\partial y'}.\end{aligned}$$

The matrix of quantities $\omega_i(A_k)$ is the following:

$$\left\| \begin{matrix} -y' & 1 & x+yy' \\ -(1+y'^2)^{3/2} & 0 & y(1+y'^2)^{3/2} + R(1+y'^2) \end{matrix} \right\|.$$

We thus have

$$\omega_s(A_3) = -\left(y + \frac{R}{\sqrt{1+y'^2}}\right)\omega_s(A_1)\left(x - \frac{Ry'}{\sqrt{1+y'^2}}\right)\omega_s(A_2).$$

Consequently we get by simple differentiations two first integrals of the given system and the general solution is provided by the formulae

$$x = \frac{Ry'}{\sqrt{1+y'^2}} + C_1,$$
$$y = \frac{R}{\sqrt{1+y'^2}} + C_2,$$

where

$$(x-C_1)^2 + (y-C_2)^2 = R^2.$$

170. Example II. — Consider the third order differential equation

$$y''' = \frac{3y'y''^2}{1+y'^2},$$

which defines plane curves of constant curvature. It is equivalent to the system

$$\omega_1 \equiv dy - y'\,dx = 0,$$
$$\omega_2 \equiv dy' - y''\,dx = 0,$$
$$\omega_3 \equiv dy'' - \frac{3y'y''}{1+y'^2}\,dy' = 0.$$

This system admits four infinitesimal transformations corresponding to a translation parallel to Ox, a translation parallel to Oy, a rotation around O and a homothety with centre O. The symbols of these transformations, considered as operating on x, y, y' and y'', are

$$A_1 f = \frac{\partial f}{\partial x},$$
$$A_2 f = \frac{\partial f}{\partial y},$$
$$A_3 f = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + (1+y'^2)\frac{\partial f}{\partial y'} + 3y'y''\frac{\partial f}{\partial y''},$$
$$A_4 f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y''\frac{\partial f}{\partial y''}.$$

Here, the matrix of quantities $\omega_i(A_k)$ is

$$\begin{Vmatrix} -y' & 1 & x+yy' & y-xy' \\ -y'' & 0 & 1+y'^2+yy'' & -xy'' \\ 0 & 0 & 0 & -y'' \end{Vmatrix}.$$

It is of rank 3 and, for example, the determinant obtained by taking the 1st, the 2nd and the 4th columns is different from zero. We deduce the relations

$$\omega_s(A_3) = -\left(y + \frac{1+y'^2}{y''}\right)\omega_s(A_1) + \left(x - y'\,\frac{1+y'^2}{y''}\right)\omega_s(A_2),$$

which lead to two first integrals

$$\begin{aligned} u &= x - y'\,\frac{1+y'^2}{y''},\\ v &= y + \frac{1+y'^2}{y''}. \end{aligned}$$

To continue the integration, choose linear combinations $\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3$ such that the principal determinant of the matrix $\overline{\omega}_i(A_k)$ is reduced to its normal form. For this, it is sufficient to take

$$\begin{aligned} \overline{\omega}_1 &= dx - \left(\frac{1}{y''} + \frac{3xy'}{1+y'^2}\right)dy' + x\,\frac{dy''}{y''},\\ \overline{\omega}_2 &= dy - \left(\frac{y'}{y''} + \frac{3xy'}{1+y'^2}\right)dy' + y\,\frac{dy''}{y''},\\ \overline{\omega}_3 &= \frac{3y'\,dy'}{1+y'^2} - \frac{dy''}{y''}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} du &= \overline{\omega}_1 + u\,\overline{\omega}_3,\\ dv &= \overline{\omega}_2 + v\,\overline{\omega}_3, \end{aligned}$$

and

$$\begin{aligned} \overline{\omega}_1{}' &= -[\overline{\omega}_1\,\overline{\omega}_3],\\ \overline{\omega}_2{}' &= -[\overline{\omega}_2\,\overline{\omega}_3],\\ \overline{\omega}_3{}' &= 0. \end{aligned}$$

Consequently $\overline{\omega}_3$ is an exact differential and we get *by one quadrature* the missing first integral. The general solution of the given equation is provided by the formulae

$$\frac{(1+y'^2)^{3/2}}{y''} = C_3,$$
$$x = \frac{C_3 y'}{\sqrt{1+y'^2}} + C_1,$$
$$y = -\frac{C_3}{\sqrt{1+y'^2}} + C_2.$$

We see here that the group G which preserves the data is

$$\overline{C}_1 = C_1, \quad \overline{C}_2 = C_2, \quad \overline{C}_3 = aC_3 \quad \left(\text{because } \overline{\varpi}_3 = \frac{dC_3}{C_3}\right)$$

with one arbitrary constant a; it is because it has only one parameter that the integration reduces to a quadrature. In the previous exercise, group G reduced to the identity transformation and the solution was obtained without integration.

171. NOTE. — In all the examples where we arrive at n invariant linear forms, we have integral invariants of all degrees by constructing with $\omega_1, \ldots, \omega_n$ an arbitrary exterior form with constant coefficients. This is how in the last exercise we have the integral invariant $\iiint \overline{\omega}_1 \overline{\omega}_2 \overline{\omega}_3$ which, if we limit ourselves to sets of states corresponding to the same value of x, reduces to

$$\iiint \frac{dy\, dy'\, dy''}{y''^2}$$

Consequently, if we consider any family of circumferences that depend on three parameters and if we cut the circles of this family by any parallel to the y-axis, the integral $\iiint \frac{dy\, dy'\, dy''}{y''^2}$ over to the family of circles considered, is independent of x. Moreover, it is equal to $\iiint \frac{dC_1\, dC_2\, dC_3}{C_3}$, where C_1 and C_2 denote the coordinates of the centre and C_3 denotes the radius.

Chapter XVII

Application of the previous theory to the *n*-body problem.

I. — *Reducing the number of degrees of freedom.*

172. We have already seen (n° **123**) how, for the canonical equations of Dynamics

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

the method of integration presented in Chapter XII arises. We assumed that the function H is arbitrary. If this function is independent of time, the function H is a first integral (n° **92**) and we are brought back to the integration of the equations

$$\frac{dq_i}{\frac{\partial H}{\partial p_i}} = \frac{-dp_i}{\frac{\partial H}{\partial q_i}}$$

whose first integrals are the solutions of the equation

$$(H\,f) = 0,$$

and with one quadrature.

173. We will study a little more closely the reduction produced in the integration of the n-body problem, by taking into account the already determined infinitesimal transformations (n° **93**) which the equations of motion admit. We will assume, which is allowed, that the system of n bodies is referred to its centre of gravity, that is, the $3n$ coordinates x_i, y_i, z_i and the $3n$ components of the velocities x'_i, y'_i, z'_i are related by the relations

$$\sum m_i x_i = 0, \qquad \sum m_i y_i = 0, \qquad \sum m_i z_i = 0,$$
$$\sum m_i x'_i = 0, \qquad \sum m_i y'_i = 0, \qquad \sum m_i z'_i = 0.$$

Let U be the force function, assumed homogeneous and of degree $-p$ with respect to the coordinates. The equations of motion admit the five infinitesimal transformations

$$\begin{aligned}
A_0 f &= \frac{\partial f}{\partial t},\\
A_1 f &= \sum \left(y_i \frac{\partial f}{\partial z_i} - z_i \frac{\partial f}{\partial y_i} + y_i' \frac{\partial f}{\partial z_i'} - z_i' \frac{\partial f}{\partial y_i'} \right),\\
A_2 f &= \sum \left(z_i \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial z_i} + z_i' \frac{\partial f}{\partial x_i'} - x_i' \frac{\partial f}{\partial z_i'} \right),\\
A_3 f &= \sum \left(x_i \frac{\partial f}{\partial y_i} - y_i \frac{\partial f}{\partial x_i} + x_i' \frac{\partial f}{\partial y_i'} - y_i' \frac{\partial f}{\partial x_i'} \right),\\
A_4 f &= \sum \left[x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i} + z_i \frac{\partial f}{\partial z_i} - \frac{p}{2} \left(x_i' \frac{\partial f}{\partial x_i'} + y_i' \frac{\partial f}{\partial y_i'} + z_i' \frac{\partial f}{\partial z_i'} \right) \right] + \left(1 + \frac{p}{2}\right) t \frac{\partial f}{\partial t}
\end{aligned}$$

On the other hand,

$$\omega' = \sum \left\{ m_i [\delta x_i' \, \delta x_i] + m_i [\delta y_i' \, \delta y_i] + m_i m_i [\delta z_i' \, \delta z_i] \right\} - [\delta H \, \delta t]$$

by putting

$$H = \frac{1}{2} \sum m_i ({x_i}'^2 + {y_i}'^2 + {z_i}'^2) - U.$$

174. The five invariant linear forms

$$\omega_i = \omega'(A_i, \delta)$$

are

$$\left.\begin{aligned}
\omega_0 &= \delta H,\\
\omega_1 &= \delta H_1,\\
\omega_2 &= \delta H_2,\\
\omega_3 &= \delta H_3,\\
\omega_4 &= -\sum m_i \left(x_i \, dx_i' + y_i \, dy_i' + z_i \, dz_i' + \frac{p}{2} x_i' \, dx_i + \frac{p}{2} y_i' \, dy_i + \frac{p}{2} z_i' \, dz_i \right)\\
&\qquad + \left(1 + \frac{p}{2}\right) t \, \delta H + pH \, \delta t,
\end{aligned}\right\} \tag{1}$$

by putting

$$\left.\begin{aligned}
H_1 &= \sum m_i (y_i' z_i - z_i' y_i),\\
H_3 &= \sum m_i (z_i' x_i - x_i' z_i),\\
H_3 &= \sum m_i (x_i' y_i - y_i' x_i).
\end{aligned}\right\} \tag{2}$$

We have finally

$$\omega'_4 = A_4(\omega') = \left(1 - \frac{p}{2}\right)\omega' .$$

The matrix of quantities $a_{ij} = \omega'(A_i, A_j)$ has already been set up in a slightly more general case (n° **95**). We reproduce it below.

