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# Lectures on The Geometry of Riemannian Spaces

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# Contents

# PREFACE

## TO THE FIRST EDITION

This book reproduces a course given during the first semester 1925-1926 at the Faculty of Sciences of the University of Paris. I have adopted in principle the same point of view as in Volume IX of *Mémorial des Sciences Mathématiques* devoted to the same subject. I have almost always used the analytic apparatus imposed by the system of coordinates in terms of which the line element, assumed given, of the space to be studied is formed. This required concepts from the absolute differential calculus, which I have tried to present by bringing out as clearly as possible the essential geometric content and by always maintaining the closest contact with Euclidean geometry. The enourmous service which has been rendered and which will still be rendered by the absolute differential calculus of Ricci and Levi-Civita should not prevent us from avoiding calculations that are altogether too formal, or the debauchery of indices masking a geometrical reality that is often very simple. It is this reality that I have sought to reveal at all times.

I have discussed at length the interesting problem of spaces which, while being locally Euclidean, differ from our ordinary space from the point of view of the *Analysis Situs*; these are the "Clifford-Klein spatial forms" of the German geometers. The prospects that the solution to this problem opens up for the foundations of elementary geometry and on certain theories of analysis seem to me to justify the space that I have devoted to it. It is a little for the same reasons that I have examined the important role played in geometry by the axiom of the plane and the axiom of free mobility, intimately related to each other. This led me quite naturally to a brief study of non Euclidean geometries, especially in two dimensions: the services that such a study can render in various fields of mathematics need no longer to be demonstrated.

The first two Notes at the end of this book take up certain concepts studied in the course of the volume, but introduce much less restrictive hypotheses about the analytic nature of the coefficients of the fundamental form. In this regard, I believe that the concept of *linear* (and *not superficial*) Riemannian curvature had not yet been reported; it would without doubt have applications in the theory of relativity. The third Note, devoted to spaces with variable, but negative, curvature is related to a longtime classic Note by Hadamard on the geodesics of surfaces with negative curvatures.

The Bibliography at the end of the volume is limited to fundamental books and memoirs related to the topics discussed.

I have left out many important problems in this volume. They will perhaps be the subject of a later book, in which I will explain the method of the moving rectangular frame, with its numerous applications.

Finally, I would like to thank the House of Gauthier-Villars, who have edited this book with their usual care and diligence.

E. Cartan.

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# PREFACE

## TO THE SECOND EDITION

This new edition differs from the first by some changes and additions, of which I will mention the most Important.

I have adopted the Einstein's now classic convention which consists of deleting the summation sign wherever there is no fear of ambiguity. The most important change is the substitution of the notation $\omega'$ to denote what I used to call the "exterior derivative" of the differential form $\omega$, by the more satisfying notation $d\omega$, due to E. Kähler, to denote what I now call the "exterior differential" of $\omega$. The symbol $D$ now serves to denote absolute differentiation, whether of an ordinary tensor or of a tensorial differential form, hence a desirable unification of notations.

The most important additions in this second edition are of three kinds. First, I have introduced a new chapter on the method of the moving frame, with applications to the properties of manifolds immersed in a Riemannian space. Second, the chapter on the Riemann normal coordinates (old Chapter IX) has been split into two by the addition of a new Chapter XI on symmetry, parallel transport and symmetric spaces. Finally, two new Chapters (XII and XIII), the first of which is very extensive, are devoted to groups of displacements of Riemannian spaces and to the conditions of applicability of two Riemannian spaces. All these new Chapters make almost exclusive use of the method of the moving frame.

Finally, two new Notes have been added to the three Notes of the first edition: one (Note IV) is devoted to properties of geodesics of a normal Riemannian space, the other (Note V) to completely integrable Pfaffian systems.

The House of Gauthier-Villars, in spite of the difficulties of the present time, as always have given their full attention to this new edition. To them I express my most sincere thanks.

E. Cartan.

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# 1 Cartesian Coordinates, Vectors, Multivectors, Tensors

## I.– Vectors, cartesian coordinates.

**1.** We assume known the classical theorems on the addition of vectors. Recall that if we denote a vector by a bold faced letter $\boldsymbol{x}$, the symbol $m\,\boldsymbol{x}$, where $m$ is a numerical coefficient, denotes a vector parallel to $\boldsymbol{x}$ whose length that of $\boldsymbol{x}$ in the ratio $m$, and which has the same direction as $\boldsymbol{x}$ or opposite direction according as $m$ is positive or negative.

Recall also that if we choose rectangular axes in an Euclidean space, of which we assume the number of dimensions equal to $n$, the square of the length of a vector $\boldsymbol{x}$, whose components are $X_1, X_2, ..., X_n$, is

$$X_1^2 + X_2^2 + \cdots + X_n^2;$$

the *scalar product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with components $X_i$ and $Y_i$ is similarly

$$\boldsymbol{x} \cdot \boldsymbol{y} = X_1Y_1 + X_2Y_2 + \cdots + X_nY_n.$$

Note that we can obtain this scalar product as the coefficient of $2\lambda$ in the expression that gives the square of the length of the vector $\boldsymbol{x} + \lambda\boldsymbol{y}$:

$$(\boldsymbol{x} + \lambda\boldsymbol{y})^2 = \boldsymbol{x}^2 + 2\lambda\boldsymbol{x} \cdot \boldsymbol{y} + \lambda^2\boldsymbol{y}^2 = \sum_i X_i^2 + 2\lambda\sum_i X_iY_i + \lambda^2\sum_i Y_i^2.$$

Scalar multiplication of two vectors is *commutative* and *distributive*, expressed by the formulas

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}, \quad (\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{z} + \boldsymbol{y} \cdot \boldsymbol{z}.$$

Finally, the cosine of the angle $V$ between the two vectors $\boldsymbol{x}, \boldsymbol{y}$ is

$$\cos V = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\sqrt{\boldsymbol{x}^2\,\boldsymbol{y}^2}} = \frac{\sum_i X_iY_i}{\sqrt{\sum_i X_i^2 \cdot \sum_i Y_i^2}}. \tag{1.1}$$

**2.** In the ordinary space we obtian the most general system of *Cartesian* coordinates by taking three coordinate axes $Ox_1, Ox_2, Ox_3$, and by choosing on each of them an unit of length. The coordinates of a point $M$ are then the algebraic measures of the projection of the vector $OM$ onto these axes, each projection being measured with the unit of length chosen on the corresponding axis, and the projection is made onto each axis parallel to the plane determined by the two other axes. This system of coordinates, more general than those which we habitually consider in analytic geometry, clearly have the property that we pass from one of these systems to another by a *linear* substitution with constant coefficients.

In fact, let $x_1^0, x_2^0, x_3^0$ be the coordinates of the origin of the new system with respect to the old; denote furthermore respectively by

$$\begin{array}{ll} (O'x'_1) \qquad\qquad & a_{11}, a_{21}, a_{31}; \\ (O'x'_2) \qquad\qquad & a_{12}, a_{22}, a_{32}; \\ (O'x'_3) \qquad\qquad & a_{13}, a_{23}, a_{33}; \end{array}$$

the projections onto the old axes of the *unitary* vectors carried by the new axes. We pass from the coordinates $x'_1, x'_2, x'_3$ of a point with respect to the new system to the coordinates $x_1, x_2, x_3$ of this point relative to the old system by means of the formulas

$$\begin{aligned} x_1 &= x_1^0 + a_{11}x'_1 + a_{12}x'_2 + a_{13}x'_3, \\ x_2 &= x_2^0 + a_{21}x'_1 + a_{22}x'_2 + a_{23}x'_3, \\ x_3 &= x_3^0 + a_{31}x'_1 + a_{32}x'_2 + a_{33}x'_3. \end{aligned}$$

Conversely, all groups of formulas of the above form define a change of Cartesian coordinates, if the determinant of the elements of the tableau of the $a_{ij}$ is not zero, and we can choose arbitrarily the old system of coordinates.

If it is a vector, the formulas of the change of coordinates reduce to

$$\begin{aligned} X_1 &= a_{11}X'_1 + a_{12}X'_2 + a_{13}X'_3 \\ X_2 &= a_{21}X'_1 + a_{22}X'_2 + a_{23}X'_3 \\ X_3 &= a_{31}X'_1 + a_{32}X'_2 + a_{33}X'_3 \end{aligned}$$

where we have denoted by $X_1, X_2, X_3$ the projections of the vector onto the old axes, and by $X'_1, X'_2, X'_3$ its projections onto the new axes.

These considerations may be generalised without difficulty in the space of $n$ dimensions.

**3.** Since the components of a vector in any system of Cartesian coordinates are deduced by a linear homogeneous substitution of its components in a rectangular system of coordinates, the square of the length of a vector with components $X_1, ..., X_n$ will be given by a quadratic form

$$\sum_{i,j} g_{ij} X_i X_j = g_{11} X_1^2 + \cdots + 2g_{12} X_1 X_2 + \cdots . \tag{1.2}$$

Knowledge of the coefficients of this form allows us to form the scalar product of two vectors $\boldsymbol{x}, \boldsymbol{y}$. We have, in fact,

$$(\boldsymbol{x}+\lambda\boldsymbol{y})^2 = \sum_{ij} g_{ij}X_iX_j + 2\lambda\sum_{ij} g_{ij}X_iY_j + \lambda^2\sum_{ij} g_{ij}Y_iY_j$$

from which ($\text{n}^{\text{o}}$ 1)

$$\boldsymbol{x}\cdot\boldsymbol{y} = \sum_{ij} g_{ij}X_iY_j = g_{11}X_1Y_1 + \cdots + g_{12}(X_1Y_2 + X_2Y_1) + \cdots \tag{1.3}$$

It is necessary to assume that the right hand side the indices $i, j$ take, independently of one another, all possible values.

We deduce from this, for the cosine of the angle between two vectors,

$$\cos V = \frac{\displaystyle\sum_{i,j} g_{ij}X_iY_j}{\sqrt{\displaystyle\sum_{i,j} g_{ij}X_iX_j}\sqrt{\displaystyle\sum_{i,j} g_{ij}Y_iY_j}}\,. \tag{1.4}$$

The coefficients $g_{ij}$ can furthermore be interpreted geometrically as the scalar products two by two of the unitary vectors on the axes. If, in fact, we denote these vectors (*basis vectors*) by $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n$, the vector $\boldsymbol{e}_i$ has all its components equal to zero, except the $i^{\text{th}}$ which is equal to 1, and we have, by applying formula (1.3),

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij} \tag{1.5}$$

**4.** From now on, we will write the components of a vector with superscript indices:

$$\boldsymbol{x} = X^1\boldsymbol{e}_1 + X^2\boldsymbol{e}_2 + \cdots + X^n\boldsymbol{e}_n$$

Let us introduce, with G. Ricci and T. Levi-Civita, $n$ new quantities $X_i$ (not to be confused with those which were represented by this notation in the preceding paragraphs). The quantity $X_i$ will be by definition the scalar product $\boldsymbol{x}\cdot\boldsymbol{e}_i$ of the given vector $\boldsymbol{x}$ by the unitary vector $\boldsymbol{e}_i$. According to formula (1.3), we have

$$X_i = g_{ij}X^k \qquad (i = 1, 2, ..., n) \tag{1.6}$$

According to a now classical convention, we have suppressed on the right hand side the summation sign $\sum$: it is therefore understood that it is necessary to form the sum of all the terms obtained by successively giving to $k$ the values $1, 2, ..., n$: we note that this index $k$ appears twice, once as a subscript, and once as a superscript, in the general term on the right hand side.

By the introduction of these new quantities, which we call the *covariant components* of the vector, whereas the ordinary components $X^i$ are called the *contravariant components*, the expression for the scalar product of two vectors $\boldsymbol{x}, \boldsymbol{y}$

can be put indifferently in one of the two simple forms

$$\boldsymbol{x} \cdot \boldsymbol{y} = X_i Y^i = X^i Y_i, \tag{1.7}$$

and we have, for the square of the length of a vector,

$$\boldsymbol{x}^2 = X_i X^i. \tag{1.8}$$

Conversely, we pass from the covariant components of a vector to its contravariant components by the formulas

$$X^i = g^{jk} X_k, \tag{1.9}$$

denoting by $g^{ij}$ the minor with respect to $g_{ij}$ of the determinant

$$\begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix},$$

this minor being divided by the value of the determinant of the same, which we shall denote by $g$.

We may note that a vector all of whose contravariant components are zero, except for the $i^{\text{th}}$, is parallel to the $i^{\text{th}}$ coordinate axis, whereas a vector all of whose covariant components are zero, except the $i^{\text{th}}$, is perpendicular to all the coordinate axes except the $i^{\text{th}}$, that is, it is perpendicular to the $(n-1)$-plane determined by these $n-1$ axes.

it is unnecessary to point out that rectangular coordinates the covariant components and the contravariant ones merge.

## II.– Bivectors, systems of bivectors.

**5.** We call a *bivector* the figure formed by two vectors $\mathbf{x}, \mathbf{y}$ arranged in a definite order.

This definition only makes sense if we define the *equality* of two bivectors.

To this end, associate to a bivector a parallelogram $OACB$ whose sides $OA$ and $OB$ are equivalent respectively to vectors $\mathbf{x}, \mathbf{y}$, the outline of this parallelogram being traversed beginning with $OA$. Two bivectors are then said to be equal if the two parallelograms associated with them are in the same plane (or in parallel planes), have the same area, and are traversed in the same direction. We also call the plane of the associated parallelogram, or any plane parallel to it, the *plane of the bivector.*

Therefore let two bivectors be defined, one by the vectors $\mathbf{x}$ and $\mathbf{y}$ and the other by the vectors $\mathbf{u}$ and $\mathbf{v}$. The equations that express that a vector $\mathbf{z}$ is perpendicular to the plane of the first bivector are

$$Z_1X^1 + Z_2X^2 + \cdots + Z_nX^n = 0,$$
$$Z_1Y^1 + Z_2Y^2 + \cdots + Z_nY^n = 0.$$

We see that the components $Z_1, Z_2, ..., Z_n$ of this vector are subject to the single relation obtained by eliminating $Z_1$, namely

$$Z_2(X^1Y^2 - X^2Y^1) + Z_3(X^1Y^3 - X^3Y^1) + \cdots + Z_n(X^1Y^n - X^nY^1) = 0.$$

Put

$$P^{ij} = X^iY^j - X^jY^i, \tag{1.10}$$

and similarly

$$Q^{ij} = U^iV^j - V^jU^i.$$

If the two equations

$$Z_iP^{1i} = 0, \quad Z_iQ^{1i} = 0$$

are to be equivalent, the quantities $P^{1i}$ and $Q^{1i} (i > 1)$ must be proportional. This means basically that the ratios $P^{ij}/Q^{ij}$ do not change when we change only one of the indices $i, j$. It follows immediately that it is the same whatever the values of the two indices. *The first condition for equality of two bivectors is thus translated by relations of the form*

$$Q^{ij} = \lambda P^{ij}$$

We note moreover that analogous reasoning would lead us to the relations

$$Q_{ij} = \mu P_{ij},$$

by setting

$$P_{ij} = X_iY_j - X_jY_i\ ; \quad Q_{ij} = U_iV_j - V_jU_i, \tag{1.11}$$

and we see easily that $\mu = \lambda$.

**6.** To express the second condition for equality, evaluate the area of the parallelogram $OACB$ associated with the first given bivector. The square of this area is

$$m^2 = \overline{OA}^{\,2}\overline{OB}^{\,2}\sin^2\widehat{AOB} = \overline{OA}^{\,2}\overline{OB}^{\,2} - (\overline{OA}\ \overline{OB}\cos\widehat{AOB})^2$$
$$= \mathbf{x}^2\mathbf{y}^2 - (\mathbf{x}\cdot\mathbf{y})^2 = \begin{vmatrix} \mathbf{x}^2 & \mathbf{x}\cdot\mathbf{y} \\ \mathbf{y}\cdot\mathbf{x} & \mathbf{y}^2 \end{vmatrix}.$$

Replacing the scalar products by their values, we have

$$m^2 = \begin{vmatrix} X_i & X_j \\ Y_i & Y_j \end{vmatrix} X^iY^j = P_{ij}X^iY^j.$$

Noting that the sum $P_{ij}X^iY^j$ is equal to the sum $P_{ji}X^jY^i = -P_{ij}Y^iX^j$, we deduce the definitive relation that we sought,

$$m^2 = \frac{1}{2}\, P_{ij}(X^iY^j - X^jY^i) = \frac{1}{2}\, P_{ij}P^{ij}. \tag{1.12}$$

It will be noted that, for example with $n = 3$, formula (1.12) expanded gives

$$m^2 = P_{12}P^{12} + P_{13}P^{13} + P_{23}P^{23}.$$

The second condition for the equality of bivectors is, in consequence of formula (1.12), $\lambda^2 = 1$, that is to say,

$$Q^{ij} = \pm P^{ij}.$$

The third condition (equality of direction) is clearly translated by the presence of the + sign and not by the − sign, and we arrive at the following theorem:

> *For two bivectors to be equal, it is necessary and sufficient that the $n(n-1)/2$ quantities $P^{ij}$ defined by formulas (1.10) be the same for the two bivectors.*

These quantities $P^{ij}$ will be called the *coordinates* of the bivector.

The conditions for equality can also be reduced to the equality of the quantities $P_{ij}$, which we call the *covariant components* of the bivector, the $P^{ij}$ being its *contravariant components.*

Let us recall finally formula (1.12) which gives the square of the measure of a bivector.

**7.** We can proceed differently to find the conditions for equality of two bivectors. The conditions that express that a vector $\boldsymbol{z}$ is parallel to the plane of the bivector defined by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are that all the determinants with three lines and three columns formed from the array

$$\begin{Vmatrix} Z^1 & Z^2 & \cdots & Z^n \\ X^1 & X^2 & \cdots & X^n \\ Y^1 & Y^2 & \cdots & Y^n \end{Vmatrix}$$

are zero. We will have, in particular, between the $Z^1, Z^2, Z^3$, the single relation

$$Z^1P^{23} + Z^2P^{31} + Z^3P^{12} = 0.$$

It follows that the equality of two bivectors leads to the relations

$$\frac{Q^{23}}{P^{23}} = \frac{Q^{31}}{P^{31}} = \frac{Q^{12}}{P^{12}}\ ;$$

this is to say basically that the ratio $Q^{ij}/P^{ij}$ does not change when we change one of the indices while conserving the other. All these ratios are thus equal to one another. Conversely, if we have

$$Q^{ij} = \lambda P^{ij},$$

the conditions that express that the vector $\boldsymbol{z}$ is parallel to the plane of the bivector are clearly the same for the two bivectors.

That said, let us note that $Q^{12}/P^{12}$ is none other than the ratio of the areas of the two parallelograms obtained by projecting onto the plane $x_1Ox_2$ the parallelograms associated with the two bivectors, the projection being made naturally in the direction of the $(n-2)$-plane formed by the axes $Ox_3, ..., Ox_n$. The two bivectors will be equal if this ratio is equal to 1, and conversely. The conditions for equality are thus

$$Q^{ij} = P^{ij}. \tag{1.13}$$

**8.** Passage from the contravariant components $P^{ij}$ of a bivector to its covariant components $P_{ij}$ is made without difficulty. We have

$$P_{ij} = \begin{vmatrix} X_i & X_j \\ Y_i & Y_j \end{vmatrix} = \begin{vmatrix} g_{ih}X_h & g_{jk}X^k \\ g_{ih}Y^h & g_{jk}Y^k \end{vmatrix} = g_{ih}g_{jk}P^{hk}.$$

We see that the square of the measure of a bivector, which we can write as

$$\frac{1}{2}\,P_{ij}P^{ij} = \frac{1}{4}\,(g_{ih}g_{jk} - g_{ik}g_{jh})P^{ij}P^{hk}, \tag{1.14}$$

is given by an expression entirely analogous to that which gives [formula (1.2)] the square of the length of a vector. The only difference is that here a bivector has $n(n{-}1)/2$ components $P^{ij} = -P^{ji}$, and that the coefficients of the corresponding quadratic form are, for the two *compound indices* $(ij)$ and $(hk)$,

$$g_{(ij)(hk)} = g_{ih}g_{jk} - g_{ik}g_{jh}\ ;$$

Note that these coefficients are still *symmetric*,

$$g_{(hk)(ij)} = g_{(ij)(hk)}\ .$$

Furthermore, forming the covariant components $P_{ij}$ from the covariant components $P^{hk}$ is done in exactly the same way as for an ordinary vector, by substituting for the coefficients $g_{ij}$ the coefficients $g_{(ij)(hk)}$. every bivector can thus be represented by a vector of an Euclidean space of dimension $n(n-1)/2$, whose metric will be defined by the coefficients $g_{(ij)(hk)}$.

**9.** These results allow us to define the *scalar product* of two bivectors by any one of two equivalent expressions

$$\frac{1}{2}\,P_{ij}Q^{ij} = \frac{1}{2}\,P^{ij}Q_{ij}. \tag{1.15}$$

This scalar product clearly has a numerical value that is independent of the choice of coordinates. To interpret it geometrically, choose *rectangular* axes whose two first ones are parallel to the plane of the first bivector. Since the components $P^{ij} = P_{ij}$ are all zero except for $P_{12}$, the scalar product reduces to

$$P_{12}Q_{12} = (X_1 Y_2 m - X_2 Y_1)(U_1 V_2 - U_2 V_1) \ :$$

*it is clearly equal to the product of the measure $P_{12}$ of the first bivector with the measure of the orthogonal projection of the second bivector onto the plane of the first.* It is zero when there exists a direction parallel to the plane of one of the bivectors and perpendicular to the plane of the other.

We check immediately that, in ordinary space, the scalar product of two bivectors is equal to the product of their measures with the cosine of the angle between their planes. One can thus define the angle $\varphi$ of the planes of the two bivectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by the formula

$$\cos\varphi = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\sqrt{\boldsymbol{a}^2}\,\sqrt{\boldsymbol{b}^2}} \ ;$$

the right hand side, as easily dawns on one, is always less than 1 in absolute value.

**10.** The vector $\boldsymbol{u}$ with components

$$U^i = P^{ki} Z_k \tag{1.16}$$

is called the *interior product* of a bivector with components $P^{ij}$ and a vector with components $Z^i$.

It is important to note that this vector has a meaning that is independent of the choice of axes. In fact, the scalar product of this vector $\boldsymbol{u}$ with an arbitrary vector $\boldsymbol{v}$ is

$$U^i V_i = P^{ki} Z_k V_i = \frac{1}{2}\, P^{ik}(Z_k V_i - Z_i V_k) \ ;$$

it is the scalar product of the given bivector and of the bivector defined by $\boldsymbol{z}$ and $\boldsymbol{v}$, a scalar product that is independent of the choice of axes. The product $\boldsymbol{u}\cdot\boldsymbol{v}$ being, whatever the fixed vector $\boldsymbol{v}$, independent of the choice of axes, it is the same of the vector $\boldsymbol{u}$.

Since the scalar product of bivectors which has just been considered is also expressible as

$$\frac{1}{2}\, P_{ki}(Z^k V^i - Z^i V^k) \ ,$$

we also have

$$U_i = P_{ki} Z^k \ . \tag{1.17}$$

To interpret geometrically this vector $\boldsymbol{u}$, choose rectangular axes where the two first are parallel to the plane of the bivector. We will have

$$U_1 = -P_{12} Z_2, \quad U_2 = P_{12} Z_1, \quad U_3 = 0, \quad \ldots, \quad U_n = 0.$$

We see immediately the following properties:

**1°** *The vector* $\boldsymbol{u}$ *is zero when the vector* $\boldsymbol{z}$ *is perpendicular to the plane of the bivector;*

**2°** *When the vector* $\boldsymbol{z}$ *is parallel to the plane of the bivector (and so we can assume it parallel to* $Ox_1$*), we obtain the vector* $\boldsymbol{u}$ *by turning the vector* $\boldsymbol{z}$ *through a right angle parallel to the plane of the bivector, and by multiplying it by the measure of the bivector.*

**3°** *In the general case, the scalar product is the same as if we were to replace the vector* $\boldsymbol{z}$ *by its orthogonal projection onto the plane of the bivector.*

In three dimensional space, the interior product of a bivector $\boldsymbol{a}$ by a vector $\boldsymbol{z}$ is equal in magnitude to the product of the measure of the bivector by the length of the vector $\boldsymbol{z}$, and by the cosine of the angle of the vector with the plane of the bivector. In the general case, we will define the angle $\varphi$ made by the direction of a vector $\boldsymbol{z}$ with the plane of a bivector $\boldsymbol{a}$ by the formula

$$\cos\varphi = \frac{\mathrm{mes}(\boldsymbol{a}\cdot\boldsymbol{z})}{\mathrm{mes}(\boldsymbol{a}).\mathrm{mes}(\boldsymbol{z})}$$

where we denote by mes $\boldsymbol{a}$ the number which measures $\boldsymbol{a}$.

**11.** A set of any number of bivectors is called a *system of bivectors.* The algebraic sum of the scalar products of the bivectors of the system by the given bivector will be called the *scalar product* of the system of bivectors with the given bivector.

Two systems of bivectors will be said to be *equal* if they have the same scalar product with an *arbitrary* bivector. It follows that a system of bivectors, if we do not regard as distinct two equal systems of bivectors, is completely defined by the $n(n-1)/2$ quantities obtained by adding the components with the same indices of the bivectors of the system. A system of bivectors has contravariant components $P^{ij}$ and covariant components $P_{ij}$.

We will define, as we did for a bivector, the scalar product of two systems of bivectors, the interior product of a system of bivectors by a vector, etc.

We often mean by the term *bivector* any system of bivectors; bivectors properly so called are then said to be *simple bivectors.* In three dimensional space, all bivectors are simple. In fact, let $P^{23}, P^{31}, P^{12}$ be three arbitrary numbers. Take two independent solutions, on the one hand $X^1, X^2, X^3$; $Y^1, Y^2, Y^3$ on the other hand, of the linear equation

$$P^{23}X^1 + P^{31}X^2 + P^{12}X^3 = 0\ ;$$

we deduce immediately that

$$\frac{X^2Y^3 - X^3Y^2}{P^{23}} = \frac{X^3Y^1 - X^1Y^3}{P^{31}} = \frac{X^1Y^2 - X^2Y^1}{P^{12}}\ ;$$

it then suffices to multiply the $Y^i$ by the same conveniently chosen factor $\rho$ for the simple bivector defined by $\boldsymbol{x}$ and $\rho\boldsymbol{y}$ to have the given components.