	0	1	2	3	4
0	0	0	0	0	$-pH$
1	0	0	H_3	$-H_2$	$(1-\frac{p}{2})H_1$
2	0	$-H_3$	0	H_1	$(1-\frac{p}{2})H_2$
3	0	H_2	$-H_1$	0	$(1-\frac{p}{2})H_3$
4	pH	$(1-\frac{p}{2})H_1$	$(1-\frac{p}{2})H_2$	$(1-\frac{p}{2})H_3$	0

175. We now know five invariant linear forms and the matrix of coefficients a_{ij} defined by the generalised Poisson bracket operation:

$$N[\omega'^{N-1}\,\omega_i\,\omega_j] = a_{ij}[\omega'^{n}].$$

Let us apply the theory of Chapter XII (n° **125**). Construct the auxiliary form

$$\Phi(\xi) = \sum_{(ij)}^{0,1,\dots,4} a_{ij}[\xi_i\,\xi_j];$$

it is written

$$\Phi(\xi) = pH[\xi_4\,\xi_0] + \frac{p-2}{2}\,[\xi_4(H_1\xi_1 + H_2\xi_2 + H_3\xi_3)] + H_1[\xi_2\,\xi_3] + H_2[\xi_3\,\xi_1] + H_3[\xi_1\,\xi_2].$$

It has rank 4 and its reduction to normal form

$$\Phi = [\xi'_4\,\xi'_0] + [\xi'_1\,\xi'_2]$$

can be made by setting

$$\begin{aligned}
\xi_0' &= pH\xi_0 + \frac{p-2}{2}(H_1\xi_1 + H_2\xi_2 + H_3\xi_3),\\
\xi_4' &= \xi_4,\\
\xi_1' &= \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3,\\
\xi_2' &= \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3,\\
\xi_3' &= \xi_0,
\end{aligned}$$

where the α_i and β_i are chosen, which is always possible, such as to have

$$\alpha_2\beta_3 - \beta_2\alpha_3 = H_1, \qquad \alpha_3\beta_1 - \beta_3\alpha_1 = H_2, \qquad \alpha_1\beta_2 - \beta_1\alpha_2 = H_3;$$

we can add the supplementary conditions

$$\begin{gathered}
\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,\\
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2 = \sqrt{H_1^2 + H_2^2 + H_3^2}\,.
\end{gathered}$$

Now, if we define five linear forms $\varpi_0, \varpi_1, \varpi_2, \varpi_3, \varpi_4$, by the identity

$$\xi_0\omega_0 + \xi_1\omega_1 + \xi_2\omega_2 + \xi_3\omega_3 + \xi_4\omega_4 = \xi_0'\varpi_0 + \xi_1'\varpi_1 + \xi_2'\varpi_2 + \xi_3'\varpi_3 + \xi_4'\varpi_4,$$

we get without difficulty

$$\begin{aligned}
\varpi_4 &= \omega_4,\\
\varpi_0 &= \frac{2}{p-2}\,\frac{H_1\,dH_1 + H_2\,dH_2 + H_3\,dH_3}{H_1^2 + H_2^2 + H_3^2},\\
\varpi_3 &= dH - \frac{2pH}{p-2}\,\frac{H_1\,dH_1 + H_2\,dH_2 + H_3\,dH_3}{H_1^2 + H_2^2 + H_3^2},\\
\varpi_1 &= \frac{\alpha_1\,dH_1 + \alpha_2\,dH_2 + \alpha_3\,dH_3}{\sqrt{H_1^2 + H_2^2 + H_3^2}},\\
\varpi_2 &= \frac{\beta_1\,dH_1 + \beta_2\,dH_2 + \beta_3\,dH_3}{\sqrt{H_1^2 + H_2^2 + H_3^2}}.
\end{aligned}$$

If the auxiliary form Φ has been reduced to $[\xi_4'\,\xi_0'] + [\xi_1'\,\xi_2']$, we have

$$\omega' = [\varpi_4\,\varpi_0] + [\varpi_1\,\varpi_2] + [\omega_5\,\varpi_3] + [\omega_6\,\omega_7] + \cdots,$$

that is, by carrying out the calculations,

$$\left.\begin{aligned}
\omega' = \frac{p-2}{2}\left[\omega_4\,\frac{H_1\,dH_1 + H_2\,dH_2 + H_3\,dH_3}{H_1^2 + H_2^2 + H_3^2}\right]\\
+\frac{H_1[dH_2\,dH_3] + H_2[dH_3\,dH_1] + H_3[dH_1\,dH_2]}{H_1^2 + H_2^2 + H_3^2} + \Omega,
\end{aligned}\right\} \qquad (3)$$

by putting

$$\Omega = \left[\omega_3 \left(dH - \frac{2pH}{p-2}\,\frac{H_1\,dH_1 + H_2\,dH_2 + H_3\,dH_3}{H_1^2 + H_2^2 + H_3^2}\right)\right] + [\omega_6\,\omega_7] + \cdots \tag{4}$$

176. If we equate the four first integrals H, H_1, H_2, H_3 to arbitrary constants, the rank of ω' is reduced by six units; it thus goes from $6n-6$ to $6n-12$, corresponding therefore to a problem with $3n-6$ degrees of freedom (3 in the case of the three-body problem). But the corresponding characteristic system contains arbitrary parameters.

There is a (theoretical) process for reducing the number of degrees of freedom while avoiding the introduction of arbitrary parameters. By anulling the exterior derivative of the right hand side of equation (3) and taking into account the relation

$$\omega_1' = \frac{2-p}{2}\,\omega',$$

we get

$$\Omega' = \left[\frac{H_1\,dH_1 + H_2\,dH_2 + H_3\,dH_3}{H_1^2 + H_2^2 + H_3^2}\;\Omega\right].$$

This relation expresses the fact that the exterior derivative of the quadratic form

$$\begin{aligned}\frac{1}{\sqrt{H_1^2+H_2^2+H_3^2}}\,\Omega &= \frac{1}{\sqrt{H_1^2+H_2^2+H_3^2}}\,\omega' - \frac{2}{p-2}\left[\omega_4\;\frac{H_1\,dH_1 + H_2\,dH_2 + H_3\,dH_3}{(H_1^2+H_2^2+H_3^2)^{3/2}}\right]\\ &\quad - \frac{H_1[dH_2\,dH_3] + H_2[dH_3\,dH_1] + H_3[dH_1\,dH_2]}{(H_1^2+H_2^2+H_3^2)^{3/2}}\\ &= \frac{2}{2-p}\left(\frac{1}{\sqrt{H_1^2+H_2^2+H_3^2}}\,\omega_4\right)' - \frac{H_1[dH_2\,dH_3] + H_2[dH_3\,dH_1] + H_3[dH_1\,dH_2]}{(H_1^2+H_2^2+H_3^2)^{3/2}}\end{aligned}$$

is zero. This property is also evident from the right hand side of the previous equality whose first term, which is an exact exterior derivative, has a zero exterior derivative. We will see that the same is true of the second term.

To interpret this second term, consider the vector (OS) of length $\gamma = \sqrt{H_1^2+H_2^2+H_3^2}$, which represents the angular momentum of the system with respect to the origin and which has projections H_1, H_2, H_3. If we imagine a surface element $d\sigma$ described by the point S, and if we call the direction cosines of the normal to this surface element $\alpha_1, \alpha_2, \alpha_3$, we have

$$[dH_2\,dH_3] = \alpha_1\,d\sigma, \qquad [dH_3\,dH_1] = \alpha_2\,d\sigma, \qquad [dH_1\,dH_2] = \alpha_3\,d\sigma,$$

and consequently

$$\frac{H_1[dH_2\,dH_3]+H_2[dH_3\,dH_1]+H_3[dH_1\,dH_2]}{(H_1^2+H_2^2+H_3^2)^{3/2}} = \frac{\alpha_1 H_1+\alpha_2 H_2+\alpha_3 H_3}{\gamma^3}\,d\sigma = \frac{\cos\psi\,d\sigma}{\gamma^3} = d\omega,$$

where ψ represents the angle between OS and the normal to the element, and $d\omega$ the solid angle subtended at the origin by this element of surface. Denoting by θ and φ the colatitude (angle with Oz) and the longitude (angle between the plane zOS and the plane xOz) of the point S, we have on the other hand

$$d\omega = \sin\theta\ [d\theta\,d\varphi];$$

consequently the form considered can be regarded as the exterior derivative of the linear form $-\cos\theta\,d\varphi$.

Thus put

$$\varpi = \frac{2}{2-p}\ \frac{\omega_4}{\gamma} + \cos\theta\,d\varphi : \tag{5}$$

we see that *the characteristic equations of the relative integral invariant* $\int\omega$ are

$$\left.\begin{aligned} &dH - \frac{2p}{p-2}\,H\,\frac{d\gamma}{\gamma} = 0,\\ &\omega_5 = 0,\ \ \omega_6 = 0,\ \ \omega_7 = 0,\ \ \dots \end{aligned}\right\} \tag{6}$$

II. — *The equations of motion referred to a moving frame.*

177. It is easy to interpret this system. For definiteness, consider the three-body problem of celestial mechanics ($p=1$). Calculate first the quantities $\omega_5(A_i), \omega_6(A_i)$, etc. For this, apply the operation corresponding to $A_i f$ to the two sides of formula (3), but writing only the terms in $dH, \omega_6, \omega_7, \dots, \omega_{11}$. We will straightway have

$$\begin{aligned} dH &= \omega_5(A_0)\,dH - \omega_7(A_0)\omega_6 + \omega_6(A_0)\omega_7 + \cdots + \omega_{10}(A_0)\omega_{11},\\ 0 &= \omega_5(A_1)\,dH - \omega_7(A_1)\omega_6 + \omega_6(A_1)\omega_7 + \cdots + \omega_{10}(A_1)\omega_{11},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ 0 &= \omega_5(A_4)\,dH - \omega_7(A_4)\omega_6 + \omega_6(A_4)\omega_7 + \cdots + \omega_{10}(A_4)\omega_{11}. \end{aligned}$$

We thus find that, for $\alpha \geq 5$, all the $\omega_\alpha(A_i)$ are zero, except $\omega_5(A_0)$ which is equal to 1.

Note on the other hand that the quantities $A_iH + 2H\dfrac{A_i\gamma}{\gamma}$ are all zero, since the function $K = H\gamma^2$ is invariant for each of the infinitesimal transformations considered. We see that *the system (6) characteristic of* $\dfrac{1}{\gamma}\Omega$ *can be defined as formed from all the linear combinations of the equations of motion which have the property of being satisfied identically when we replace the symbol of indeterminate differentiation by the symbol of any one of the infinitesimal transformations* A_1f, A_2f, A_3f, A_4f.

178. This is the result which will allow us to interpret system (6) geometrically.

For this, consider the various possible systems of reference, where each system of reference is defined by three rectangular coordinate axes, an origin of time and the units of length, time and mass. We will fix once and for all the unit of mass and we will impose on the other units the condition that the constant of universal attraction has a fixed numerical value. The unit of length is still arbitrary. Finally we will fix the origin of the axes, which will be the centre of gravity of the system of three bodies, as well as the origin of time.

The available systems of reference depend on four arbitrary parameters; three of them determine the orientation of the axes and a fourth determine the units.

To each *state* of the three bodies (defined by their positions, their velocities and time, and depending on 13 variables) there corresponds a moving system of reference, according to a law determined in advance, in such a way as to reduce by 4 units the number of quantities which determine the state of the three bodies with respect to this system of reference. *For example*, we can choose for the x-axis the straight line which joins the centre of gravity O to the body A_1, the plane of the three bodies for the x,y-plane, and the distance OA_1 for the unit of length: the state of the three bodies is then defined by the two coordinates of A_2, the six projections onto the three axes of the velocities of A_1 and A_2, and finally the time t. We could fix the choice of the moving system of reference corresponding to a given state by another law; by always taking Oz perpendicular to the plane of the three bodies, we could take Ox parallel to A_1A_2 and take the side A_1A_2 as the unit of length. We could also choose the axes according to one of the two previous laws, but choose the units in such a way that the angular momentum OS is measured by the number 1. We could also take Oz perpendicular to the plane of the three bodies, take the zOS-plane as the xz-plane and choose the units in such a way as to measure OS by the number 1. In this last hypothesis the nine quantities which would determine the state of the three bodies with respect to the moving system of reference would be the given two coordinates of A_1, the two coordinates of A_2, the time, and finally the six components of the velocities of A_1 and A_2, which would make 11 quantities, but related by the two relations $\gamma = 1, \varphi = 0$.