In a space of four dimensions, the theorem is no longer true. Moreover, it can be shown immediately that a simple bivector, with components $P^{ij}$, in a space with any number of dimensions, is characterised by all the equations of the form

$$P^{ij}P^{kl} + P^{ik}P^{lj} + P^{il}P^{jk} = 0 \qquad (i, j, k, l = 1, 2, ..., n).$$

Every bivector is clearly equal to a system of $n(n-1)/2$ simple bivectors parallel respectively to the planes formed by two of the coordinate axes: it is sufficient, in fact, to choose the bivectors where all the components zero, except one $P^{ij}$.

**12.** Forming a simple bivector from two given vectors can be regarded as a type of *multiplication.* If we agree to denote by the notation $[\boldsymbol{xy}]$ the bivector formed from the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we verify immediately the formulas,

$$[\boldsymbol{xy}] = -[\boldsymbol{yx}] \ ,$$
$$[(\boldsymbol{x}+\boldsymbol{y})\boldsymbol{z}] = [\boldsymbol{xz}] + [\boldsymbol{yz}] \ ;$$

the multiplication is *distributive*, but it is not commutative, the product changing sign with the order of the factors. We say that the bivector $[\boldsymbol{xy}]$ is the *exterior product* of the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

If we have

$$\boldsymbol{x} = X^i\boldsymbol{e}_i \ , \quad \boldsymbol{y} = Y^i\boldsymbol{e}_i,$$

we get, by applying the rules of exterior multiplication,

$$[\boldsymbol{xy}] = X^iY^j[\boldsymbol{e}_i\boldsymbol{e}_j] = \frac{1}{2}\,(X^iY^j - X^jY^i)[\boldsymbol{e}_i\boldsymbol{e}_j] = \frac{1}{2}\,P^{ij}[\boldsymbol{e}_i\boldsymbol{e}_j],$$

which returns the decomposition of a bivector into $n(n-1)/2$ simple bivectors carried by the $n(n-1)/2$ coordinate planes.

## III.– Trivectors.

**13.** We call a *simple trivector* the figure formed by three vectors arranged in a given order. If at a point $O$ we take three vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ equivalent to three given vectors, the plane manifold of three dimensions (or *triplane*) determined by the points $O, A, B, C$, is called the *triplane* of the trivector; its direction is defined only in the sense that it can moved parallel to itself.

Two simple trivectors will be said to be equal if their triplanes are parallel, if the parallelepipeds constructed on the three vectors transported to the same point have the same volume, and finally if the orientation of these two parallelepipeds are the same.

In the space of three dimensions the volume $V$ of the parallelepiped constructed on the three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ is related to the determinant

$$P^{123} = \begin{vmatrix} X^1 & X^2 & X^3 \\ Y^1 & Y^2 & Y^3 \\ Z^1 & Z^2 & Z^3 \end{vmatrix}$$

of the components of these three vectors. In fact, the ratio $P^{123}/V$ does not change when we add to the vector $\boldsymbol{z}$ a sum of multiples of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. The volume $V$ does not change, since the base (constructed on $\boldsymbol{x}$ and $\boldsymbol{y}$) and the height do not change. The determinant does not change either, because we are adding to the last line sums of multiples of the first two. We can therefore, without changing $P^{123}/V$, reduce it to the case where the vector $\boldsymbol{x}$ lies on $Ox_3$; we then reduce similarly to the case where the vector $\boldsymbol{x}$ lies on $Ox_1$, and the vector $\boldsymbol{y}$ lies on $Ox_2$. We will then have

$$\frac{P^{123}}{V} = \frac{X^1 Y^2 Z^3}{V}$$

and the right hand side is clearly equal to the volume $V_0$ of the parallelepiped constructed on the three unitary vectors of the coordinate system.

To calculate the volume $V_0$, note that in rectangular coordinates we have, for any parallelepiped,

$$V = \begin{vmatrix} X^1 & X^2 & X^3 \\ Y^1 & Y^2 & Y^3 \\ Z^1 & Z^2 & Z^3 \end{vmatrix} ,$$

and consequently, multiplying the determinant on the right hand side by itself,

$$V^2 = \begin{vmatrix} X_i^2 & X_i Y_i & X_i Z_i \\ Y_i X_i & Y_i^2 & Y_i Z_i \\ Z_i X_i & Z_i Y_i & Z_i^2 \end{vmatrix} = \begin{vmatrix} \boldsymbol{x}^2 & \boldsymbol{x} \cdot y & \boldsymbol{x} \cdot z \\ \boldsymbol{y} \cdot x & \boldsymbol{y}^2 & \boldsymbol{y} \cdot z \\ \boldsymbol{z} \cdot x & \boldsymbol{z} \cdot y & \boldsymbol{z}^2 \end{vmatrix} .$$

By applying this result to the parallelepiped constructed on the three basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, we have

$$V_0^2 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = g$$

Consequently, the volume $V$ of a simple trivector is

$$V = V_0 P^{123} = \sqrt{g}\, P^{123} \tag{1.18}$$

As regards the covariant components,

$$P_{123} = \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix},$$

we find without difficulty, by replacing the $X_i, Y_i, Z_i$ by their values, and by proceeding as in n° 8,

$$P_{123} = gP^{123};$$

so in particular,

$$V^2 = P_{123}P^{123}. \tag{1.19}$$

**14.** In the space of dimension greater than three, we will show, as we did for bivectors, that the condition for equality of two simple trivectors is the equality of the contravariant components

$$P^{ijk} = \begin{vmatrix} X^i & X^j & X^k \\ Y^i & Y^j & Y^k \\ Z^i & Z^j & Z^k \end{vmatrix}$$

of these two trivectors, or furthermore the equality of their covariant components

$$P_{ijk} = \begin{vmatrix} X_i & X_j & X_k \\ Y_i & Y_j & Y_k \\ Z_i & Z_j & Z_k \end{vmatrix}.$$

The square of the measure $m$ of a trivector is

$$m^2 = \frac{1}{6}\, P_{ijk}P^{ijk}.$$

We define the scalar product of two trivectors by means of one or other of the two expressions

$$\frac{1}{6}\, P_{ijk}Q^{ijk} = \frac{1}{6}\, P^{ijk}Q_{ijk},$$

which we interpret as for bivectors.

We define finally general *trivectors* and we will show that the formation of a simple trivector can be regarded as an exterior multiplication:

$$[\boldsymbol{xyz}] = [\boldsymbol{yzx}] = [\boldsymbol{zxy}] = -[\boldsymbol{xzy}] = -[\boldsymbol{zyx}] = -[\boldsymbol{yxz}]$$

## IV.– Multivectors.

**15.** Generalisation of the above notions is done naturally if we have defined, in a space of dimension $n > 3$, the volume of a parallelepiped of $n$-dimensions constructed on $n$ given vectors. We define this volume in rectangular coordinates by the determinant of the projections of the $n$ vectors; in Cartesian coordinates, the volume constructed on the $n$ unitary vectors of the coordinates is equal to the square root of the determinant $g$ of the quantities $g_{ij}$.

If $p$ is an integer less than or equal to $n$, a simple $p$-vector is defined by the set of $p$ vectors arranged in a given order. Two $p$-vectors are equal if they have the same contravariant components

$$P^{i_1 i_2 \dots i_p} = \begin{vmatrix} X^{i_1} & X^{i_2} & \dots & X^{i_p} \\ Y^{i_1} & Y^{i_2} & \dots & Y^{i_p} \\ \dots & \dots & \dots & \dots \\ U^{i_1} & U^{i_2} & \dots & U^{i_p} \end{vmatrix}$$

or equivalently the same covariant components

$$P_{i_1 i_2 \dots i_p} = \begin{vmatrix} X_{i_1} & X_{i_2} & \cdots & X_{i_p} \\ Y_{i_1} & Y_{i_2} & \cdots & Y_{i_p} \\ \dots & \dots & \dots & \dots \\ U_{i_1} & U_{i_2} & \cdots & U_{i_p} \end{vmatrix}$$

where we have denoted by $\boldsymbol{x}, \boldsymbol{y}, ..., \boldsymbol{u}$ the $p$ vectors that define the $p$-vector considered.

The square of the measure of a $p$-vector is

$$\frac{1}{p!}\, P_{i_1 i_2 \dots i_p} P^{i_1 i_2 \dots i_p}$$

We will define the scalar product of two $p$-vectors, systems of $p$-vectors, etc.

If $p = n$, the magnitude of an $n$-vector is

$$\sqrt{g}\; P^{i_1 i_2 \dots i_p} = \frac{1}{\sqrt{g}}\, P_{i_1 i_2 \dots i_p} = \sqrt{P_{i_1 i_2 \dots i_p} P^{i_1 i_2 \dots i_p}}$$

## V.– Supplementary multivectors.

**16.** We shall call *supplementary multivector* of a given simple $p$-vector $\boldsymbol{a}$ that simple $(n-p)$-vector $\boldsymbol{b}$ that satisfies the following three conditions:

1° *The $(n-p)$-plane of $\boldsymbol{b}$ is totally normal to the $p$-plane of $\boldsymbol{a}$, that is to say that each of the $p$ vectors that define $\boldsymbol{a}$ is perpendicular to each of the $n-p$ vectors that define $\boldsymbol{b}$;*

2° *The multivectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same measure;*

3° *The $n$-vector formed by the $p$ vectors of $\boldsymbol{a}$ and the $n-p$ vectors of $\boldsymbol{b}$ has a direct orientation.*

This last condition assumes that we have *oriented the space*, that is to say that we have decided *a priori* that such an $n$-vector will be positive or negative. We shall assume that the unitary vectors of the coordinates have been chosen in such a way that on taking them in the natural order of their indices, they define a positive $n$-vector.

Consider first the simplest case $n = 3$, and let us try to determine the vector $\boldsymbol{z}$ supplementary to a given simple bivector $[\boldsymbol{xy}]$.

We will have first

$$Z_1Y^1 + Z_2Y^2 + Z_3Y^3 = 0,$$
$$Z_1X^1 + Z_2X^2 + Z_3X^3 = 0,$$

from which

$$\frac{Z_1}{P^{23}} = \frac{Z_2}{P^{31}} = \frac{Z_3}{P^{12}} = \lambda\ ;$$

an analogous calculation will give

$$\frac{Z^1}{P_{23}} = \frac{Z^2}{P_{31}} = \frac{Z^3}{P_{12}} = \mu\,.$$

To determine $\lambda$ and $\mu$, begin from the equality

$$Z_iZ^i = \frac{1}{2}\,\lambda\mu P_{ij}P^{ij},$$

from which, according to the second condition of the definition,

$$\lambda\mu = 1.$$

Finally, the trivector [ xyz] has as its contravariant component

$$\begin{vmatrix} X^{i_1} & X^{i_2} & X^{i_p} \\ Y^{i_1} & Y^{i_2} & Y^{i_p} \\ Z^{i_1} & Z^{i_2} & Z^{i_p} \end{vmatrix} = Z^1P_{23} + Z^2P^{31} + Z^3P^{12} = \frac{1}{2}\,\mu P_{ij}P^{ij}\ ;$$

the measure of this trivector, which is

$$\frac{1}{2}\,\sqrt{g}\;\mu P_{ij}P^{ij},$$

must be equal to the square of the measure of the bivector, from which we deduce

$$\mu = \frac{1}{\sqrt{g}}, \quad \lambda = \sqrt{g}\,.$$

*The supplementary vector of the given bivector thus has components*

$$\left.\begin{array}{lll} Z_1 = \sqrt{g}\; P^{23}, & Z_2 = \sqrt{g}\; P^{31}, & Z_3 = \sqrt{g}\; P^{12} \\ Z^1 = \dfrac{1}{\sqrt{g}}\; P_{23}, & Z^2 = \dfrac{1}{\sqrt{g}}\; P_{31}, & Z^3 = \dfrac{1}{\sqrt{g}}\; P_{12} \end{array}\right\} \qquad (1.20)$$

This vector is none other than that which is calle from habit the *vector product* of the vectors $\boldsymbol{x}, \boldsymbol{y}$.

Conversely, the bivector $[\boldsymbol{xy}]$ is the supplement of the vector $\boldsymbol{z}$.

**17.** The interior product $\boldsymbol{v}$ of the bivector $[\boldsymbol{xy}]$ and a vector $\boldsymbol{u}$ has been defined (n° 10) by the formulas

$$\begin{aligned} V_1 &= P_{21}U^2 + P_{31}U^3, \\ V_2 &= P_{12}U^1 + P_{32}U^3, \\ V_3 &= P_{13}U^1 + P_{23}U^2. \end{aligned}$$

By introducing the vector $\boldsymbol{z}$ supplementary to $[\boldsymbol{xy}]$, we obtain

$$V_1 = \sqrt{g}(Z^2U^3 - Z^3U^2), \quad V_2 = \sqrt{g}(Z^3U^1 - Z^1U^3), \quad V_3 = \sqrt{g}(Z^1U^2 - Z^2U^1);$$

the vector $\boldsymbol{v}$ is thus the vector supplement of the bivector defined by the vector $\boldsymbol{z}$ and the vector $\boldsymbol{u}$. With the ordinary vector notation, we have

$$[\boldsymbol{xy}] \cdot \boldsymbol{u} = (\boldsymbol{x} \wedge \boldsymbol{y}) \wedge \boldsymbol{u}.$$

It is easy to prove that if we regard a bivector as the geometric sum of many others, the vector supplementary to the given bivector is the geometric sum of the vectors supplementary to the constituent bivectors.

**18.** In the general case, the components $Q$ of the $(n-p)$-vector supplementary to a given $p$-vector, with components $P$, are

$$\left.\begin{array}{lcl} Q_{j_1 j_2 \ldots j_{n-p}} & = & \sqrt{g}\; P^{i_1 i_2 \ldots i_p}, \\ Q^{j_1 j_2 \ldots j_{n-p}} & = & \dfrac{1}{\sqrt{g}}\; P_{i_1 i_2 \ldots i_p}\,. \end{array}\right\} \qquad (1.21)$$

We assume, in these formulas, that the indices $i_1, i_2, ..., i_p, j_1, j_2, ..., j_{n-p}$ are, up to order, all the indices $1, 2, ..., n$, and that the permutation formed is *even.*

We see that if $\boldsymbol{b}$ is the multivector supplementary to the $p$-vector $\boldsymbol{a}$, the $p$-vector $\boldsymbol{a}'$ supplementary to $\boldsymbol{b}$ is $\boldsymbol{a}$ or $-\mathbf{a}$; the latter does not occur unless if $p$ and $n-p$ are odd (and consequently $n$ is even).

We define the *multivector* supplementary to a given non-simple $p$-vector by the set of multivectors supplements of the simple $p$-vectors of which the given $p$-vector is the sum.

In the case $n = 4$ and $p = 2$, the bivector supplementary to a bivector with components $P^{ij}$ or $P_{ij}$ has as components

$$\begin{gathered} Q_{23} = \sqrt{g} P^{14}, \quad Q_{31} = \sqrt{g} P^{24}, \quad Q_{12} = \sqrt{g} P^{34}, \\ Q_{14} = \sqrt{g} P^{23}, \quad Q_{24} = \sqrt{g} P^{31}, \quad Q_{34} = \sqrt{g} P^{12}; \end{gathered}$$

or

$$\begin{gathered} Q^{23} = \frac{1}{\sqrt{g}} P_{14}, \quad Q^{31} = \frac{1}{\sqrt{g}} P_{24}, \quad Q^{12} = \frac{1}{\sqrt{g}} P_{34}, \\ Q^{14} = \frac{1}{\sqrt{g}} P_{23}, \quad Q^{24} = \frac{1}{\sqrt{g}} P_{31}, \quad Q^{34} = \frac{1}{\sqrt{g}} P_{12}; \end{gathered}$$

we find that the scalar product of a bivector by its supplement is zero except when this bivector is simple and conversely.

## VI.– Sliding or applied multivectors.

**19.** As we distinguish *free* vectors and *sliding* vectors, we can distinguish *free* multivectors and *sliding* multivectors. A sliding simple $p$-vector is formed from $p$ vectors *situated in the same $p$-plane*, and two sliding $p$-vectors are equal only if they are situated on the same $p$-plane. The components previously considered of a $p$-vector are not sufficient to determine a sliding $p$-vector.

Give us a point $x^1, ..., x^n$ of the $p$-plane which contains the $p$-vector. To fix ideas, take $p = 3$. The equations of the 3-plane that contains a trivector $[\boldsymbol{xyz}]$ are obtained by setting to zero all determinants of degree 5 of the matrix

$$\left\| \begin{matrix} \xi^1 & \xi^2 & ... & \xi^n & 1 \\ x^1 & x^2 & ... & x^n & 1 \\ X^1 & X^2 & ... & X^n & 0 \\ Y^1 & Y^2 & ... & Y^n & 0 \\ Z^1 & Z^2 & ... & Z^n & 0 \end{matrix} \right\| ;$$

we have denoted by $\xi^1, \xi^2, ..., \xi^n$ the running coordinates. We see that the quantities that will be involved in these equations will be, in addition to the quantities $P^{ijk}$, the quantities

$$p^{ijkl} = \left| \begin{matrix} x^i & x^j & x^k & x^l \\ X^i & X^j & X^k & X^l \\ Y^i & Y^j & Y^k & Y^l \\ Z^i & Z^j & Z^k & Z^l \end{matrix} \right| .$$

The quantities $P^{ijk}$ themselves can be written in the form $p^{0ijk}$ by agreeing to put $x^0 = 1, X^0 = Y^0 Z^0 = 0$. The sliding trivector can then be regarded as an exterior product $[M\boldsymbol{xyz}]$, denoting by $M$ a point taken arbitrarily in the three-plane of the trivector. The law of formation of these components is the same as that for a free quadrivector by supposing that we are in a space of $n+1$ dimensions, the $(n+1)^{\text{th}}$ coordinate of $M$ being 1 and the $(n+1)^{\text{th}}$ coordinates of the of the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ being zero.

If $p = 1$ and $n = 3$, we recover the six classical coordinates of a sliding vector.

We define similarly a *system of sliding p-vectors.*

## VII.– Application to the motion of a rigid body with one fixed point.

**20.** When a solid body moves continuously, having a fixed point which we can assume taken as the origin of coordinates, the velocity field of its various points has the fundamental property that the velocities of any two points $M$ and $M'$ have the same projection on the line $MM'$.

If we denote by $x^i$ and $y^i$ the coordinates of $M$ and $M'$, by $u_i$ and $v_i$ the covariant components of their velocities, we have

$$(v_i - u_i)(y^i - x^i) = 0.$$

On the other hand, the velocity of $M$ being perpendicular to $OM$ and that of $M'$ perpendicular to $OM'$, we have

$$u_i x^i = v_i y^i = 0,$$

from which the relation

$$u_i y^i + v_i x^i = 0.$$

By taking for $M'$ the point at the extremity of the vector $\boldsymbol{e}_i$, this relation shows that $u_i$ is a combination with constant coefficients of the coordinates $x^k$,

$$u_i = a_{ik} x^k, \tag{1.22}$$

and the perpendicularity of the velocity $\boldsymbol{u}$ with the line $OM$ gives

$$u_i x^i = a_{ik} x^k x^i = 0.$$

The quantities $a_{ij}$ are thus antisymmetric ($a_{ji} = -a_{ij}$). They define a bivector[1]: *the velocity of a point $M$ is the interior product of the bivector with covariant components $a_{ij}$ and the vector $\overrightarrow{OM}$.*

We shall say that the solid performs an instantaneous rotation represented by the bivector with components $a_{ij}$.

We will have similarly

$$u^i = a^{ik} x_k, \tag{1.23}$$

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[1] See, in n° 23, note (1).

by introducing the contravariant components of the bivector.

The rotation is said to be *simple* when the bivector is simple; in this case, the points which are in the $(n-2)$-plane of the perpendiculars raised at $O$ to the plane of the simple bivector all have zero velocity; *the rotation is performed around this* $(n-2)$*-plane.* The angular velocity of rotation is moreover equal to the number that measures the bivector.

For $n = 3$ all rotations are simple. The most general rotation is decomposable into three rotations around the perpendiculars to the three coordinate planes, having respectively angular velocities

$$a^{23}\sqrt{g_{22}g_{33}-g_{23}^2}, \qquad a^{31}\sqrt{g_{33}g_{11}-g_{31}^2}, \qquad a^{12}\sqrt{g_{11}g_{22}-g_{12}^2}.$$

If we introduce the vector $\boldsymbol{c}$ supplementary to the bivector representing the rotation, this too can be decomposed into three rotations around the perpendiculars to the three coordinate planes, having respective angular velocities

$$\sqrt{g_{11}}\; c^1 = \sqrt{\frac{g_{11}}{g}}\; a_{23}, \qquad \sqrt{g_{22}}\; c^2 = \sqrt{\frac{g_{22}}{g}}\; a_{31}, \qquad \sqrt{g_{33}}\; c^1 = \sqrt{\frac{g_{33}}{g}}\; a_{12}.$$

In the general case a rotation can also be represented by the $(n-2)$-vector supplementary to the bivector $a_{ij}$, in accordance with that which we usually do in ordinary space.

## VIII.– Tensors. Tensor algebra.

**21.** Vectors, bivectors, multivectors, or rather *the systems of numbers that define them analytically*, are all special cases of *tensors*. We call in a general way a *tensor* a system of numbers that define analytically a geometrical (or physical) entity of such a kine that, by a change of Cartesian coordinates, the components of the tensor undergo a *linear* transformation whose coefficients do not depend on the numerical values of these components, but only on the two systems of coordinates, or, in a way that is more exact and precise, on the coefficients of the transformation undergone by the contravariant components of a vector by the change of coordinates considered.

We shall say that tensors whose components obey the same transformation law for the same change of coordinates form a *space*;[2] we shall also say that they are of the same kind. But it is important to note that the nature of a space of tensors is uniquely defined by the law of transformation of their components independently of their geometrical, mechanical or physical significance of the entities represented analytically by these components.

From this point of view, the space of contravariant vectors and the space of

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[2] *Fr. corps*, literally a body. We have no appropriate word in English to translate this, so I have translated it as *space.*

covariant vectors are to be regarded as two distinct tensor spaces, because the transformation undergone by the contravariant components of a vector under a given change of coordinates is not the same as that undergone by the covariant components.

In the space of contravariant vectors (or covariant) the components are linearly independent, that is, they are not related by any linear homogeneous relation with constant coefficients (the same for all the tensors of the space). But it is not always so; nevertheless the tensors which we will have to consider will always form part of a more extended space with linearly independent components.

The *tensor algebra* provides the rules for forming tensors. We point out first an important theorem that enables us to recognise the tensor character of certain entities represented analytically by a certain finite number of components.

**Theorem** *Let $(\mathcal{C})$ be a space of tensors with $r$ linearly independent components $u_1, u_2, ..., u^r$. Suppose on the other hand that there is an entity that is capable of being represented analytically, in a system of Cartesian coordinates, by $r$ quantities $v^1, v^2, ..., v^r$, in such a way that the sum $u_i v^i$, where the $u_i$ are the components of an arbitrary tensor of the space $(\mathcal{C})$, is independent of the choice of coordinates. Under these conditions*

1°. *the quantities $v^i$ form a tensor;*

2°. *the nature of this tensor is uniquely determined by the nature of the space $(\mathcal{C})$.*

The proof is straight forward. Let

$$(u_i)' = \lambda_k{}^i u_k$$

be the linear transformation undergone by the components of a generic tensor of the space $(\mathcal{C})$ under a change of Cartesian coordinates. If we denote by $(v^i)'$ the quantities that define the entity considered in the new system of coordinates, we have the relation

$$\lambda_k{}^i u_k (v^i)' = u_k v^k,$$

from which, *because of the linear independence of the $u_k$*,

$$v^k = \lambda_k{}^i (v^i)';$$

this last relation defines unambiguously the linear transformation which enables us to pass from the $v^i$ to the $(v^i)'$; both parts of the theorem are therefore proved[3].

An altogether special case of this theorem makes evident the tensor character of covariant vectors: it is sufficient to take as the space $(\mathcal{C})$ the space of contravariant vectors: everything that can be represented by $n$ components $u_i$ such that the sum $u_i X^i$ is independent of the choice of coordinates, where the $X^i$ denote the

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[3] The two linear substitutions undergome by the $u_i$ and the $v^i$ are said to be *contragredient.*

contravariant components of a generic vector, is a tensor, and this tensor is of the same nature as a covariant vector.

**22.** Tensor algebra includes various operations that enable us to pass from two given tensors to a third tensor.

The first of these operations is the *addition* which allows us to pass from two tensors of the same nature to a third tensor of the same nature, whose components result from the addition of the corresponding components of the two given tensors. The geometric addition of contravariant vectors, and those of covariant vectors, are particular cases of this operation.

A second operation is that of *general multiplication.* Given two tensors, of the same nature or not, the quantities obtained by multiplying in all possible ways the components of the first tensor by those of the second tensor form a third tensor whose nature depends only on the natures of the two first ones; if a change of coordinates causes the components $u_i$ of the first tensor to undergo the linear transformation

$$(u_i)' = \lambda_i{}^k u_k$$

and the components of the second tensor the linear transformation

$$(v_i)' = \mu_i{}^k v_k,$$

this causes the products $u_i v_j$ to undergo the transformation

$$(u_i v_j)' = \lambda_i{}^k \mu_j{}^h (u_k v_h).$$

A very simple example is provided by the general product of two contravariant vectors $X^i, Y^i$ whose components are the products $X^i Y^j$; the general product of a contravariant vector $X^i$ and a covariant vector $Y_i$ has as components the products $X^i Y_j$.

The above operation, combined with the operation of addition (or that of subtraction), leads to a simple contravariant bivector $P^{ij}$ which is the difference between the tensor $X^i Y^j$ and the tensor *of the same kind* $Y^i X^j$; by additions of simple bivectors, we arrive then at the most general bivector. We see in this way that *the components $P^{ij}$ of a bivector transform as the components of the general product of two contravariant vectors.*

**23.** The foregoing considerations lead us us to the most commonly used tensors: there are the tensors with multiple indices, some of which are superscripts (contravariant indices), others subscripts (covariant indices). We can, for example, characterise a tensor with two covariant indices $a_{ij}$ by the property that *the sum $a_{ij} X^i Y^j$, where the $X^i$ and $Y^j$ are the contravariant components of any two vectors, has a numerical value that is independent of the choice of coordinates*[4].