Suppose now that we have chosen one of the preceding laws or any other conceivable law that makes each state of the three bodies correspond to a moving system of reference, and let

$$q_1,\ q_2,\ \dots,\ q_9$$

be the nine quantities which determine the state of the three bodies with respect to the corresponding moving system of reference. The state of the three bodies will be determined with respect to a fixed reference system if we know, in addition to $q_1,\ldots,q_9$, the four parameters u_1,u_2,u_3,u_4 which define the position of the moving system of reference with respect to the fixed system of reference; these four parameters will be, for example, the three parameters on which the nine direction cosines and the ratio of the moving unit of length to the fixed unit of length depend. Finally the quantities (19 in number, but reducing to 13):

$$x_i,\ y_i,\ z_i,\ x_i',\ y_i',\ z_i',\ t,$$

which determine the state of the three bodies with respect to the fixed system of reference are well defined functions of the 13 quantities

$$q_1,\ q_2,\ \ldots,\ q_9;\ u_1,\ u_2,\ u_3,\ u_4.$$

Conversely, the latter are well defined functions of the former. Now, *the 9 quantities* $q_1,\ldots,q_9$*, considered as functions of* $x_i,y_i,z_i,x_i',y_i',z_i',t$*, are clearly invariants under each of the infinitesimal transformations* $A_1f,\ldots,A_4f$, because performing one of these transformations is the same as changing the fixed system of reference thus altering the quantities u_1,u_2,u_3,u_4, which define the relation between the moving system of reference and the fixed system of reference, but without altering the quantities q_i which define the state of the three bodies with respect to the moving system of reference.

We can also say that if we look for all the linear differential forms in $dx_i,dy_i,\ldots,dz_i',dt$ which have the property of becoming zero when we replace the symbol d by one of the symbols $A_1f,\ldots,A_4f$, we find all the linear combinations in $dq_1,\ldots,dq_9$, and only those.

In particular, the equations (6) of the characteristic system of Ω have their left hand sides linear in $dq_1,\ldots,dq_9$. Since they are 8 in number, these equations (6) can be put in the form

$$dq_i - C_i\,dq_9 = 0 \qquad (i = 1,2,\ldots,8);$$

and, since they are completely integrable, the C_i depend only on the q_i. In other words, system (6) is a system of ordinary differential equations in $q_1,\ldots,q_9$: *it thus defines the motion of the three bodies with respect to the moving system of reference*.

179. It is now easy actually to form the equations of system (6). For this, begin from the relative integral invariant $\int \omega$, where we have put

$$\varpi = \frac{2}{\gamma}\,\omega_4 + \cos\theta\,d\varphi,$$

and imagine that we have expressed all the quantities $x_i,y_i,\ldots,z_i',t$ by means of $q_1,\ldots,q_9,u_1,\ldots,u4$. First, we know that the differentials du_1,du_2,du_3,du_4, must not appear in the final expression for ϖ' which must be formed only by means of the linear forms $dq_i - C_i\,dq_9$. Consequently, to calculate ϖ', we can regard $u_1,\ldots,u^4$ as fixed parameters. Moreover, the coefficients of the form ϖ', expressed in terms of $dq_1,\ldots,dq_9$, must not contain the variables $u_1,\ldots,u^4$, without this the exterior derivative of ϖ' would not be zero: to do the calculation, we can thus not only look at $u_1,\ldots,u_4$

as fixed parameters, but *we can also give them arbitrary numerical values*. Thus, in particular, we can give them the numerical values which would correspond to the case where the fixed system of reference coincides with the moving system of reference. In other words, *to form ϖ', we can give to the quantities $x_i, y_i, \dots, z_i, t$ in ϖ the values $X_i, Y_i, \dots, Z_i, T$ which define the state of the three bodies with respect to the moving system of reference*; these thirteen quantities $X_i, Y_i, \dots, Z_i, T$ reduce, as we have seen, to nine.

180. Consider in particular the case where the moving unit of length is chosen in such a way as to reduce γ to unity (the moving unit of length is then fixed). In this case we have

$$\varpi = 2\omega_4 + \cos\theta\, d\varphi;$$

by adding an exact differential, we have the relative integral invariant $\int \omega + \cos\theta\, d\varphi$ for the differential equations that we seek.

If the z-axis is assumed to be normal to the plane of the three bodies and the x-axis parallel to A_1A_2, for example, the position of the triangle depends on three quantities ξ_1, ξ_2, ξ_3. Now, we have

$$\begin{aligned} \omega + \cos\theta\, d\varphi &= \sum m_i (X'\, dX_i + Y_i\, dY_i) - H\, dT + \cos\theta\, d\varphi \\ &= \eta_1\, d\xi_1 + \eta_2\, d\xi_2 + \eta_3\, d\xi_3 + \eta_4\, d\xi_4 - H\, dT, \end{aligned}$$

by putting

$$\xi_4 = \varphi, \qquad \eta_4 = \cos\theta.$$

The equations of relative motion we seek are then

$$\frac{d\xi_i}{dt} = \frac{\partial H}{\partial \eta_i}, \qquad \frac{d\eta_i}{dt} = -\frac{\partial H}{\partial \xi_i}, \qquad (i = 1, 2, 3, 4).$$

They are canonical and admit the first integral $H =$ constant.

For example, we could take for ξ_1, ξ_2, ξ_3 the lengths of the three sides of the triangle $A_1A_2A_3$.

Had we assumed plane motion, θ would be zero, and there would only be six unknown functions $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$.

181. Once the motion of the three bodies relative to the moving system of reference is known, the absolute motion can be determined by a quadrature. First, in fact, knowing the projections of the angular momentum OS onto the moving axes, we would have the ratio of the moving units to the fixed units by giving ourselves the constant number C which measures OS relative to the fixed system of reference. We could then take OS as a fixed z-axis, the position of the fixed axes depending on an unknown angle. This angle would be given by a quadrature: in fact, it is sufficient to note that the invariant form ω_4 (expressed by means of fixed coordinates) is an exact differential when we take into account the relations assumed obtained which define the relative motion: the formula

$$2\omega_4' = \omega' = 2\left[\frac{d\gamma}{\gamma}\,\omega_4\right] + \frac{H_1[dH_2\,dH_3] + H_2[dH_3\,dH_1] + H_3[dH_1\,dH_2]}{\gamma^2} + \Omega$$

shows in fact that under these conditions ω_4' is zero (since H_1 and H_2 are zero). The integration is therefore completed by means of the formula

$$\int \omega_4 = \text{ Constant.}$$

It may be noted that this quadrature can be performed when we have determined the relative motion from the geometric point of view alone before we have found the time (by a quadrature, as we know). In other words, the two quadratures which give the time (in relative motion) and the final orientation of the fixed axes in relation to the moving axes *can be performed independently of each other.*

III. — *Case where the area constants are all zero.*

182. The preceding theory essentially assumes that $H_1^2 + H_2^2 + H_3^2 \neq 0$. Consider the motions for which the areal constants would all three be zero. In this case, we must suppose the 18 quantities

$$x_i,\ y_i,\ z_i \quad x_i',\ y_i',\ z_i'$$

not only related by the relations

$$\sum m_i x_i = 0, \quad \sum m_i y_i = 0, \quad \sum m_i z_i = 0,$$
$$\sum m_i x_i' = 0, \quad \sum m_i y_i' = 0, \quad \sum m_i z_i' = 0,$$

but also by the relations

$$\sum m_i(y_i' z_i - z_i' y_i) = 0, \quad \sum m_i(z_i' x_i - x_i' z_i) = 0, \quad \sum m_i(x_i' y_i - y_i' x_i) = 0.$$

It is easy to see that the plane of the three-body triangle remains fixed, because the components of the three velocities that are normal to this plane are all zero, at least if the three bodies are not in a straight line.

We can thus assume that the z_i and the z_i' are all zero, and that between the 12 quantities x_i, y_i, x_i', y_i', are related by the five relations

$$\sum m_i x_i = 0, \qquad \sum m_i y_i = 0,$$
$$\sum m_i x_i' = 0, \qquad \sum m_i y_i' = 0,$$
$$\sum m_i (x_i' y_i - y_i' x_i) = 0.$$

There are therefore seven dependent variables in all and one independent variable (time). Now ω', which has even rank, cannot have rank equal to the number of differential equations of motion; *the characteristic system of* ω' *does not merge with that of the equations of motion*.

We have here three infinitesimal transformations $A_0 f, A_3 f, A_4 f$ with

$$\omega'(A_0, \delta) = \delta H, \qquad \omega'(A_3, \delta) = 0, \qquad \omega'(A_4, \delta) = \omega_4.$$

The seven differential equations of motion can be put into the form of Pfaffian equations

$$\varpi_1 = 0, \quad \varpi_2 = 0, \quad \ldots, \quad \varpi_7 = 0,$$

and we can assume that

$$\varpi_1(A_3) = \cdots = \varpi_6(A_3) = 0, \qquad \varpi_7(A_3) = 1.$$

The form ω', which can be expressed by means of $\varpi_1, \ldots, \varpi_7$, *certainly does not contain varpi*$_7$, or else the form $\omega'(A_3, \delta)$ would not be identically zero. Consequently, *the characteristic system of* ω' *is the completely integrable system*

$$\varpi_1 = \varpi_2 = \cdots = \varpi_6 = 0:$$

it gives the motion of the three bodies *independently of the orientation of the triangle of the three bodies around their centre of gravity*. Once this system is integrated, *the orientation is given by a quadrature*. The relation $\varpi_7(A_3) = 1$ in fact guarantees for ϖ_7 the property of being an invariant form for the differential equation $\varpi_7 = 0$.

Return now to the ω' form of rank 6. Its characteristic system admits the two infinitesimal transformations $A_0 f$ and $A_4 f$, which give rise to two invariant linear forms $\omega_0 = \delta H$, which we will denote by ϖ_1 and ω_4, which we will denote by ϖ_2. We will assume, which is allowed, that for each of the forms $\varpi_3, \ldots, \varpi_6$, we have $\varpi(A_0) = \varpi(A_4) = 0$. A calculation similar to that performed in the general case gives

$$\omega' = -\frac{1}{H}[\varpi_1\, \varpi_2] + \Omega = -\frac{1}{H}[\delta H\, \varpi_2] + \Omega,$$

where Ω is of rank 4 and formed from $\varpi_3, \varpi_4, \varpi_5, \varpi_6$.

As was seen above, we have $\omega' = 2\omega_4' = 2\omega_2'$, consequently

$$\Omega = 2\varpi_2' + \left[\frac{\delta H}{H}\, \varpi_2\right] = \frac{2}{\sqrt{H}}(\sqrt{H}\, \varpi_2)'.$$

The form $\frac{1}{2} sqrtH\,\Omega$ is thus an exact derivative; it thus has as characteristic system the equations

$$\varpi_3 = \varpi_4 = \varpi_5 = \varpi_6 = 0. \tag{7}$$

This system can be integrated by equations of orders

$$4, \quad 2, \quad 0.$$

Finally, the equations of motion will be given by operations of orders 4 and 2 followed by two quadratures.