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[4] This is what happened with the quantities $a_{ij}$ that represent the velocity field of a rigid

The theorem of n° 21 shows us in fact that the components $a_{ij}$ define a tensor whose nature depends only on the nature of the tensor $X^iY^j$, a tensor whose components are linearly independent. Since the quantities $U_iV_j$, where $U_i$ and $V_j$ are the covariant components of two vectors, satisfy the condition stated above that the product $a_{ij}X^iY^j = U_iV_jX^iY^j$ has a numerical value independent of the choice of coordinates, it follows that the tensor $a_{ij}$ is of the same nature as the product of two covariant vectors, or better that under a change of coordinates the components $a_{ij}$ undergo the same linear transformation as the products $U_iV_j$.

A contravariant tensor with two indices $a^{ij}$ is similarly characterised by the property that the sum $a^{ij}X_iY_j$, where $X_i$ and $Y_j$ are the covariant components of any two vectors, has a numerical value that is independent of the choice of Cartesian coordinates. The components $a^{ij}$ transform like the components $U^iV^j$ of the general product of two contravariant vectors.

Finally a mixed tensor with two indices ${a_i}^j$ is characterised by the property that the sum ${a_i}^jX^iY_j$ is independent of the choice of coordinates, the $X^i$ and $Y_i$ denoting the components of any two vectors, the first contravariant, the second covariant. The components ${a_i}^j$ transform like the components $U_iV^j$ of the general product of a covariant vector with a contravariant vector.

We characterise similarly tensors with three indices, contravariant, covariant or mixed, $a^{ijk}, a_{ijk}, {a_i}^j{}_k$, etc. It is necessary to suppose that the order in which the three indices are arranged is involved in the definition of the tensor; *in certain special cases*, the components do not change in numerical value if, for example, we swap the order of two indices. In the general case, the tensor has $n^3$ linearly independent components, but it may happen that they are related linear equations with constant coefficients: it then necessary that the coefficients of these relations be the same whatever Cartesian system of coordinates is used: it is necessary in this case, to consider that we have in fact a number of components smaller than $n^3$.

Suppose for example that we have an antisymmetric tensor $a_{ijk}$: this is to say that the numerical value of the components $a_{ijk}$ does not change if we perform an even permutation of the three indices, and changes sign if we perform an odd permutation on these indices. It is important to note that *if this property is valid in a particular system of Cartesian coordinates, it is valid in all systems of coordinates.* In fact, the $a_{ijk}$ transform like the products $X_iY_jZ_k$, which involves the components of three arbitrary vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. Consequently they transform equally like the quantities

$$X_iY_jZ_k + Y_iZ_jX_k + Z_iX_jY_k - Y_iX_jZ_k - Z_iY_jX_k - X_iZ_jY_k$$

or like the components of a covariant trivector; in place of $n^3$ linearly independent

body rotating about the origin (n° 20), because the scalar product of a vector $\boldsymbol{y}$ and the velocity of the tip of a vector $\boldsymbol{x}$ is, according to formula (1.22), $a_{ki}X^kY^i$ and its numerical value is independent of the choice of coordinates; the $a_{ki}$ are thus components of a tensor with two indices, and because these components are antisymmetric, it is a bivector. This legitimises the assertion made in n° 20.

components, there are in fact only $n(n-1)(n-2)/6$. An antisymmetric tensor with three indices is thus a trivector.

Similarly, a symmetric tensor such as $a_{ijk}$ admits $n(n{+}1)(n{+}2)/6$ independent components; they transform like the components $X_i X_j X_k$ of the product of three identical covariant vectors.

A tensor with multiple indices can be put analytically in several different forms; it is sufficient to introduce, for example in the case of three indices, the quantities $a^i{}_{jk}, a_i{}^j{}_k, ..., a^{ij}{}_k, ..., a^{ijk}$ defined by the identities

$$a^i{}_{jk} X^i Y^j Z^k = a^i{}_{jk} X_i Y^j Z^k = a^{ij}{}_k X_i Y_j Z^k = a^{ijk} X_i Y_j Z_k$$

Care must be taken, when we pass in this way an index from below to above, to respect the order of the indices; we sometimes do this by placing points above lower indices and below upper indices, by writing $a^i_{\bullet jk}$ in place of $a^i{}_{jk}$.

An important tensor is that which has as its components $g_{ij}$; it is called the *fundamental tensor*. It is indeed a tensor since the sum

$$g_{ij} X^i Y^j$$

has a numerical value that is independent of the choice of coordinates; moreover it is the scalar product of the vectors $\boldsymbol{x}, \boldsymbol{y}$. Its mixed components $g_i{}^j$ or $g^i{}_j$ are defined by the identities

$$g_i{}^j X^i Y_j = g^i{}_j X_i Y^j = X^i Y_i = X_i Y^j;$$

we have therefore

$$g_i{}^j = g^i{}_j = 1, \quad \text{if } i = j; \qquad g_i{}^j = g^i{}_j = 0, \quad \text{if } i \neq j. \tag{1.24}$$

The contravariant components $g^{ij}$ of the fundamental tensor are defined by the identity

$$X_i Y^j = g^{ij} X_i Y_j,$$

from which

$$Y^j = g^{ij} Y_j;$$

these are the quantities which have already been introduced with this notation (n° 4).

Raising an index from below to above is performed furthermore by formulas such as the following

$$a_i{}^j = g^{jk} a_{ij};$$

we have similarly

$$a_i{}^j X^i Y_j = g_{ik} a^{kj}.$$

**24.** From a two index tensor $a_{ij}$ we can always deduce a scalar tensor (that is, a *constant* independent of the system of coordinates) by the operation called *saturation of indices*. This scalar tensor, called the *contracted tensor* of the given tensor, is ${a_i}^i$.

The proof is immediate. The components of the mixed tensor ${a_i}^j$ transform among themselves like the components of $X_iY^j$ of the general product of a covariant vector $\boldsymbol{x}$ and a contravariant vector $\boldsymbol{y}$. Now, the invariance of the sum $X_iY^i$, which represents the scalar product of the two vectors, results in the tensor property of the sum ${a_i}^i$.

It is important to note a second contracted tensor, namely ${a^i}_i$; but *it is equal to the first*. In fact, we have, by performing the calculation,

$${a_i}^i = g_{ik}a^{ki}, \quad {a^i}_i = g_{ik}a^{ik};$$

the property of symmetry ($g_{ik} = g_{ki}$) of the fundamental tensor shows the equality of the two contracted tensors.

The contracted tensor of a bivector is zero, because we have

$${a_i}^i = \frac{1}{2}\, g_{ik}(a^{ki} + a^{ik}) = 0$$

Tensors with more than two indices also admit contracted tensors. Consider, for example, the tensor $a_{ijkh}$; let us raise the last index so as to obtain the mixed tensor ${a_{ijk}}^h$: its components transform like the components of the general product $X_iY_jZ_kU^h$ of four tensors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u}$, where the first three are covariant, and the fourth contravariant; now the sum $X_iY_jZ_kU^k$, the product of $X_iY_j$ with the scalar product of the vectors $\boldsymbol{z}, \boldsymbol{u}$, transforms as $X_iY_j$; consequently it is the same for the quantity

$$b_{ij} = {a_{ijk}}^k;$$

the $b_{ij}$ are thus the components of a covariant tensor with two indices: it is the given tensor *contracted by saturation of the two last indices.*

We could consider others by saturating two indices of different ranks.

If $p \geq 4$, all contracted tensors give rise in turn to other contracted tensors, and so on.

The contracted tensors of a multivector are all zero.

**25.** One last operation is frequently used, it is the *contracted multiplication*; it consists of contracting one or more times the general product of two given tensors.

The products once contracted of the fundamental tensor with any tensor, such as $a_{ijk}$, are of one of the forms

$${g^\alpha}_i a_{\alpha jk}, \quad {g^\alpha}_j a_{i\alpha k}, \quad ..., \quad {g_i}^\alpha a_{\alpha jk}, \quad ...;$$

given the numerical values of the mixed components of the fundamental tensor, all these contracted products give the tensor $a_{ijk}$ itself. *The fundamental tensor, in these operations, plays the role of the identity.*

One of the contracted products of a bivector $a_{ij}$ and a vector $X^i$ is

$$a_{ki}X^k;$$

this is exactly the *interior product* of the bivector and the vector (n° 10).

More generally, we can define the interior product of a $p$-vector $a_{i_1 i_2 \dots i_p}$ with a $q$-vector $b_{j_1 j_2 \dots j_q}$ $(q \leq p)$ as the $q$-times contracted product

$$c^{i_{q+1} \dots i_p} = a^{k_1 \dots k_q i_{q+1} \dots i_p} b_{k_1 \dots k_q}$$

or rather as its quotient by $q!$.

This interior product will allow us to define and to form the cosine of the angle of a $p$-plane and a $q$-plane.

**26.** We end this Section as an application of the preceding theorems, by showing how we can find the components of the supplementary multivector of a given multivector.

Note first the tensor character of the square root $\sqrt{g}$ of the discriminant of the fundamental form. In the space of $n$-dimensions, the $n$ vector with single contravariant component $P^{12\dots n}$ is a tensor (but it is not a *scalar* tensor because its single component has a numerical value that changes with the system of coordinates); on the other hand, the fact that $\sqrt{g}\ P^{12\dots n}$, the volume of the $n$-vector, is independent of the choice of coordinates, shows, by virtue of the theorem in n° 21, the *tensor character of* $\sqrt{g}$ and shows at the same time that $\sqrt{g}$ transforms as a covariant $n$-vector.

It follows from the above that, given a contravariant bivector with components $P^{ij}$, the quantities $\sqrt{g}\ P^{ij}$ are the components of a tensor. Suppose, for definiteness, that $n = 5$. According to what was said above, the quantities $\sqrt{g}\ P^{ij}$ transform as the quantities $Q_{12345}P^{ij}$, denoting by $Q_{12345}$ the components of a covariant 5-vector; but $Q_{12345}P^{45}$ for example is one of the components of the contracted product $Q_{ijkhl}P^{hl}$ of the covariant 5-vector and the bivector; now, these components transform like those of a covariant trivector. Consequently *the tensor* $\sqrt{g}\ P^{ij}$ *is of the same kind as a covariant trivector*, to the component $\sqrt{g}\ P^{ij}$ corresponding the component $Q_{hkl}$ of the trivector such that the permutation $(ijkhl)$ is even. To the bivector $P^{ij}$ is thus associated a well defined trivector.

To interpret geometrically this trivector, in the case where the bivector $P^{ij}$ is simple, choose rectangular coordinates with five basis vectors $\boldsymbol{e}_i$, where the two first ones $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are taken in the biplane of the bivector; this bivector will then have only one non-zero component, namely $P^{12}$, equal to the measure $m$ of the bivector. The trivector we seek will consequently hve only one non-zero component, $Q_{345}$, equal to $m$; it is thus a simple trivector whose triplane is completely normal to the biplane of the bivector and having the same measure as the bivector.

We find again the definition of the trivector supplementary to the given bivector, as was given in Section V (p 16).[5]

The tensor $\sqrt{g}\, P^{ij}$ provides an example of tensors which do not present themselves in the guise of tensors with multiple indices which were the subject of n$^{\text{os}}$ 21 to 25, but which nevertheless can be represented analytically in such a way as to fit into this general class.

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[5] P 16 of the French text.

# 2 Curvilinear Coordinates in Euclidean Geometry

### I. – The line element of space in Cartesian coordinates.

**27.** We have seen that the square of the length of a vector in any system of Cartesian coordinates is expressed by a positive definite quadratic form

$$g_{ij}X^iX^j \tag{2.1}$$

of the (contravariant) components of this vector.

Conversely, given *a-priori* a positive definite quadratic form of $n$ variables $X^1, ..., X^n$, this form can always be regarded as defining, in a suitably chosen system of Cartesian coordinates, the square of the length of a vector whose components with respect to this system are $X^1, ..., X^n$.

In fact, if the form (2.1) is positive definite, it is reducible to a sum of $n$ independent squares, that is, there exist $n$ linear forms $Y_1, ..., Y_n$ of the $X^i$ that satisfy the identity

$$Y_1^2 + Y_2^2 + \cdots + Y_n^2 = g_{ij}X^iX^j. \tag{2.2}$$

That said, take any system of *rectangular* coordinates in space and consider the vector $\boldsymbol{x}$ whose components with respect to this system are $Y_1, Y_2, ..., Y_n$. We have, by hypothesis, formulae such that

$$Y_i = \alpha_{ik}X^k;$$

consider the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n$ whose projections onto the axes are respectively, for $\boldsymbol{e}_i$,

$$\alpha_{1i}, \alpha_{2i}, ..., \alpha_{ni}.$$

If we then consider a system of Cartesian coordinates for which the $n$ basis vectors are precisely $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$, we see immediately that the contravariant components of the vector $\boldsymbol{x}$ are $X^1, X^2, ..., X^n$ in this system of coordinates. Identity (2.2) then shows that the given quadratic form indeed represents the square of the length of the vector $\boldsymbol{x}$ in this system of Cartesian coordinates.

Since the system of rectangular coordinates chosen originally is arbitrary, we see that *the system of Cartesian coordinates corresponding to the given quadratic form is well defined, up to a displacement or up to a displacement and a symmetry.*

*For a quadratic differential form with constant coefficients*

$$g_{ij}du^i du^j$$

*to be regarded as the square of the distance of two infinitely close points in a system of Cartesian coordinates, it is necessary and sufficient that this form be positive definite.*

**28.** In ordinary analytic geometry, when we consider oblique coordinates, we usually choose an unit vector on each axis whose length is the unit of length itself. The quadratic form that gives the square of the length of a vector $(X, Y, Z)$ is then

$$X^2 + Y^2 + Z^2 + 2\cos\lambda\; YZ + 2\cos\mu\; ZX + 2\cos\nu\; XY, \tag{2.3}$$

where we denote the faces of the coordinate trihedron[1] by $\lambda, \mu, \nu$. *The necessary and sufficient condition for the three angles $\lambda, \mu, \nu$, contained between 0 and $\pi$, to be regarded as the faces of a trihedron is thus that the quadratic form (2.3) is positive definite.* The decomposition into squares gives

$$(X + Y\cos\nu + Z\cos\mu)^2 + \left(Y\sin\nu + Z\;\frac{\cos\lambda - \cos\mu\cos\nu}{\sin\nu}\right)^2 \\ + \frac{\sin^2\mu\sin^2\nu - (\cos\lambda - \cos\mu\cos\nu)^2}{\sin^2\nu}\;Z^2.$$

The condition is thus

$$(\cos\lambda - \cos\mu\cos\nu)^2 - \sin^2\mu\sin^2\nu < 0$$

or

$$[\cos(\mu+\nu) - \cos\lambda][\cos\lambda - \cos(\mu-\nu)] > 0.$$

If we suppose that $\lambda$ is the largest of the given angles, the second factor of the left hand side of this inequality is itself negative, and the condition is

$$\cos\lambda > \cos(\mu+\nu)$$

or, equivalently,

$$\lambda < \mu + \nu < 2\pi - \lambda.$$

We find the classic conditions that each face is smaller than the sum of the other two, and that the sum of the faces is less than four right angles. We see that these conditions are sufficient.

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[1] Translator's note : From formula (2.3), it is clear that $\lambda$ is the angle between the coordinate vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, $\mu$ is the angle between the coordinate vectors $\boldsymbol{e}_3$ and $\boldsymbol{e}_1$, and $\nu$ is the angle between the coordinate vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Cartan's "faces" are thus the three planes of the coordinate trihedron that meet at the origin which he labels them by the angles formed by their edges, which are the coordinate axes.

## II. – The fundamental theorem of metric geometry.

**29.** The two main problems in the theory of curvilinear coordinates in Euclidean geometry are the following:

**1$^o$.** *Given the line element of space in a system of curvilinear coordinates, determine the nature of these coordinates*; or, equivalently, transform these coordinates into rectangular coordinates.

**2$^o$.** *What conditions must the coefficients of an* a priori *given line element satisfy for this line element to be considered as that of Euclidean space in a suitably chosen system of curvilinear coordinates?*

These two problems were treated for the first time by Lamè in the case of systems of *orthogonal* coordinates.[2]

Before we deal with the first of the two, it is very important to be aware *a priori* of the number of solutions that it admits.

First, consider the following useful remark.

Let

$$ds^2 = g_{ij} du^i du^j \tag{2.4}$$

be the line element of space in any system of coordinates. To any point $\boldsymbol{M}$ with coordinates $u^1, u^2, \ldots, u^n$, we can attach a system of Cartesian coordinates whose origin is the point $M$ and whose basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are chosen so that the coordinates of the point

$$M'(u^i + du^1, ..., u^n + du^n),$$

infinitely close to $M$, are precisely $du^1, \ldots, du^n$. For this, it is sufficient that the vector $\boldsymbol{e}_i$ is tangent to the $i^{\text{th}}$ coordinate curve (obtained by varying the single coordinate $u^i$) and, more precisely, which represents the *velocity* of a point that moves along this curve when we regard the variable coordinate $u^i$ as representing time.

In this system of Cartesian coordinates, which defines what we will call the *natural* system of reference attached to the point $\boldsymbol{M}$, the scalar product of the two vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ is $g_{ij}$, and consequently the angle $\varphi_{ij}$ between the $i^{\text{th}}$ and $j^{\text{th}}$ coordinate curves is given by

$$\cos\varphi_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}}\ . \tag{2.5}$$

More generally, the cosine of the angle $\varphi$ of two directions at $\boldsymbol{M}$ defined by the two symbols of differentiation $d$ and $\delta$ is

$$\cos\varphi = \frac{g_{ij} du^i \delta u^j}{\sqrt{g_{ij} du^i du^j}\sqrt{g_{jj}\delta u^i \delta u^j}}\ .$$

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[2] G. Lamè, *Leçons sur les coordonnées curvilignes*, Paris, 1859.

**30.** That said, suppose first that, in ordinary three dimensional space, the line element of the space takes the form

$$ds^2 = du^2 + dv^2 + dw^2. \tag{2.6}$$

It is clear that this is the case in any system of rectangular Cartesian coordinates. *We will prove that there is no other system of coordinates that gives the line element the form (2.6).*

In fact, if we search for the straight lines of the space using their characteristic property of being the shortest path from one point to another, we will have to search for the minimum of the integral $\int \sqrt{du^2 + dv^2 + dw^2}$. When $u, v, w$ are rectangular coordinates, the problem of analysis to which it reduces leads to lines defined by a system of two equations of first degree in $u, v, w$; but the result of the calculation, which depends only on the given expression $\sqrt{du^2 + dv^2 + dw^2}$, is general and the straight lines are, *whatever the nature of the coordinates* $u, v, w$, defined by linear equations. The coordinate curves are thus all straight lines, respectively rectangular to each other according to formula (2.5). The *coordinate surfaces*, $w =$ constant, are planes, because any straight line that has two of its points in one of them is completely contained within it. The straight lines with coordinate $(w)$ are all perpendicular to the plane $w = 0$, so are parallel to each other. From this, we can deduce easily that $u, v, w$ are the distances of a point in space to the three fixed rectangular planes $u = 0, v = 0, w = 0$. Consequently, *the coordinates considered are necessarily Cartesian and rectangular.*

We can present this reasoning in another form, *which has the advantage of applying to any form of the line element.*

Consider two systems of coordinates, curvilinear or not, that give the line element of ordinary space the same form

$$ds^2 = g_{ij} du^i du^j.$$

Let $\boldsymbol{M}$ and $\boldsymbol{M'}$ be the two points in space that have the same given coordinates $u^1, u^2, u^3$ in these two systems. The correspondence thus obtained between points in space defines a *point transformation* (the one that takes us from point $\boldsymbol{M}$ to point $\boldsymbol{M'}$) which has the property that the distance between any two infinitely close points in space is conserved by this transformation. (A transformation of this kind is called *isometric.*)

The transformation considered changes an arc of a curve into another arc of a curve *which obviously has the same length* (because it is given by the same integral). Consequently the distance, in the elementary sense of the word, between any two points $\boldsymbol{M}$ and $\boldsymbol{N}$ is preserved by the transformation. Now, we prove in elementary geometry that two figures that correspond point by point with conservation of distances are equal or symmetric. Consequently, *the transformation* *that takes us from* $\boldsymbol{M}$ *to* $\boldsymbol{M}'$ *is either a displacement, or a displacement followed by a symmetry.*

If we imagine in space the network of the three families of coordinate lines, where each of these lines is *labelled* (by the numerical values of the two non-variable coordinates on this line), we see that *the two networks attached to two systems of coordinates giving the same* $ds^2$ *can be deduced each other by a displacement, followed or not by a symmetry.*

We can also state this result by saying that, whatever the coordinates system that provides a given $ds^2$, the natural systems of reference attached to the different points in space (as defined in n° 29), always have the same position relative to each other; when the system of reference attached to a given set of values $u^i = a^i$ has been chosen, *all other systems of reference are defined by that one.*

The preceding theorem shows that *all the geometric properties of space are virtually contained in its line element* (with however a restriction concerning the *orientation* of space). This is *the fundamental theorem of metric geometry.*

**31.** It is of interest to point out that the only hypotheses made in the preceding reasoning on the coordinates $u^i$ are those that guarantee that the square of the distance of two infinitely close points is a quadratic differential form with respect to the $du^i$. *It is not sufficient* for this to assume that the coordinates of two infinitely close points are infinitely close; on the other hand, it is sufficient to suppose that the $u^i$ are functions of the rectangular coordinates which admit continuous partial derivatives of the first order; the same property will belong to rectangular coordinates, considered as functions of the $u^i$.

The previous proofs were carried out for the case of the space of three dimensions. The effective resolution of the first problem stated in n° 29 will show that the fundamental theorem of metric geometry is true whatever the number of dimensions of the space.

## III. – Local reconstruction of space from its line element.

**32.** We now turn to the first fundamental problem stated in n° 29. Knowledge of the line element

$$ds^2 = g_{ij} du^i du^j$$

allows us to imagine at each point $\boldsymbol{M}\,(u^1, ..., u^n)$ a *natural* system of reference of reference $(R)$ of given size and shape (up to a symmetry) that has this point as origin. First, we propose to position, with respect to the natural system of reference $(R)$ at a point $\boldsymbol{M}$, the natural system of reference $(R')$ at an infinitely close point $\boldsymbol{M}'$.

The coordinate vectors $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ of the system of reference $(R)$ are none other than the vectors $\dfrac{\partial \boldsymbol{M}}{\partial u^1}, ..., \dfrac{\partial \boldsymbol{M}}{\partial u^n}$, as expressed by the equality

$$d\boldsymbol{M} = du^1\boldsymbol{e}_1 + du^2\boldsymbol{e}_2 + \cdots + du^n\boldsymbol{e}_n. \tag{2.7}$$

On the other hand, the assumptions made about the coordinates $u^i$ show that the vectors $\boldsymbol{e}'_1, ..., \boldsymbol{e}'_n$ attached at $\boldsymbol{M}'$ differ infinitely little from the vectors $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$. To define them analytically, we must assume that the coordinates $u^i$, considered as functions of the rectangular coordinates, have continuous *second order* partial derivatives.

The direction parameters, that is, the contravariant components of the vector $\boldsymbol{e}'_i$ with respect to the system of reference $(R)$, are then of the form

$$\omega_i^1, \omega_i^2, ..., \omega_i^{i-1},\ 1 + \omega_i^i,\ ..., \omega_i^n\ ,$$

where the $\omega_i^k$ are linear expressions with respect to the differentials $du^1, ..., du^n$

$$\omega_i{}^k = \Gamma_i{}^k{}_1 du^1 + \Gamma_i{}^k{}_2 du^2 + \cdots + \Gamma_i{}^k{}_n du^n, \tag{2.8}$$

and, in general, we can write

$$d\boldsymbol{e}_i = \omega_i{}^k \boldsymbol{e}_k = \Gamma_i{}^k{}_r\, du^r \boldsymbol{e}_k. \tag{2.9}$$

Our first task is to determine the $n^3$ quantities $\Gamma_i{}^k{}_r$. Once these are determined, all the systems of reference $(R')$ infinitely close to the given system of reference $(R)$ are positioned relative to $(R)$: *the Euclidean space has been reconstructed in the neighbourhood of the origin* $\boldsymbol{M}$ *of the system of reference* $(R)$.

**33.** Before we solve the problem at issue, introduce a new notation. Note that the $\omega_i{}^k$, where the index $i$ remains fixed and the index $k$ varies, are the contravariant components with respect to $(R)$ of the vector $d\boldsymbol{e}_i$. Introduce the *covariant* components

$$\omega_{ij} = g_{jk}\omega_i{}^k \tag{2.10}$$

of this vector, and put

$$\omega_{ij} = \Gamma_{ijr} du^r,$$

which is equivalent to putting

$$\Gamma_{ijr} = g_{jk}\Gamma_i{}^k{}_r. \tag{2.11}$$

With these notations well understood, we have a first group of relations for determining the coefficients of the expressions $\omega_i{}^k$; these are those which state that the natural system of reference always has the size and shape imposed by the line element of the space, namely

$$e_i \cdot e_j = g_{ij}.$$

Differentiation of these relations gives

$$g_{jk}\omega_i{}^k + g_{ik}\omega_j{}^k = dg_{ij} \qquad ,$$

or, according to (2.10),

$$\omega_{ij} + \omega_{ji} = dg_{ij}. \tag{I}$$

There are $\dfrac{n^2(n+1)}{2}$ relations (I): *they are therefore not sufficient to determine the unknown coefficients.*

We will have other relations by expressing the *conditions of integrability* of equations (2.7) and (2.9)

$$\begin{aligned} d\boldsymbol{M} &= du^i \boldsymbol{e}_i \\ d\boldsymbol{e}_i &= \omega_i{}^k \boldsymbol{e}_k. \end{aligned}$$

For the moment, we state only the integrability conditions for equations (2.7); by expressing the vector $\dfrac{\partial^2 M}{\partial u^i \partial u^j}$ in two different ways, we obtain

$$\begin{aligned} \frac{\partial^2 M}{\partial u^i \partial u^j} &= \frac{\partial}{\partial u^j}\left(\frac{\partial M}{\partial u^i}\right) = \frac{\partial \boldsymbol{e}_i}{\partial u^j} = \Gamma_i{}^k{}_j \boldsymbol{e}_k, \\ \frac{\partial^2 M}{\partial u^i \partial u^j} &= \frac{\partial}{\partial u^i}\left(\frac{\partial M}{\partial u^j}\right) = \frac{\partial \boldsymbol{e}_j}{\partial u^i} = \Gamma_j{}^k{}_i \boldsymbol{e}_k. \end{aligned}$$

The relations provided by these integrability conditions are

$$\Gamma_i{}^k{}_j = \Gamma_j{}^k{}_i \ , \tag{II}$$

or, according to (2.11)

$$\Gamma_{ikj} = \Gamma_{kji} \ . \tag{II$'$}$$

Leave aside for the moment the integrability conditions for equations (2.9), and note that equations (II) are $\dfrac{n^2(n-1)}{2}$ in number; this number, added to the number $\dfrac{n^2(n+1)}{2}$ of equations (I), gives precisely $n^3$, the number of unknowns.