Note that the form $\sqrt{H}\,\varpi_2$, which plays the role of relative invariant for system (7), is equal, according to expression (1) for $\omega_4 = \varpi_2$, to

$$\sqrt{H}\,\varpi_2 = -\sqrt{H}\sum m_i\left(x_i\,\delta x_i' + y_i\,\delta y_i' + \frac{1}{2}x_i'\,\delta x_i + \frac{1}{2}y_i'\,\delta y_i\right) + \delta(H^{3/2}t).$$

System (7) is easy to interpret: it gives the motion of the three bodies relative to a moving system of reference, which can be made to correspond, according to a specific law, to each state of the three bodies *where the origin of time is no longer necessarily fixed*. For example, we can choose the actual time as the moving origin of time, and fix the unit of length by the condition that the energy H has a given fixed numerical value. The equations of the system are obtained by starting from the form $\sqrt{H}\,\varpi_2$, in which we use the *moving* coordinates: in the hypothesis considered, we can obviously substitute for it the form

$$\sum m_i(x_i'\,\delta x_i + y_i'\,\delta y_i).$$

Here, the quantities of motion of the three bodies form a system of vectors equivalent to zero: we can thus consider the quantity of motion of body A_i as the resultant of two vectors u_j and u_k directed along the sides A_iA_k and A_iA_j and counted positive in the extensions of A_kA_i and A_jA_i. We have then, denoting by r_1, r_2, r_3 the three sides of the triangle,

$$\omega = u_1\,\delta r_1 + u_2\,\delta r_2 + u_3\,\delta r_3.$$

We have furthermore

$$H = \frac{1}{2m_i}\left(u_2^2 + u_3^2 + u_2u_3\,\frac{r_2^2 + r_3^2 - r_1^2}{r_2r_3}\right) + \cdots - f\left(\frac{m_2m_3}{r_1} + \frac{m_3m_1}{r_2} + \frac{m_1m_2}{r_3}\right) = h.$$

The equations of relative motion are thus

$$\frac{dr_1}{\frac{\partial H}{\partial u_1}} = \frac{dr_2}{\frac{\partial H}{\partial u_2}} = \frac{dr_3}{\frac{\partial H}{\partial u_3}} = \frac{-du_1}{\frac{\partial H}{\partial r_1}} = \frac{-du_2}{\frac{\partial H}{\partial r_2}} = \frac{-du_3}{\frac{\partial H}{\partial r_3}}.$$

IV. — *Case where the constant of the vis viva is zero.*

183. The preceding theory implicitly assumes that the constant *vis viva*[1,2] is non-zero. If we assume it to be zero, the variables are subject to a new relationship

$$\frac{1}{2}\sum m_i({x_i'}^2+{y_i'}^2)-U=0;$$

there are now only six dependent variables and one independent variable. The invariant form $\omega'(A_0,\delta)$ is identically zero here, as is the form $\omega'(A_3,\delta)$.

The system of the equations of motion can be put into the form

$$\varpi_1=\varpi_2=\cdots=\varpi_6=0,$$

and we can assume (n° **163**)

$$\begin{aligned}&\varpi_1(A_0)=\varpi_2(A_0)=\cdots=\varpi_4(A_0)=0,\qquad \varpi_5(A_0)=1,\qquad \varpi_6(A_0)=0,\\ &\varpi_1(A_3)=\varpi_2(A_3)=\cdots=\varpi_4(A_3)=0,\qquad \varpi_5(A_3)=0,\qquad \varpi_6(A_3)=1.\end{aligned}$$

The form ω', expressed in terms of the ϖ_i, clearly contains neither ϖ_5 nor ϖ_6. Assume finally that

$$\varpi_2(A_4)=\varpi_3(A_4)=\varpi_4(A_4)=0,\qquad \varpi_1(A_4)=1,$$

and $\omega'(A_4,\delta)=\varpi_2$. We will have

$$\omega'=2\varpi_2'=[\varpi_1\,\varpi_2]+[\varpi_3\,\varpi_4].$$

The form ϖ_2 is of the second type and the equations

$$\varpi_2=\varpi_3=\varpi_4=0$$

form a completely integrable system, characteristic of the equation $\varpi_2=0$: this defines the motion of the three bodies with respect to a moving system of reference, where the time origin is possibly variable. For example, we can assume here that we have chosen as the unit of length the side r_3 of the triangle. The equations to be integrated then are the characteristic system of the Pfaffian *equation*

$$2\,d(u_1r_1+u_2r_2+u_3)-u\,dr_1-u_2\,dr_2=0,$$

[1] Fr. *Forces vives*.

[2] TRANSLATOR'S NOTE. — The term *forces vives*, which I have translated as *vis viva*, is used inconsistently in the literature. Sometimes it refers to the quantity mv^2 or $2T$, where T is the kinetic energy of the system. Here however, it is clear from the first equation of n° **183** that Cartan means the *total energy* of the system. Recall that U is the "force function", which is the negative of the potential energy (see Footnote 3 of n° **1**).

where the quantities u_1, u_2, u_3, r_1, r_2 are related by the relation

$$\frac{1}{2m_1}(u_2^2+u_3^2+2u_2u_3\cos A_1)+\cdots-f\left(\frac{m_2m_3}{r_1}+\frac{m_3m_1}{r_2}+m_1m_2\right)=0.$$

Putting

$$r_1=x,\quad r_2=y,\quad u_1r_1+u_2r_2=z,\quad u_1=2p,\quad u_2=2q,$$

it comes down to integrating the first order partial differential equation

$$\begin{aligned}2\left(\frac{1}{m_2}+\frac{1}{m_3}\right)p^2+2\left(\frac{1}{m_1}+\frac{1}{m_3}\right)q^2+\frac{2}{m_3}\,\frac{x^2+y^2-1}{xy}\,pq\\ +(z-2px-2qy)\left(\frac{p}{m_2}\,\frac{x^2+1-y^2}{x}+\frac{q}{m_1}\,\frac{y^2+1-x^2}{y}\right)\\ +\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)(z-2px-2qy)^2-f\left(\frac{m_2m_3}{x}+\frac{m_1m_3}{y}+m_1m_2\right)=0.\end{aligned}$$

After integrating this equation, we get the general solution of the characteristic system of ω' by differentiations because, since ϖ_2 is put into the form $Z_1\,dY_1+Z_2\,dY_2$, we deduce by differentiations the first integrals Y_1, Y_2, Z_1, Z_2 of this system.

But the equations of motion are not yet fully integrated. completely; we still have to integrate the equations

$$\varpi_5=\varpi_6=0.$$

They form a system of differential equations which admit the two infinitesimal transformations A_0f, A_3f, and the matrix

$$\left\|\begin{matrix}\varpi_5(A_0) & \varpi_5(A_3)\\ \varpi_6(A_0) & \varpi_6(A_3)\end{matrix}\right\|$$

is precisely reduced to its normal form

$$\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\|$$

On the other hand, the two transformations A_5f, A_3f commute with each other, since we have

$$A_0(A_3f)-A_3(A_0f)=0,$$

we get

$$\begin{aligned}\varpi_5'&=0,\\ \varpi_6'&=0.\end{aligned}$$

Consequently the integration is done by means of two independent quadratures; one gives the orientation of the triangle $A_1A_2A_3$, the other gives the time.

Chapter XVIII

Integral invariants and the calculus of variations

I. — *Extremals attached to a relative integral invariant.*

184. We have already seen in the first Chapter (n° **9**) that the differential equations of the extremals of the integral

$$I = \int F(q_1, \ldots, q_n; q'_1, \ldots, q'_n; t)dt$$

coincide with the characteristic equations of the relative integral invariant ω, by putting

$$\omega = \sum_{i=1}^{i=n} \frac{\partial F}{\partial q'_i}\,\delta q_i - \left(\sum_{i=1}^{i=n} q'_i \frac{\partial F}{\partial q'_i} - F \right) \delta t,$$

and where we regard $q_1, \ldots, q_n, q'_1, \ldots, q'_n, t$ as $2n+1$ independent variables.

In the calculus of variations we regard $q_1, \ldots, q_n$ as arbitrary functions of t, and $q'_1, \ldots, q'_n$ as their derivatives. In the $(n+1)$-dimensional space $(q_1, \ldots, q_n, t)$, any extremal has the property that the integral I, over a given arc of this curve, is stationary with respect to infinitely close curve arcs of curves that have the same initial and the same end point. But we can also place ourselves in a $(2n+1)$-dimensional space $(q_1, \ldots, q_n,\, q'_1, \ldots, q'_n,\, t)$: an extremal curve then has the property that the integral I over a given arc of this curve is stationary with respect to infinitely close curves for which the initial values and final values of only the coordinates $(q_1, \ldots, q_n, t)$, are the same as for the given extremal. Adopting this second point of view, $(q'_1, \ldots, q'_n)$ are functions of t that have no *a priori* connection with the derivatives of $(q_1, \ldots, q_n)$ with respect to t.

185. More generally, begin from a linear differential form ω with $2n+1$ variables; assume the form ω' is of rank $2n$ and, finally, suppose that that n of the coefficients of the differentials in ω are identically zero. We can thus put

$$\omega = a_1\,\delta x_1 + a_2\,\delta x_2 + \cdots + a_n\,\delta x_n - b\,\delta t,$$

where the quantities $a_1,\ldots,a_n,b$ are functions of $2n+1$ independent variables $x_1,\ldots,x_n,y_1,\ldots,y_n,t$.

The characteristics of the relative integral invariant ω, or equivalently, of the exterior quadratic form ω', are given by a system of ordinary differential equations

$$\frac{dx_i}{dt} = X_i, \qquad \frac{dy_i}{dt} = Y_i \tag{1}$$

by assuming, as we do, that t is not a first integral of the characteristic equations.

That said, consider in the $(2n+1)$-dimensional space an arc of curve arc that goes from the point $M_0\,(x_i^0, y_i^0, t^0)$ to the point $M_1\,(x_i^1, y_i^1, t^1)$ and the integral

$$I = \int_{M_0}^{M_1} a_1\,dx_1 + a_1\,dx_1 + \cdots - b\,dt.$$

Calculate the variation of this integral when we pass from the arc of the curve considered to an infinitely close arc of a curve that goes from the point $(x_i^0 + \delta x_i^0, y_i^0 + \delta y_i^0, t^0 + \delta t^0)$ to the point $(x_i^1 + \delta x_i^1, y_i^1 + \delta y_i^1, t^1 + \delta t^1)$. We will get

$$\delta I = [\omega_\delta]_0^1 + \int_{M_0}^{M_1} (\delta\omega_d - d\omega_\delta) = [\omega_\delta]_0^1 + \int_{M_0}^{M_1} \omega'(d,\delta).$$

If we want the integral to be stationary with respect to any arcs of curve that are infinitely close to the given arc of curve, when moving along the given arc of curve it is necessary that we have

$$\omega'(d,\delta) = 0$$

whatever $\delta x_i, \delta y_i, \delta t$ may be; in other words, *it is necessary that the arc of curve belongs to a characteristic of the form* ω'. The value of the integral will be stationary for all infinitely close arcs of curve for which ω_δ is zero at the initial and at the end point, that is, for which $x_1,\ldots,x_n,t$ have the same initial and final values as for the given arc of curve.

186. Suppose now that we restrict the field of curves infinitely close to the given curve to those curves for which the functions x_i, y_i, t satisfy the first n characteristic equations

$$\frac{dx_i}{dt} = X_i. \tag{2}$$

We will assume that the n functions $X_1,\ldots,X_n$ are independent of the $y_1,\ldots,y_n$, which allows us to take arbitrary functions of t for the x_i. Finally, we will assume that the initial and final values of $x_1,\ldots,x_n,t$ are the same for the varied curves as for the original curve. Under these conditions, we have

$$\delta I = -\int \omega'(d,\delta).$$

It is easy to see that in $\omega'(\delta, d)$ there are no terms in $\delta y_1, \delta y_2, ..., \delta y_n$. In fact, the coefficient of δy_1 would be

$$\frac{\partial a_1}{\partial y_1} dx_1 + \frac{\partial a_2}{\partial y_1} dx_2 + \cdots + \frac{\partial a_n}{\partial y_1} dx_n - \frac{\partial b}{\partial y_1} dt;$$

this coefficient is necessarily zero by taking the characteristic equations (1) into account, and then by taking equations (2) into account; it is thus zero when we move along the extremal. Since, in the expression for δI, only $\delta x_1, ..., \delta x_n, \delta t$ enter under the $\int$ sign, the coefficients of $\delta x_1, ..., \delta x_n, \delta t$ must be zero. Consequently, *the extremals are given by the characteristic equations of* ω'.

II. — *The principle of least action of Maupertuis.*

187. Suppose that Hamilton's function H is time independent. Consider the set of all motions for which the function H has a given constant value h. The corresponding trajectories are the characteristics of the linear integral invariant $\int \omega$, with

$$\omega = \sum_{i=1}^{i=n} p_i \, dx_i - h \, dt,$$

or equivalently of the invariant integral $\int \varpi$, with

$$\varpi = \sum_{i=1}^{i=n} p_i \, dx_i;$$

in fact the form ϖ differs from ω only by an exact differential. This form ϖ is constructed from $2n$ variables related by the relation

$$H = h,$$

and only n of the coefficients are not zero. The characteristic equations are

$$\frac{dq_1}{\frac{\partial H}{\partial p_1}} = \cdots = \frac{dq_n}{\frac{\partial H}{\partial p_n}} = \frac{-dp_1}{\frac{\partial H}{\partial q_1}} = \cdots \frac{-dp_n}{\frac{\partial H}{\partial q_n}}$$

Consequently, in the $(2n-1)$-dimensional space (q_i, p_i), the trajectories are the extremals of the integral

$$\int p_1\,dq_1 + \cdots + p_n\,dq_n,$$

whether we consider all curves for which the initial and final values of $q_1, ..., q_n$ are given, or we consider only those of these curves which satisfy the equations

$$\frac{dq_1}{\frac{\partial H}{\partial p_1}} = \cdots = \frac{dq_n}{\frac{\partial H}{\partial p_n}}$$

and, of course, the condition $H = h$.

188. Adopt, for example, the second point of view. Assume that q_i are the position parameters of the system, and p_i are the components of the momenta. If we denote the *vis viva* by $2T$ and that we decompose it into its second-degree, first-degree and zeroth-degree terms in $q'_1, ..., q'_n$, we have

$$H = \sum q'_i \frac{\partial T}{\partial q'_i} - T - U = T_2 - T_0 - U.$$

Substitute the variables q'_i for the variables p_i. We have by assumption

$$T_2(q') = T_0 + U + h,$$

$$\varpi = \sum_{i=1}^{i=n} \frac{\partial T}{\partial q'_i}\,\delta q_i = \sum_{i=1}^{i=n} \frac{\partial T_2}{\partial q'_i}\,\delta q_i + T_1(\delta q_i).$$

Finally, suppose that we have

$$\frac{dq_1}{q'_1} = \frac{dq_2}{q'_2} = \cdots = \frac{dq_n}{q'_n} = \frac{\sqrt{T_2(dq)}}{\sqrt{T_2(q)}} = \frac{\sqrt{T_2(dq)}}{\sqrt{T_0 + U + h}}.$$

The quantity under the $\int$ sign in the integral I is

$$\varpi_\delta = \sum q'_i \frac{dT_2}{q'_i} \cdot \frac{\sqrt{T_2(dq)}}{\sqrt{T_0 + U + h}} + T_1(dq) = \sqrt{2(T_0 + U + h)\cdot 2T_2(dq)} + T_1(dq).$$

If therefore we put

$$2T = \sum a_{ij} q'_i q'_j + 2\sum b_i q'_i + 2T_0,$$

we arrive at the following theorem, which is the *principle of least action of Maupertuis*:

The trajectories are the extremals of the integral

$$\int \left(\sqrt{2(T_0+U+h)\cdot\sum a_{ij}dq_idq_j}+\sum b_idq_i\right)$$

with respect to all infinitely close trajectories that are required to have the same the same initial position and the same final position of the system, and to satisfy the vis viva *theorem $H = h$, with a given constant of the* vis viva*.*

We recover the principle in its classic form when $T_1 = T_0 = 0$.

189. EXAMPLE. — In the case of an unconstrained[1] moving point referred to fixed axes, the trajectories are the extremals of the integral

$$\int \sqrt{2(U+h)}\,ds.$$

If the point is referred to axes rotating around oz with a constant angular velocity α, and if furthermore the time independent force field is dragged along with the axes, we have

$$\begin{aligned} 2T &= m[(x'+\alpha y)^2+(y'-\alpha x)^2+z'^2] \\ &= m(x'^2+y'^2+z'^2)-2m\alpha(xy'-yx')+m\alpha^2(x^2+y^2). \end{aligned}$$

For a point of mass 1, the trajectories are the extremals of the integral

$$\int \sqrt{\alpha^2(x^2+y^2)+2U+2h}\;ds-\alpha(x\,dy-y\,dx).$$

III. — *Generalisations.*

190. What we have just done in the case where time does not enter H explicitly can also be done in the case where H does not contain one of the other variables q_i and p_i. For definiteness, take the case of an unconstrained[2] material point of mass 1 subjected to a central force function that depends on distance. Consider all motions occuring in the given plane, which we will take to be the xy-plane, and obeying the law of areas with a given constant C. The actual motions are given by a system of differential equations that admit the relative integral invariant

$$\int \omega = \int \left[r'\,\delta r+r^2\theta'\,\delta\theta-\left(\frac{1}{2}r'^2+\frac{1}{2}r^2\theta'^2-U\right)\delta t\right],$$

[1] Fr. *libre*.

[2] Fr. *libre*.

which here clearly reduces to

$$\int \varpi = \int r'\,\delta r - \left(\frac{1}{2}r'^2 + \frac{1}{2}\frac{C^2}{r^2} - U\right)\delta t.$$

The form ϖ depends only on the variables r, r' and t, and one of its characteristic equations is

$$\frac{dr}{dt} = r'.$$

It follows that if we take as initial conditions the values r_0 and t_0, and as final conditions the values r_1 and t_1, the actual motion that satisfies these conditions is the one which makes stationary the integral

$$I = \int_{t_0}^{t_1} r'\,dr - \left(\frac{1}{2}r'^2 + \frac{1}{2}\frac{C^2}{r^2} - U\right)dt = \int_{t_0}^{t_1} \left[\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{1}{2}\frac{C^2}{r^2} + U\right]dt$$

with respect to all infinitely close motions that satisfy the same boundary conditions and satisfying the area law with area constant C.

IV. — *Application to the propagation of light in an isotropic medium.*

191. Consider an isotropic medium whose index of refraction n is known at each point. Fermat's principle leads us to define light rays as extremals of the integral

$$\int n\,ds = \int n\sqrt{dx^2 + dy^2 + dz^2}\,.$$

By introducing an auxiliary variable t, this is the case of an integral

$$\int F(x, y, z; x', y', z'; t),$$

with

$$F(x, y, z; x', y', z'; t) = n\sqrt{dx^2 + dy^2 + dz^2}\,.$$

The relative linear integral invariant $\int \omega$ whose light rays are the characteristics is defined by the formula

$$\omega = \frac{\partial F}{\partial x}\,\delta x + \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial z}\,\delta z - \left(x'\,\frac{\partial F}{\partial x'} + y'\,\frac{\partial F}{\partial y'} + z'\,\frac{\partial F}{\partial z'} \right) \delta t,$$

which here becomes

$$\omega = n \left(\frac{x'}{\sqrt{x'^2+y'^2+z'^2}}\,\delta x + \frac{y'}{\sqrt{x'^2+y'^2+z'^2}}\,\delta y + \frac{z'}{\sqrt{x'^2+y'^2+z'^2}}\,\delta z \right),$$

or again

$$\omega = n\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)\,,$$

by denoting by α, β, γ the direction cosines of an arbitrary direction. The form ω thus really depends on 5 variables. It would be easy to form its characteristic equations and to show that they contain in particular the equations

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{\gamma}\,.$$

The direction (α, β, γ) is clearly nothing other than that of the tangent to the light ray considered.

192. The property of ω of being a relative integral invariant leads to the property of a bundle of light rays that, if we draw a closed curve (C) encircling the bundle, the integral $n\cos\theta\,ds$ over this curve is independent of the curve (C) chosen, where we have denoted the angle made by the tangent at a point M of (C) with the tangent to the light ray passing through M by θ. We can easily demonstrate that the necessary and sufficient condition for the rays of a congruence all to be normal to the same surface is that this integral is zero for any bundle of rays taken in the congruence. This corresponds to Malus' theorem, according to which the rays of a congruence normal to a surface are normal to an infinite number of surfaces. The condition for this to be the case is that the exterior quadratic form ω' is zero, or, more precisely, that the alternating bilinear form $\omega'(\delta, \delta')$ is zero when δ is the symbol of differentiation with respect to one of the parameters of a congruence, and δ' as the symbol of differentiation with respect to the the other parameter.

The light rays propagating in the medium considered depend on four parameters u_1, u_2, u_3, u_4. A *Malus transformation* is a transformation performed on these parameters which changes any congruence of rays normal to a surface into another congruence of rays normal to a surface. As we know, the form ω' is expressible by means of the u_i and their differentials. The most general Malus transformation is obviously defined by the equation

$$\omega'(u', du') = k\,\omega'(u, du),$$

where k is an unknown function. The exterior derivative of the two sides gives immediately

$$[dk\,\omega'] = 0;$$

since ω' is of rank 4, this is possible only if $dk = 0$, that is, if k is a *constant*. Consequently, the transformations that we are looking for can be obtained by expressing the fact that the linear form

$$\omega'(u', du') - k\,\omega'(u, du)$$

is an exact differential:

$$n(x', y', z')(\alpha'\,dx' + \beta'\,dy' + \gamma'\,dz') = kn(x, y, z)(\alpha\,dx + \beta\,dy + \gamma dz) + dV.$$

For example, define a light ray by the coordinates (x_0, y_0) of the point where it intersects the x, y-plane and denote the direction cosines of the tangent at this point by $(\alpha_0, \beta_0, \gamma_0)$. We will have

$$n(x'_0, y'_0, 0)(\alpha'_0\,dx'_0 + \beta'_0\,dy'_0) = kn(x_0, y_0, 0)(\alpha_0\,dx_0 + \beta_0\,dy_0) = dV.$$

1st *Case. — There is no relation between* x'_0, y'_0, x_0, y_0. In this case, V is a specific function of x_0, y_0, x'_0, y'_0, and we have

$$-kn(x_0, y_0, 0)\,\alpha_0 = \frac{\partial V}{\partial x_0}, \qquad -kn(x_0, y_0, 0)\,\beta_0 = \frac{\partial V}{\partial y_0},$$
$$n(x'_0, y'_0, 0)\,\alpha'_0 = \frac{\partial V}{\partial x'_0}, \qquad n(x'_0, y'_0, 0)\,\beta'_0 = \frac{\partial V}{\partial y'_0}.$$

The first two equations give x'_0 and y'_0; the last two then give α'_0 and β'_0.