To solve equations (I) and (II'), we start from the equation

$$\Gamma_{ijk} + \Gamma_{jik} = \frac{\partial g_{ij}}{\partial u^k} \ ,$$

deduced from from (I), and change, using (II'), $\Gamma_{jik}$ into $\Gamma_{kij}$,

$$\Gamma_{kij} + \Gamma_{ijk} = \frac{\partial g_{ij}}{\partial u^k} \ .$$

A cyclic permutation, performed twice in a row on the letters $i, j, k$, gives two new equations

$$\begin{aligned} \Gamma_{ijk} + \Gamma_{jki} &= \frac{\partial g_{jk}}{\partial u^i}, \\ \Gamma_{jki} + \Gamma_{kij} &= \frac{\partial g_{ki}}{\partial u^j}. \end{aligned}$$

We find easily that

$$\Gamma_{jki} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right) = \left[\begin{array}{c} ij \\ k \end{array}\right] ,$$

by introducing a notation due to E. B. Christoffel[3] (*Christoffel symbols of the first kind*).

Conversely, we verify easily that the values thus found for the quantities $\Gamma_{jki}$ indeed satisfy equations (I) and (II'); equations (II') in fact follow immediately from the symmetry of the Christoffel symbols with respect to the upper indices[4].

Finally, we have the formulas we sought,

$$\Gamma_{ikj} = \Gamma_{jki} = \left[\begin{array}{c} ij \\ k \end{array}\right] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right). \tag{2.12}$$

We now pass from the $\Gamma_{ikj}$ to the $\Gamma_i{}^k{}_j$ by inverting formulae (2.11), which gives

$$\Gamma_i{}^k{}_j = \Gamma_j{}^k{}_i = g^{kh}\Gamma_{ihj} = g^{kh}\left[\begin{array}{c} ij \\ k \end{array}\right] = \left\{\begin{array}{c} ij \\ k \end{array}\right\} . \tag{2.13}$$

The quantities $\left\{\begin{array}{c} ij \\ k \end{array}\right\}$ are *Christoffel's symbols of the second kind.*

Formulas (2.12) or (2.13) completely solve the problem posed. We see that the solution is unique, in agreement with the fundamental theorem of metric geometry.

Since we have thus accomplished the local reconstruction of the space from its line element, it is conceivable that we will be able, by integration, completely to position the natural systems of reference attached to different points of space with respect to one another. We will return to this later.

## IV. – Absolute differentiation. Kinematic applications. Lagrange's equations.

**34.** Consider a field of vectors in space. At each point $\boldsymbol{M}$ the vector of the field has given contravariant components $X^1, X^2, ..., X^n$ with respect to the natural system of reference attached to this point. Let us determine the elementary geometric variation $\boldsymbol{x'} - \boldsymbol{x}$ of the vector of the field as we pass from the point $\boldsymbol{M}$ to an infinitely close point $\boldsymbol{M'}$. The equality

$$\boldsymbol{x} = X^i \boldsymbol{e}_i,$$

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[3] E. B. Christoffel, *Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades* (*J. de Crelle*, Vol. 70, 1869, p 48 and 49).

[4] Translator's note : Cartan here means the indices $i, j$; in Christoffel's notation, they are in the upper position. In Cartan's $\Gamma$-notation, they occupy the lower position.

gives, when we take into account formulae (2.9),

$$d\boldsymbol{x} = dX^i \boldsymbol{e}_i + X^i {\omega_i}^k \boldsymbol{e}_k = (dX^i + X^k {\omega_k}^i)\boldsymbol{e}_i.$$

We see that the elementary geometric variation of the vector, or *its absolute differential*, has components, with respect to the frame of reference attached to the point $\boldsymbol{M}$,

$$DX^i = dX^i + X^k {\omega_k}^i. \tag{2.14}$$

The quantities $DX^i$ define the *absolute differential* of the vector of the given field; they are its contravariant components. In particular, if the field of vectors is *uniform*, the absolute differential is zero, and we have

$$dX^i + X^k {\omega_k}^i = 0. \tag{2.15}$$

We can say that if we transport a vector from a point $\boldsymbol{M}$ to an infinitely close point in such that it remains equal[5,6] to itself (equal transport)[7], we have relations (2.15).

It is important to know how to calculate the *covariant* components of the absolute differential of a vector with covariant components $X_i$. For this, introduce an arbitrary, but fixed, *uniform* field of vectors with contravariant components $Y^i$. The elementary variation of the scalar product

$$\boldsymbol{x} \cdot \boldsymbol{y} = X_i Y^i$$

can be obtained in two ways. First, it is equal to the scalar product of the elementary variation of the vector $\boldsymbol{x}$ with the *fixed* vector $\boldsymbol{y}$, which gives

$$DX_i \; Y^i.$$

Second, we can perform the calculation directly, which gives

$$dX_i \; Y^i + X_i \; dY^i;$$

if we take into account the equations similar to (2.15) that are satisfied by the uniform field $\boldsymbol{y}$, the second sum can be written as

$$-X_i Y^k {\omega_k}^i = -X_k Y^i {\omega_i}^k;$$

we thus have finally

$$DX_i Y^i = (dX_i - X_k {\omega_i}^k) Y^i.$$

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[5] Fr. *equipollent*

[6] Translator's note : The word *equipollent* is not used in English, nor does it mean *parallel*. It is a combination of two Latin words that mean *equi-valent*, that is, having the same value. It could be translated as *equivalent* or, more simply, as *equal*. Equipollent vectors are equal vectors, that is, vectors that have the same direction and magnitude.

[7] Translator's note : Technically, the correct English term here is *parallel transport*. Cartan's term, *equipollent transport* correctly emphasises that both magnitude and direction are preserved in this transport. The English term suggests (incorrectly) that only the direction of the vector is preserved. In later occurrences of this term, I shall translate it as *parallel transport*.

Since this relation is valid *whatever the uniform field* $\boldsymbol{y}$, we have the required equations

$$DX_i = dX_i - X_k {\omega_i}^k. \tag{2.16}$$

**35.** We can adopt a slightly more general point of view. Attach to each point $\boldsymbol{M}$ of the space a point $P$ which has coordinates $x^1, ..., x^n$ with respect to the natural system of reference $(R)$ at $\boldsymbol{M}$. We thus define a *field of points.* We look for the *absolute* elementary displacement of the point $P$ as we pass from $\boldsymbol{M}$ to $\boldsymbol{M}'$. We can write

$$\overrightarrow{P - M} = x^i \boldsymbol{e}_i,$$

from which

$$d(\overrightarrow{P - M}) = dx^i\ \boldsymbol{e}_i + x^k {\omega_k}^i \boldsymbol{e}_i;$$

consequently, taking into account (2.7),

$$dP = [dx^i + du^i + x^k {\omega_k}^i]\boldsymbol{e}_i.$$

In analogy with that which we did for vector fields, we will then put

$$Dx^i = dx^i + du^i + x^k {\omega_k}^i, \tag{2.17}$$

which formula defines the absolute differential of a field of points.

**36.** The preceding results ($\mathrm{n}^{\mathrm{os}}$ 33 and 34) allow us to determine easily the velocity and acceleration of a moving point $\boldsymbol{M}$. Suppose that the curvilinear coordinates $u^i$ of this point are given functions of the time $t$; the velocity of the point will clearly have contravariant components, with respect to the natural system of reference attached at this point, given by

$$v^i = \frac{du^i}{dt}. \tag{2.18}$$

The acceleration is the quotient by $dt$ of the absolute differential of the vector velocity

$$\gamma^i = \frac{Dv^i}{dt} = \frac{d^2u^i}{dt^2} + {\Gamma_k}^i{}_h \frac{du^k}{dt}\frac{du^h}{dt}. \tag{2.19}$$

Note that since the functions $u^i$ of $t$ are known, it is sufficient to know the numerical values of the Christoffel symbols of the second kind at $\boldsymbol{M}$ to know the contravariant components of the acceleration.

Formula (2.19) generalises the theorem of the composition of accelerations.

We find the accelerations of different orders similarly.

**37.** Formula (2.19) immediately provides the means for finding the equations of straight lines in the system of coordinates considered. It is sufficient to integrate the differential equations of second order

$$\frac{d^2u^i}{dt^2} + \Gamma_k{}^i{}_h \frac{du^k}{dt}\frac{du^h}{dt} = 0;$$

we can take as the independent variable $t$ the abscissa $s$ of a point on the straight line, where the abscissa is calculated on this straight line from a fixed origin, and we then have

$$\frac{d^2u^i}{ds^2} + \Gamma_k{}^i{}_h \frac{du^k}{ds}\frac{du^h}{ds} = 0; \tag{2.20}$$

We would have arrived at the same result by searching for the lines which minimise the integral $\int ds$.

To Lagrange we owe a general method in analytical mechanics which allows us find preceding result directly. In fact, if we form the expression $2T$ which gives the square of the velocity of a point

$$2T = g_{ij}(u^i)'(u^j)',$$

the equations of motion of this point, which is assumed to be subject to no force, are

$$\frac{d}{dt}\frac{\partial T}{\partial (u^i)'} - \frac{\partial T}{\partial u^i} = 0.$$

But the theory of the Lagrange equations in Mechanics gives us something better than the differential equations of straight lines. The quantities

$$P_i = \frac{d}{dt}\frac{\partial T}{\partial (u^i)'} - \frac{\partial T}{\partial u^i}$$

allow us in fact to calculate the elementary work of the vector acceleration for an arbitrary displacement $\delta u^i$ of the point: this work is

$$\gamma_i \, \delta u^i = P_i \, \delta u^i;$$

*the quantities $P_i$ are thus none other than the covariant components of the acceleration* of any moving point.

The proof is easy. First calculate these covariant components directly, starting from the contravariant components $\gamma^i$ given by formulae (2.19). Taking into account (2.11), we easily get

$$\gamma_i = g_{ik}\,\frac{d^2u^k}{dt^2} + \Gamma_{kih}\frac{du^k}{dt}\frac{du^h}{dt}.$$

On the other hand, calculating Lagrange's quantity $P_i$ gives

$$\begin{aligned} P_i &= \frac{d}{dt}\left(g_{ik}\,\frac{du^k}{dt}\right) - \frac{1}{2}\frac{\partial g_{kh}}{\partial u^i}\frac{du^k}{dt}\frac{du^h}{dt} \\ &= g_{ik}\,\frac{d^2u^k}{dt^2} + \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^h} + \frac{\partial g_{ih}}{\partial u^k}\right)\frac{du^k}{dt}\frac{du^h}{dt} - \frac{1}{2}\frac{\partial g_{kh}}{\partial u^i}\frac{du^k}{dt}\frac{du^h}{dt} \\ &= g_{ik}\,\frac{d^2u^k}{dt^2} + \begin{bmatrix} k\,h \\ i \end{bmatrix}\frac{du^k}{dt}\frac{du^h}{dt}. \end{aligned}$$

The two results agree.

**38.** The fundamental theorem of analytical mechanics, which follows from Lagrange's equations, namely that *the essential mechanical properties of a system are virtually contained in the analytical expression of its* vis viva,is to be compared, as we can see, with the fundamental theorem of metric geometry, which is in fact a special case.