2nd *Case. — There is one and only one relation between* x'_0, y'_0, x_0, y_0. Let this relation be

$$F(x_0, y_0; x'_0, y'_0) = 0.$$

Denoting by V an arbitrary function of $x_0, y_0; x'_0, y'_0$ and introducing an auxiliary parameter λ, we have

$$-kn(x_0, y_0, 0)\,\alpha_0 = \frac{\partial V}{\partial x_0} + \lambda\,\frac{\partial F}{\partial x_0}, \qquad -kn(x_0, y_0, 0)\,\beta_0 = \frac{\partial V}{\partial y_0} + \lambda\,\frac{\partial F}{\partial y_0},$$
$$n(x'_0, y'_0, 0)\,\alpha'_0 = \frac{\partial V}{\partial x'_0} + \lambda\,\frac{\partial F}{\partial x'_0}, \qquad n(x'_0, y'_0, 0)\,\beta'_0 = \frac{\partial V}{\partial y'_0} + \lambda\,\frac{\partial F}{\partial y'_0}.$$

The first two of these four equations, together with the equation $F = 0$, give x'_0 and y'_0; the last two then give α'_0 and β'_0.

3nd *Case. —* x'_0 *and* y'_0 *are functions of* $x_0 b$ *and* y_0:

$$x'_0 = f(x_0, y_0), \qquad y'_0 = g(x_0, y_0);$$

V is then a function of x_0, y_0 and we have

$$n(x'_0, y'_0, 0)\left(\alpha'_0 \frac{\partial f}{\partial x_0} + \beta'_0 \frac{\partial g}{\partial x_0}\right) = kn(x_0, y_0, 0)\,\alpha_0 + \frac{\partial V}{\partial x_0},$$
$$n(x'_0, y'_0, 0)\left(\alpha'_0 \frac{\partial f}{\partial y_0} + \beta'_0 \frac{\partial g}{\partial y_0}\right) = kn(x_0, y_0, 0)\,\beta_0 + \frac{\partial V}{\partial y_0},$$

which equations determine α'_0 and β'_0.

193. The form ω' is invariant, and we have seen above the characteristic property of ray congruences for which this form is identically zero. The invariant form $\frac{1}{2}\,\omega'^2$ has applications in Optics; its expanded expression is

$$\frac{1}{2}\,\omega'^2 = n\,[\delta n\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)\,(\delta\alpha\,\delta x + \delta\beta\,\delta y + \delta\gamma\,\delta z)]$$
$$- n^2\,([\delta\beta\,\delta\gamma\,\delta y\,\delta z] + [\delta\gamma\,\delta\alpha\,\delta z\,\delta x] + [\delta\alpha\,\delta\beta\,\delta x\,\delta y])\,.$$

For example, take all the light rays that pass through a given surface element $d\sigma$ and whose tangents at the crossing points are parallel to straight lines inside a cone of infinitely small aperture $d\omega$. The rays considered depend on four parameters u_1, u_2, u_3, u_4, where for example the first two define the position of the crossing point on the element $d\sigma$ and the last two the orientation of the tangent at this point. Take on each light ray the *state* characterised by the corresponding crossing point (x, y, z) and the direction cosines (α, β, γ) of the tangent at this point. Since the three first quantities (x, y, z) depend only on u_1 and u_2, any exterior cubic form in $\delta x, \delta y, \delta z$ is zero; consequently the invariant $\frac{1}{2}\,\omega'^2$ reduces, up to sign, to

$$\frac{1}{2}\,\omega'^2 = n^2\,([\delta\beta\,\delta\gamma\,\delta y\,\delta z] + [\delta\gamma\,\delta\alpha\,\delta z\,\delta x] + [\delta\alpha\,\delta\beta\,\delta x\,\delta y])\,.$$

Now, denoting the direction cosines of the normal to the element $d\sigma$ by λ, μ, ν, we have

$$[\delta y\,\delta z] = \lambda\,d\sigma, \qquad [\delta z\,\delta x] = \mu\,d\sigma, \qquad [\delta x\,\delta y] = \nu\,d\sigma;$$
$$[\delta\beta\,\delta\gamma] = \alpha\,d\omega, \qquad [\delta\gamma\,\delta\alpha] = \beta\,d\omega, \qquad [\delta\alpha\,\delta\beta] = \gamma\,d\omega;$$

consequently

$$\frac{1}{2}\,\omega'^2 = n^2\,(\lambda\alpha + \mu\beta + \nu\gamma)\,d\sigma\,d\omega = n^2\cos\theta\,d\sigma\,d\omega,$$

where θ denotes the angle between the normal to the surface and the (mean) direction of the light rays that cross the surface.

That said, if we consider any set of light rays that depends on four parameters, we can take on each ray the point where the ray pierces a given surface (S); all the rays that pass through this same point form a solid cone and the invariant integral $\frac{1}{2}\,\omega'^2$ relative to the given set can be given by the formula

$$I = \int n^2 \cos\theta \, d\sigma \, d\omega,$$

where $d\sigma$ denotes the element of the surface S, $d\omega$ the opening of an elementary cone of rays starting from the same point of S and making an angle θ with the normal to S.

For example, take the set of all light rays which cross a volume bounded by a close surface (S) and take each ray at the point where it *exits* from the volume. We will have, for this set,

$$I = \int n^2 \, d\sigma \int \cos\theta \, d\omega.$$

Now the integral $\int \cos\theta \, d\omega$, by taking for coordinates the longitude φ and the colatitude θ on the sphere of radius 1, is equal to

$$\iint \sin\theta \cos\theta \, d\theta \, d\varphi$$

over the hemisphere $0 \leq \theta \leq \frac{\pi}{2}$; it is thus equal to π. So we have

$$I = \pi \int n^2 \, d\sigma.$$

If the medium is of index 1, the rays are rectilinear and the integral I is equal to the product of the area of the surface with π.

194. As an application of the general integration methods described in in Chapter XVI, we propose to determine the propagation of light rays in an isotropic medium where the refractive index n depends only on one of the rectangular on one of the rectangular coordinates z. Here we know the invariant form ω', as well as three infinitesimal transformations corresponding to a translation parallel to Ox, a translation parallel to Oy and a rotation around of Oz:

$$A_1 f = \frac{\partial f}{\partial x}, \quad A_2 f = \frac{\partial f}{\partial y}, \quad A_3 f = x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} + \alpha \frac{\partial f}{\partial \beta} - \beta \frac{\partial f}{\partial \alpha}.$$

Since these three transformations leave in variant the form

$$\omega_\delta = n(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z),$$

the three linear invariant forms $\omega'(\delta, A_i)$ reduce to

$$\begin{aligned}\delta\omega(A_1) &= \delta(n\alpha),\\ \delta\omega(A_2) &= \delta(n\beta),\\ \delta\omega(A_3) &= \delta[n(\beta x - \alpha y)].,\end{aligned}$$

We thus have three first integrals

$$n\alpha, \quad n\beta, \quad n(\beta x - \alpha y).$$

Put

$$n\alpha = a, \quad n\beta = b, \quad \beta x - \alpha y = c;$$

the last relation shows that any light ray is in a plane parallel to Oz. We thus have

$$\omega_\delta = \delta\left(ax + by + \int \sqrt{n^2 - a^2 - b^2}\, dz\right) - \left(x - a\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}}\right)\delta a - \left(y - b\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}}\right)\delta b.$$

As a result, we have for the trajectories of the light rays

$$x = a\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}} + a', \quad y = b\int \frac{dz}{\sqrt{n^2 - a^2 - b^2}} + b'.$$

Chapter XIX

Fermat's principle and the invariant Pfaffian equation of optics

I. — *Fermat's principle.*

195. In the previous Chapter we considered an invariant integral from the optics of isotropic media, assuming that the refractive index was *time independent*.

Take now any medium in which we assume the propagation of light waves is determined by a Monge equation

$$F(x,y,z,t;\ dx,dy,dz,dt) = 0 \tag{1}$$

which is homogeneous in dx,dy,dz,dt. This means that the wave emanating from a light signal emitted at time t at the point (x,y,z) at time $t+dt$ has as equation

$$F(x,y,z,t;\ X-x, Y-y, Z-z, dt) = 0.$$

The wave surface relative to point (x,y,z) and at time t has equation, as we know,

$$F(x,y,z,t;\ X-x, Y-y, Z-z, 1) = 0.$$

In such a medium a light ray is defined by taking for x,y,z three functions of t that satisfy equation (**??** and, also an additional condition which is what we call *Fermat's principle*. Among all the curves that satisfy equation (1) or, as we say, among all the *integral curves* of Monge's equation (1), the ray of light emanating from a given point (x_0,y_0,z_0) at time t_0 and which passes through the given point (x_1,y_1,z_1) is that curve which minimises the time t_1-t_0 necessary for light to travel from the first point to the second. In other words *the light rays are the extremals of Mayer's problem defined by Monge's equation* (1).

196. Briefly recall how Fermat's principle leads to the formation of the differential equations that define light rays. Imagine a light ray starting from point (x_0, y_0, z_0) at time t_0 and ending at point (x_1, y_1, z_1) at time t_1. Given any integral curve of equation (1) infinitely close to the light ray, we can assume that x, y, z, t are, both for the light ray and for the integral curve, expressed as a function of a parameter u where the values 0 and 1 of this parameter correspond, for the light beam, to time t_0 and time t_1. Let

$$x+\delta x, \quad y+\delta y, \quad z+\delta z, \quad t+\delta t$$

be the functions of u for the varied curve. Denote by x', y', z', t' the derivatives of x, y, z, t with respect to u. Writing equation (1) in the form

$$F(x,y,z,t;\ x',y',z',t') = 0 \tag{2}$$

and by varying this equation,

$$\frac{\partial F}{\partial x}\,\delta x + \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial z}\,\delta z + \frac{\partial F}{\partial t}\,\delta t + \frac{\partial F}{\partial x'}\,\delta x' + \frac{\partial F}{\partial y'}\,\delta y' + \frac{\partial F}{\partial z'}\,\delta z' + \frac{\partial F}{\partial t'}\,\delta t' = 0.$$

Multiply the left hand side of this last equation by $\lambda\, du$, where λ is an undetermined function of u, and integrate from 0 to 1; we will get

$$\int_0^1 \Big[\lambda \frac{\partial F}{\partial x}\,\delta x + \lambda \frac{\partial F}{\partial y}\,\delta y + \lambda \frac{\partial F}{\partial z}\,\delta z + \lambda \frac{\partial F}{\partial t}\,\delta t \\ + \lambda \frac{\partial F}{\partial x'}\,\delta \frac{dx}{du} + \lambda \frac{\partial F}{\partial y'}\,\delta \frac{dy}{du} + \lambda \frac{\partial F}{\partial z'}\,\delta \frac{dz}{du} + \lambda \frac{\partial F}{\partial t'}\,\delta \frac{dt}{du}\Big]\, du = 0,$$

or, integrating by parts,

$$\left[\lambda\left(\frac{\partial F}{\partial x'}\,\delta x + \frac{\partial F}{\partial y'}\,\delta y + \frac{\partial F}{\partial z'}\,\delta z + \frac{\partial F}{\partial t'}\,\delta t\right)\right]_0^1 \\ \int_0^1 \left\{\left[\lambda \frac{\partial F}{\partial x} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial x'}\right)\right]\delta x + \cdots + \left[\lambda \frac{\partial F}{\partial t} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial t'}\right)\right]\delta t\right\} du = 0. \tag{3}$$

If the integral curve close to a light ray satisfies the imposed initial and final conditions, we will have

$$(\delta x)_0 = (\delta y)_0 = (\delta z)_0 = (\delta t)_0 = (\delta x)_1 = (\delta y)_1 = (\delta z)_1 = 0,$$

and consequently

$$\lambda_1 \left(\frac{\partial F}{\partial t'}\right)_1 (\delta t)_1 + \int_0^1 \left\{\left[\lambda \frac{\partial F}{\partial x} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial x'}\right)\right]\delta x + \cdots + \left[\lambda \frac{\partial F}{\partial t} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial t'}\right)\right]\delta t\right\} du = 0.$$

We can choose the functions $\delta x, \delta y, \delta z$ arbitrarily provided that they are zero at the limits of the interval; let us then determine the function λ by the condition that the coefficient of δt in the quantity under the $\int$ sign is zero. For $(\delta t)_1$ to be zero whatever the varied integral curve, it is necessary and sufficient that the coefficients of $\delta x, \delta y, \delta z$ in the quantity under the $\int$ sign are also zero.

In other words, *by introducing an auxiliary quantity λ, the light rays are given by adding equation (2) to the equations*

$$\left.\begin{aligned}
\lambda \frac{\partial F}{\partial x} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial x'}\right) &= 0,\\
\lambda \frac{\partial F}{\partial y} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial y'}\right) &= 0,\\
\lambda \frac{\partial F}{\partial z} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial z'}\right) &= 0,\\
\lambda \frac{\partial F}{\partial t} - \frac{d}{du}\left(\lambda \frac{\partial F}{\partial t'}\right) &= 0.
\end{aligned}\right\} \tag{4}$$

Moreover, elimination of λ gives, besides equation (2), the three equations

$$\frac{\dfrac{\partial F}{\partial x} - \dfrac{d}{du}\left(\dfrac{\partial F}{\partial x'}\right)}{\dfrac{\partial F}{\partial x'}} = \frac{\dfrac{\partial F}{\partial y} - \dfrac{d}{du}\left(\dfrac{\partial F}{\partial y'}\right)}{\dfrac{\partial F}{\partial y'}} = \frac{\dfrac{\partial F}{\partial z} - \dfrac{d}{du}\left(\dfrac{\partial F}{\partial z'}\right)}{\dfrac{\partial F}{\partial z'}} = \frac{\dfrac{\partial F}{\partial t} - \dfrac{d}{du}\left(\dfrac{\partial F}{\partial t'}\right)}{\dfrac{\partial F}{\partial t'}}, \tag{4'}$$

to which we must add

$$\frac{dx}{du} = x', \qquad \frac{dy}{du} = y', \qquad \frac{dz}{du} = z', \qquad \frac{dt}{du} = t.$$

We see immediately that equation (2) derived with respect to u and equation (4') give $\frac{dx'}{du}, \frac{dy'}{du}, \frac{dz'}{du}, \frac{dt'}{du}$ through four equations of the first degree and the values we deduce do not depend on u. The parameter is therefore only involved superficially, as is natural, in the final equations, which are of the form

$$\frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'} = \frac{dt}{t'} = \frac{dx'}{X} = \frac{dy'}{Y} = \frac{dz'}{Y} = \frac{dt'}{T},$$

where X, Y, Z, T are specific functions of $x, y, z, t, x', y', z', t'$, homogeneous of the second degree in x', y', z', t', and satisfying

$$x' \frac{\partial F}{\partial x} + y' \frac{\partial F}{\partial y} + z' \frac{\partial F}{\partial z} + t' \frac{\partial F}{\partial t} + X \frac{\partial F}{\partial x'} + Y \frac{\partial F}{\partial y'} + Z \frac{\partial F}{\partial z'} + T \frac{\partial F}{\partial t'} = 0.$$

In reality, the differential equations for light rays are ordinary differential equations of the first order in $x, y, z, t, \frac{x'}{t'}, \frac{y'}{t'}, \frac{z'}{t'}$ where these seven quantities are assumed to be related by relation (2).

II. — The invariant Pfaffian equation of optics.

197. *Consider now a family of light rays that depend on a parameter α and take each of these light rays in a time interval (t_0, t_1) that depend on α and corresponding to a starting point (x_0, y_0, z_0) that varies with α, and an end point (x_1, y_1, z_1), that also varies with α. If we denote, for each light ray, the auxiliary function that appears in the equations (4) by λ, and if we apply the formula (3), we get*

$$\lambda_1 \left[\left(\frac{\partial F}{\partial x'}\right)_1 \delta x_1 + \left(\frac{\partial F}{\partial y'}\right)_1 \delta y_1 + \left(\frac{\partial F}{\partial z'}\right)_1 \delta z_1 + \left(\frac{\partial F'}{\partial t}\right)_1 \delta t_1 \right]$$
$$= \lambda_0 \left[\left(\frac{\partial F}{\partial x'}\right)_0 \delta x_0 + \left(\frac{\partial F}{\partial y'}\right)_0 \delta y_0 + \left(\frac{\partial F}{\partial z'}\right)_0 \delta z_0 + \left(\frac{\partial F}{\partial t'}\right)_0 \delta t_0 \right]$$

From this it follows that the differential system for light rays*, considered as a system of first-order differential equations in $x, y, z, t, \frac{x'}{t'}, \frac{y'}{t'}, \frac{z'}{t'}$, related by (2),* admits the invariant Pfaffian equation

$$\omega_\delta = \frac{\partial F}{\partial x'} \delta x + \frac{\partial F}{\partial y'} \delta y + \frac{\partial F}{\partial z'} \delta z + \frac{\partial F}{\partial t'} \delta t = 0.$$

This Pfaffian equation, which also depends only on the mutual ratios of x', y', z', t', is really a system in six variables; its characteristic system is a system of ordinary differential equations, which can therefore only be identical to the equations for light rays.

We thus come to the conclusion that *light rays are the characteristics of the Pfaffian equation*

$$\frac{\partial F}{\partial x'} \delta x + \frac{\partial F}{\partial y'} \delta y + \frac{\partial F}{\partial z'} \delta z + \frac{\partial F}{\partial t'} \delta t = 0; \qquad (5)$$

this is the invariant Pfaffian equation of Optics.

198. In practice, Monge's equation (1) is written in the form

$$\Omega\left(x, y, z, t; \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = F\left(x, y, z, t; \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1\right) = 0.$$

By putting

$$\frac{dx}{dt} = \dot{x}, \qquad \frac{dy}{dt} = \dot{y}, \qquad \frac{dz}{dt} = \dot{z},$$

it is easy to form the invariant Pfaffian equation. In fact, we have

$$x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'} + z' \frac{\partial F}{\partial z'} + t' \frac{\partial F}{\partial t'} = 0;$$

consequently equation (5) becomes

$$t' \frac{\partial F}{\partial x'} \delta x + t' \frac{\partial F}{\partial y'} \delta y + t' \frac{\partial F}{\partial z'} \delta z - \left(x' \frac{\partial F}{\partial x'} + y' \frac{\partial F}{\partial y'} + z' \frac{\partial F}{\partial z'} \right) \delta t = 0;$$

Since the left hand side is homogeneous in x', y', z', t', we can replace these arguments by $\dot{x}, \dot{y}, \dot{z}, 1$ respectively. We thus have, for the invariant Pfaffian equation, the form

$$\frac{\partial \Omega}{\partial \dot{x}} \delta x + \frac{\partial \Omega}{\partial \dot{y}} \delta y + \frac{\partial \Omega}{\partial \dot{z}} \delta z - \left(\dot{x} \frac{\partial \Omega}{\partial \dot{x}} + \dot{y} \frac{\partial \Omega}{\partial \dot{y}} + \dot{z} \frac{\partial \Omega}{\partial \dot{z}} \right) \delta t = 0. \tag{6}$$

For example, take a medium in which the wave surface is a sphere, and let $\dfrac{c}{n}$ be the speed of light, where c is the speed in a vacuum and n is the refractive index (a function of x, y, z, t). The Monge equation here is

$$n^2 \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] - c^2 = 0,$$

and the invariant Pfaffian equation is

$$n^2 (\dot{x}\, \delta x + \dot{y}\, \delta y + \dot{z}\, \delta z) - c^2\, \delta t = 0.$$

If we put

$$\alpha = \frac{n\dot{x}}{c}, \qquad \beta = \frac{n\dot{y}}{c}, \qquad \gamma = \frac{n\dot{z}}{c},$$

it becomes

$$n(\alpha\, \delta x + \beta\, \delta y + \gamma\, \delta z) - c\, \delta t = 0;$$

α, β, γ are thus the direction cosines of the tangent to the light ray.

If n does not depend on time, the laws of propagation of light admit the infinitesimal transformation $\dfrac{\partial f}{\partial t}$ and consequently the differential equations which give the light rays admit the invariant form

$$\delta t - \frac{n}{c}(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z).$$

The differential equations which give the (geometric) curves followed by consequently admit the relative integral invariant

$$\int n(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z);$$

we find again the point of view of the previous Chapter (n° **191**).

199. The characteristic equations of the invariant Pfaffian equation of optics can be reduced, as we know (n° **152**), to the characteristic equations of a first-order partial differential equation (the converse is also true, but we will not reciprocal is also true, but we will not dwell on this).

The existence of an integral invariant will be guaranteed whenever the law of propagation of light propagation of light admits an infinitesimal transformation; in all In these cases, we can reduce the search for light rays to an *ordinary* problem in the calculus of variations.

For example, take the case where the law of propagation of light is given by Monge's equation

$$n^2(dx^2 + dy^2 + dz^2) - c^2\,dt^2 = 0,$$

where the refractive index can depend on x, y, t, but *does not depend on* z. We then have the infinitesimal transformation

$$Af = \frac{\partial f}{\partial z};$$

the form

$$\frac{\omega(\delta)}{\omega(A)} = \frac{n(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) - c\,\delta t}{n\gamma} = \delta z + \frac{\alpha}{\gamma}\,\delta x + \frac{\beta}{\gamma}\,\delta y - \frac{c}{n\gamma}\,\delta t$$

is an invariant form. Once we know the coordinates x and y as a function of t, we will get z by a quadrature. As for the differential equations that give x and y as a function of t, they admit the relative integral invariant

$$\int \frac{\alpha}{\gamma}\,\delta x + \frac{\beta}{\gamma}\,\delta y - \frac{c}{n\gamma}\,\delta t$$

or, equivalently, the integral invariant

$$\int \xi\,\delta x + \eta\,\delta y - \zeta\,\delta t,$$

where ξ, η, ζ are three quantities related by the relation

$$1+\xi^2+\eta^2=\frac{n^2}{c^2}\,\zeta^2.$$

The characteristic equations include in particular the equations

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dt}{\dfrac{n^2}{c^2}\,\zeta}=\frac{\sqrt{dt^2-\dfrac{n^2}{c^2}(dx^2+dy^2)}}{\dfrac{n}{c}}.$$

Light rays thus render stationary the integral

$$\int -\xi\,\delta x-\eta\,\delta y+\zeta\,\delta t=\int\sqrt{\frac{c^2}{n^2}\,dt^2-dx^2-dy^2}\,.$$

III. — *Fermat's principle independent of space-time system of reference*

200. It is important to note that the invariant Pfaffian equation of Optics is related to Monge's equation which defines the law of propagation of light in a way that is *independent of the system of reference chosen for space and time. space and time. In other words, the equation*

$$\frac{\partial F}{\partial x'}\,\delta x+\frac{\partial F}{\partial y'}\,\delta y+\frac{\partial F}{\partial z'}\,\delta z+\frac{\partial F}{\partial t'}\,\delta t=0$$

is a covariant *of the equation*

$$F(z,y,z,t;x',y',z',t')=0$$

with respect to any change of variables performed on x,y,z,t. Basically, this follows from Fermat's principle itself; but we can also find this equation in the following way, where nothing differentiates the variables x,y,z,t from one another.

Consider the Pfaffian system

$$\frac{dx}{x'}=\frac{dy}{y'}=\frac{dz}{z'}=\frac{dt}{t'}, \tag{7}$$

where $x,y,z,t,\dfrac{x'}{t'},\dfrac{y'}{t'},\dfrac{z'}{t'}$ are assumed to be related by equation (2), and let us look for the *derived* system of (7). This is what we call the system formed by the Pfaffian equations which are linear combinations of equation (7) and which have the property that the exterior derivative of their left hand side is zero when we take into account equations (7). Putting, as above,

$$\frac{x'}{\dot{x}} = \frac{y'}{\dot{y}} = \frac{z'}{\dot{z}} = \frac{t'}{1},$$

any linear combination of equations (7) is of the form

$$u(dx - \dot{x}\,dt) + v(dy - \dot{y}\,dt) + w(dz - \dot{z}\,dt) = 0.$$

If we take equations (7) into account, the exterior derivative of the left hand side reduces to

$$[dt\ (u\,d\dot{x} + v\,d\dot{y} + w\,d\dot{z})];$$

the condition that it be zero when we take (**??**) and the differentiated equation (2) into account is

$$[dt\ (u\,d\dot{x} + v\,d\dot{y} + w\,d\dot{z})\ (dx - \dot{x}\,dt)\ (dy - \dot{y}\,dt)\ (dz - \dot{z}\,dt)\ dF] = 0,$$

or, simplifying,

$$\left[dx\,dy\,dz\,dt\ (u\,d\dot{x} + v\,d\dot{y} + w\,d\dot{z})\ \left(\frac{\partial F}{\partial \dot{x}}\,d\dot{x} + \frac{\partial F}{\partial \dot{y}}\,d\dot{y} + \frac{\partial F}{\partial \dot{z}}\,d\dot{z}\right)\right] = 0.$$

This gives

$$\frac{u}{\dfrac{\partial F}{\partial \dot{x}}} = \frac{v}{\dfrac{\partial F}{\partial \dot{y}}} = \frac{w}{\dfrac{\partial F}{\partial \dot{z}}}\,.$$

The derived system of system (7) is very simply the Pfaffian equation

$$\frac{\partial F}{\partial \dot{x}}(dx - \dot{x}\,dt) + \frac{\partial F}{\partial \dot{y}}(dx - \dot{y}\,dt) + \frac{\partial F}{\partial \dot{z}}(dx - \dot{z}\,dt) = 0,$$

whose characteristics are the light rays.

It follows from this that, even in Optics, the time coordinate does not play an essentially different role from that played by space coordinates. The fundamental laws of Optics are not related necessarily to the classical concepts of space and time, and carry over as they are into the theory of relativity.

201. For example, by choosing a suitable system of reference for the universe (space-time), the laws of propagation of light in the gravitational field produced by a single mass (reduced to a point) are given by Schwarzschild's equation

$$\frac{dr^2}{1 - \dfrac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) - \left(1 - \frac{2m}{r}\right)dt^2 = 0.$$

These laws admit the infinitesimal transformation $\frac{\partial f}{\partial t}$; the light rays, considered solely from the point of view of space, are therefore defined as realising the extremum of the integral

$$\int \sqrt{\frac{dr^2}{\left(1-\frac{2m}{r}\right)^2}+\frac{r^2(d\theta^2+\sin^2\theta\, d\varphi^2)}{-\frac{2m}{r}}}$$

Propagation takes place in a plane passing through the centre of attraction and if we assume that this plane is defined by $\varphi = 0$, we have to achieve the extremum of the integral

$$\int \sqrt{\frac{dr^2}{\left(1-\frac{2m}{r}\right)^2}+\frac{r^2\, d\theta^2}{-\frac{2m}{r}}}$$

Using the existence of the infinitesimal transformation $\frac{\partial f}{\partial t}$, the integration is straightforward and gives

$$\dot{\theta} = \int \frac{C\,\frac{2m}{r^2}}{\sqrt{1-\frac{C^2\left(1-\frac{2m}{r}\right)}{r^2}}}\,.$$

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E. Cartan. — *Sur l'intégration des systèmes différentiels complètement intégrables*; C. R. Acad. des Sc. Paris, Vol. CXXXV, (1902), p. 1415-1417 ; 1564-1566.

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On the so called symbolic calculus applicable to exterior differential forms, and on certain related symbolic calculi, one could consult, in addition to the preceding memoirs:

E. Cartan. — *Sur certaines expressions différentielles et le problme de Pfaff*; Ann. Ec. Norm., (3), Vol. XVI, (1899), p. 239-332.

A. Buhl. — *Sur les transformations et extensions de la formule de Stokes*; Ann. Fac. Sc. Toulouse, Vol. IV, (1912); Vol. VI, (1914); Vol. VII, (1915).

Finally, on the related subject of integral invariants of continuous groups of transformations, where the point of view is a little different from that of H. Poincar, see:

S. Lie. — *Die Theorie der Integralinvarianten ist ein Corollar der Theorie der Differentialinvarianten*; Ber. Sächs. Gesellsch., Leipzig, 1897, p. 343-357.

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Glossary of terms

All references in this glossary are to the second edition.

Absolute integral invariant. p 25-26, "An integral invariant is said to be *absolute* if its property of invariance is valid whatever may be the domain of integration; it is called *relative* if the property of invariance is valid only for a domain of integration that is closed." More explicitly, an integral invariant is *absolute* if the integral remains invariant as we slide its domain of integration along the state-space trajectories of the system whatever the nature of the domain of the integration, that is, whether the domain of integration is open or closed.

Associated system of equations. The definition of the system of equations associated with a differential form F is found on p 58, "If the form is of degree p, the associated system is obtained by setting to zero all the partial derivatives of F of order $p-1$." The 'partial derivatives of F' are defined on page 58, following this definition. In effect, these are the inner product of the form F with the vectors $\boldsymbol{e}_i$ of a vector basis. Cartan first introduces the concept of an associated system of equations on p 49-50 in connection with an ordinary quadratic form, that is, a symmetric quadratic form, and extended to the case of an exterior quadratic form in §58, p 53. The general definition is then given on p 58.

Initially, Cartan defines the associated system of equations only in connection with single *algebraic* exterior forms. He then extends it to *systems* of algebraic exterior forms. This definition is carried over without change to the case of differential exterior forms, and to systems of exterior differential forms, in Chapter 8 and the following chapters.

Though Cartan defines the associated system of equations in terms of partial derivatives, which are equivalent to inner products with the vectors of a chosen basis, his definition is equivalent to the modern definition, which can be stated as follows: Let F be a given exterior form (algebraic or differential) and let $\boldsymbol{X}$ be an arbitrary vector; then the associated equation is given by

$$\boldsymbol{X} \rfloor F = 0$$

This equation, generally called the *characteristic equation*, when expressed as a set of scalar equations by choosing a basis, produces Cartan's system of equations associated with F. Similarly, the set of equations associated with a *system* $F, G, \ldots H$ of forms is obtained from the equations

$$\boldsymbol{X} \rfloor F = 0, \quad \boldsymbol{X} \rfloor G = 0, \; \ldots, \; \boldsymbol{X} \rfloor H = 0$$

Integral invariant. p 25, "H Poincare gives the name *integral invariant* to an integral (simple or multiple) which, extended over an arbitrary set of *simultaneous* points (that is to say, all corresponding to the same value of t), does not change value when we move the points of this set along the corresponding trajectories to another instant t'."

It is important to understand the context in which this definition is made. Poincare appears to have been working in the context of an n-dimensional space with points $(x^1, \ldots, x^n)$. In time, a fluid point changes its position in this space, following a trajectory $x^i(t)$ which is a solution to the system of equations (1) on p 25, namely

$$\frac{dx^i}{dt} = X^i(x^1, \ldots, x^n, t)$$

In this n dimensional space, imagine at time t a curve C consisting of fluid points. This curve may be open or closed. A line integral over this curve would give a certain value I if evaluated at time t. The same fluid points would lie on a different curve C' at a later time t'. The same line integral evaluated of the *same material points*, i.e. over the displaced curve C', at time t' would yield a new value I'. If we have $I' = I$ *for all times* t', then the integral I is said to be an invariant integral. In classical notation,

$$\frac{DI}{Dt} = 0$$

The integral need not be a line integral; it might be a double or triple integral, or one of any dimension at all.

Cartan's view is different from that of Poincare. Cartan works in the *extended* space with coordinates $(x^1, \ldots, x^n; t)$. The different times t and t' imagined by Poincare become, in Cartan's picture, different time slices $t =$constant and $t' =$ constant of the extended space. The trajectories of the fluid points are curves that are transverse to these time slices, and the material curve C becomes a curve consisting of *simultaneous points* $(x^1(\alpha), \ldots, x^n(\alpha); t)$ at time t. The curve C' is then the curve formed by the intersection of the trajectories through these points and the time slice $t' =$ constant. This curve consists of simultaneous points at the new time t'.

An integral invariant is then an integral whose value does not change as we slide the points of C along the trajectories in such a way as to keep all of the points *simultaneous.*

Invariant differential form. §31, p 29, "We shall agree to say that a differential form that can be expressed by means of the first integrals of the system (1) and their differentials is an *invariant form* for system (1)." Equations (1) are, in index notation, with $\boldsymbol{x} = (x^1, \ldots, x^n)$,

$$\frac{dx^i}{dt} = X^i(\boldsymbol{x}, t)$$

Force vive. p 189, Cartan says, *on désigne par* $2T$ *la force vive*, or "call $2T$ the *force vive*". In other words, for Cartan, the *force vive* is twice the kinetic energy.

Relative integral invariant. p 25-26, "An integral invariant is said to be *absolute* if its property of invariance is valid whatever may be the domain of integration; it is called *relative* if the property of invariance is valid only for a domain of integration that is closed." More explicitly, an integral invariant is *relative* if the integral remains invariant as we slide its domain of integration along the state-space trajectories of the system *only if the domain of the integration is closed*, that is, a closed curve, or a closed surface, etc., but does not necessarily remain invariant if the domain of integration is open.