More practically, the Lagrange's algorithm is very convenient for calculating the Christoffel symbols. In fact, the $\Gamma_{kih}$, *for a given index $i$, are the coefficients of the quadratic form with respect to the first derivatives $(u^k)'$ that enter the expression*

$$P_i = \frac{d}{dt}\frac{\partial T}{\partial (u^i)'} - \frac{\partial T}{\partial u^i}.$$

For example, consider the line element of the space referred to polar coordinates $r, \theta, \varphi$:

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2.$$

Assign the numbers 1, 2, 3 respectively to the variables $r, \theta, \varphi$. We have

$$\begin{aligned} 2T &= r'^2 + r^2\,\theta'^2 + r^2\sin^2\theta\,\varphi'^2, \\ \frac{d}{dt}\frac{\partial T}{\partial r'} - \frac{\partial T}{\partial r} &= r'' - r\theta'^2 - r\sin^2\theta\ \varphi'^2, \\ \frac{d}{dt}\frac{\partial T}{\partial \theta'} - \frac{\partial T}{\partial \theta} &= r^2\theta'' + 2rr'\theta' - r^2\sin\theta\cos\theta\ \varphi'^2, \\ \frac{d}{dt}\frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} &= r^2\sin^2\theta\ \varphi'' + 2r\sin^2\theta\ r'\varphi' + 2r^2\sin\theta\cos\theta\ \theta'\varphi'. \end{aligned}$$

It follows that, omitting the zero Christoffel symbols,

$$\begin{bmatrix} 2\,2 \\ 1 \end{bmatrix} = -r, \qquad \begin{bmatrix} 3\,3 \\ 1 \end{bmatrix} = -r\sin^2\theta;$$

$$\begin{bmatrix} 1\,2 \\ 2 \end{bmatrix} = r, \qquad \begin{bmatrix} 3\,3 \\ 2 \end{bmatrix} = -r^2\sin\theta\cos\theta;$$

$$\begin{bmatrix} 1\,3 \\ 3 \end{bmatrix} = r\sin^2\theta, \qquad \begin{bmatrix} 2\,3 \\ 3 \end{bmatrix} = r^2\sin\theta\cos\theta.$$

It is easy to go from there to the symbols of the second kind by dividing the equations in the first, second and third lines respectively by $1, r^2$ and $r^2 \sin^2 \theta$:

$$\begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix} = -r, \qquad \begin{Bmatrix} 3\,3 \\ 1 \end{Bmatrix} = -r \sin^2 \theta;$$

$$\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} = \frac{1}{r}, \qquad \begin{Bmatrix} 3\,3 \\ 2 \end{Bmatrix} = -\sin\theta \cos\theta;$$

$$\begin{Bmatrix} 1\,3 \\ 3 \end{Bmatrix} = \frac{1}{r}, \qquad \begin{Bmatrix} 2\,3 \\ 3 \end{Bmatrix} = \frac{\cos\theta}{\sin\theta}.$$

**39.** The curvature of a left handed curve is easy to calculate. If the curvilinear coordinates of one of its points are expressed as functions of a parameter $t$, which we can regard as time, we know that the bivector defined by the velocity and the acceleration has as measure $\frac{v^3}{\rho}$, where $v$ denotes the speed and $\rho$ the radius of curvature. The speed is given by

$$v^2 = g_{ij} \frac{du^i}{dt} \frac{du^j}{dt};$$

the acceleration has been determined; so we can calculate the measure of the bivector considered, and deduce it to be $\frac{1}{\rho}$. Note only that, since the equations of the curve are assumed given, calculating its curvature at a point $\boldsymbol{M}$ requires knowledge only of the numerical values *at this point* of the quantities $g_{ij}$ and $\Gamma^k_{ij}$, that is, of the quantities $g_{ij}$ and of their partial derivatives of first order.

Calculating the torsion, on the other hand, would involve the partial derivatives of the first two orders of the $g_{ij}$.

On the other hand, given a surface of known equation, the normal curvature of the various curves on this surface through one of its points $\boldsymbol{M}$ involves only the $g_{ij}$ and their first order partial derivatives; the same is true for the principal curvatures of the surface at this point, the directions of the asymptotic tangents, the principal tangents, etc.

## V. – Tensor analysis.

**40.** The calculation (n° 34) to determine the components of the absolute differential of a vector can be generalised to any *tensor field*. Consider, for example, a field of mixed tensors with two indices ${a_i}^j$. The geometric variation of the tensor, when we pass from a point $\boldsymbol{M}$ in space to a point infinitely close to it, is an infinitely small tensor, whose components with respect to the natural system of reference at $\boldsymbol{M}$ will be denoted by $Da_i{}^j$. To calculate these quantities, introduce two arbitrary *uniform* vector fields $\boldsymbol{x}$ and $\boldsymbol{y}$, and consider the sum

$$a_i{}^j X^i Y_j;$$

the elementary variation of this sum is clearly

$$Da_i{}^j\, X^i Y_j;$$

on the other hand, by direct calculation, it is equal to

$$da_i{}^j\ X^i Y_j + a_i{}^j\ dX^i\ Y_j + a_i{}^j X^i\ dY_j;$$

taking into account the *uniformity* of the two vector fields $\boldsymbol{x}$ and $\boldsymbol{y}$, we find

$$\left[da_i{}^j - a_k{}^j \omega_i{}^k + a_i{}^k \omega_k{}^j\right] X^i\ Y_j.$$

We deduce immediately the formulae we seek

$$Da_i{}^j = da_i{}^j - a_k{}^j \omega_i{}^k + a_i{}^k \omega_k{}^j. \tag{2.21}$$

We see easily what would be the result if we had any number of indices, both superscript and subscript.

If we apply the preceding to the *fundamental tensor* $g_{ij}$, we arrive at *Ricci's theorem*, according to which *the absolute differential of the fundamental tensor is zero*. This theorem is obvious, since for two arbitrary *uniform* vector fields $\boldsymbol{x}$ and $\boldsymbol{y}$, the sum $g_{ij} X^i Y^j$ is *constant*: it is the scalar product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. We thus have

$$Dg_{ij}\ X^i Y^j = 0.$$

We can also check this calculation. We have

$$Dg_{ij} = dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k,$$

and the right hand side is zero according to the same equations (I) that were used to determine the forms $\omega_i{}^j$.

The calculation of the absolute differential of a tensor leads to a remarkable result when applied to an $n$-vector $a^{12\ldots n}$. We have

$$\begin{aligned} Da^{12\ldots n} &= da^{12\ldots n} + a^{i2\ldots n}\omega_i{}^1 + \cdots + a^{12\ldots i}\omega_i{}^n, \\ &= da^{12\ldots n} + a^{12\ldots n}\left(\omega_1{}^1 + \omega_2{}^2 + \cdots + \omega_n{}^n\right). \end{aligned}$$

In particular, if we assume that the $n$-vector field is uniform, that is, if we assume that the volume $V$ of the $n$-vector is constant, we get

$$\frac{da^{12\ldots n}}{a^{12\ldots n}} = \frac{d\dfrac{V}{\sqrt{g}}}{\dfrac{V}{\sqrt{g}}} = -\frac{d\sqrt{g}}{\sqrt{g}} = -\omega_i{}^i;$$

we thus have the remarkable formula

$$\frac{d\sqrt{g}}{\sqrt{g}} = \omega_i{}^i = \Gamma_i{}^i{}_k du^k\,, \tag{2.22}$$

which can also be proved easily by direct calculation of $dg$.

**41.** Absolute differentiation leads in tensor analysis to a new operation, the *derivation of tensors.* The coefficients $a_{ijk}$ of the absolute differential of a tensor such as $a_{ij}$

$$Da_{ij} = a_{ijk} du^k,$$

in fact form a new tensor[8]. To see this, it is sufficient to show that the quantities

$$b_{ij} = a_{ijk} X^k,$$

where we have chosen an arbitrary vector field $\boldsymbol{x}$, define a tensor. To do this, imagine each *trajectory* of the vector field, defined by the differential equations

$$\frac{du^1}{X^1} = \frac{du^2}{X^2} = \cdots = \frac{du^n}{X^n}\,,$$

traversed by a moving point in such a way that its velocity is equal to the corresponding vector of the field. We then have

$$b_{ij} = \frac{Da_{ij}}{dt}\,,$$

a formula which makes clear the tensor character of $b_{ij}$. Note that the new index $k$ introduced is covariant in nature.

Apply this operation to a vector field $X^i$ (contravariant components), or $X_i$ (covariant components). We deduce a tensor with two indices, with covariant components $X_{ij}$. From the formula

$$DX_i = dX_i - \omega_i{}^k X_k\,,$$

we deduce

$$X_{ij} = \frac{\partial X_i}{\partial u^j} - \Gamma_i{}^k{}_j X_k\,.$$

The anti-symmetric tensor

$$X_{ji} - X_{ij} = \frac{\partial X_j}{\partial u^i} - \frac{\partial X_i}{\partial u^j} \tag{2.23}$$

is well known: it is the *curl* of the vector field. It can also be obtained directly by calculating the bilinear covariant

$$d\omega(\delta) - \delta\omega(d)$$

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[8] The index $k$ is an *index of derivation*; it is sometimes distinguished from the others by preceding it with a vertical bar and by writing $a_{ij|k}$ in place of $a_{ijk}$.

of the *invariant* differential form

$$\omega(d) = X_1\, du^1 + X_2\, du^2 + \cdots + X_n\, du^n\,;$$

we find

$$\begin{aligned} d\omega(\delta) - \delta\omega(d) &= \frac{\partial X_i}{\partial u^j}(du^j \delta u^i - \delta u^j du^i) \\ &= \frac{1}{2}\left(\frac{\partial X_j}{\partial u^i} - \frac{\partial X_i}{\partial u^j}\right)(du^i \delta u^j - du^j \delta u^i); \end{aligned}$$

the right hand side is the scalar product of the curl (considered as a bivector) and the bivector determined by the two infinitesimal vectors $d\boldsymbol{M}$ and $\delta\boldsymbol{M}$.

Another important tensor derived from a vector field is the *divergence* of the field: it is the contracted tensor

$$X^i{}_{|i}\,.$$

Now, from the formula

$$DX^i = dX^i + X^k \omega_k{}^i,$$

we deduce

$$X^i{}_{|i} = \frac{\partial X^i}{\partial u^i} + X^k \Gamma_k{}^i{}_i\,.$$

Taking into account formula (2.22), we can also put the divergence into the form

$$\operatorname{div} \boldsymbol{x} = \frac{\partial X^i}{\partial u^i} + \frac{\partial \sqrt{g}}{\sqrt{g}\,\partial u^k}\; X^k = \frac{1}{\sqrt{g}}\;\frac{\partial(\sqrt{g}\,X^i)}{\partial u^i}\;. \tag{2.24}$$

This very simple expression can be obtained in another way. For the sake of simplicity, we confine ourselves to three-dimensional space. In rectangular coordinates $x, y, z$, the divergence $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z}$ of a vector field $(X, Y, Z)$ is introduced most often in the calculation of the *flux of a vector field* across a closed surface, according to the formula

$$\iint X\; dy\, dz + Y\; dz\, dx + Z\; dx\, dy = \iiint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dx\, dy\, dz. \tag{2.25}$$

To write this formula in terms of any curvilinear system of coordinates, consider an element of the surface of integration as a bivector whose plane is tangent to the surface and whose magnitude is equal to the area of the element of surface; this bivector must be oriented in such a way that its supplementary vector is exterior to the volume bounded by the surface. The element of the double integral which appears on the left hand side of equation (2.15) is then the magnitude of the trivector defined by the bivector considered and the given vector $(X, Y, Z)$. As regards the element of the triple integral on the right hand side, it is the product of the divergence with the volume element of the space.

This leads to the general formula

$$\iint \sqrt{g}\ (X^1\ du^2\, du^3 + X^2\ du^3\, du^1 + X^3\ du^1\, du^2)$$
$$= \iiint \operatorname{div} \boldsymbol{x}\ \sqrt{g}\ du^1\, du^2\, du^3\,,$$

and, applying the formula of Ostrogradsky, we find directly

$$\operatorname{div} \boldsymbol{x} = \frac{1}{\sqrt{g}}\, \frac{\partial(\sqrt{g}\ X^i)}{\partial u^i}\ .$$

**42.** Consideration of the scalar fields[9] and of their derived tensors leads to very important concepts in geometry and in mathematical physics. A scalar field is simply a function of points $V(u^1, ..., u^n)$ defined independently of any system of reference. The derived tensor

$$V_i = \frac{\partial V}{\partial u^i}$$

is the *gradient* of the function $V$; it thus defines a covariant vector field. The curl of this field is identically zero.

The square of the length of the gradient, namely

$$g^{ij}\, \frac{\partial V}{\partial u^i}\, \frac{\partial V}{\partial u^j}\,, \tag{2.26}$$

is Beltrami's *differential parameter of first order* $\Delta_1 V$.

As for the divergence of the gradient, this is Beltrami's *differential parameter of second order*,

$$\Delta_2 V = \frac{1}{\sqrt{g}} \left( \frac{\partial \sqrt{g}\, V^i}{\partial u^i} \right) = \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial u^i} \left( \sqrt{g}\, g^{ik}\, \frac{\partial V}{\partial u^k} \right). \tag{2.27}$$

In rectangular coordinates, we have

$$\Delta_2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

In orthogonal curvilinear coordinates in three-dimensional space, we have

$$\Delta_2 V = \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial u^i} \left( \sqrt{g}\, g^{ik}\, \frac{\partial V}{\partial u^k} \right)$$
$$= \frac{1}{\sqrt{g_{11} g_{22} g_{33}}} \left[ \frac{\partial \left( \sqrt{\dfrac{g_{22} g_{33}}{g_{11}}}\, \dfrac{\partial V}{\partial u^1} \right)}{\partial u^1} + \frac{\partial \left( \sqrt{\dfrac{g_{33} g_{11}}{g_{22}}}\, \dfrac{\partial V}{\partial u^2} \right)}{\partial u^2} + \frac{\partial \left( \sqrt{\dfrac{g_{11} g_{22}}{g_{33}}}\, \dfrac{\partial V}{\partial u^3} \right)}{\partial u^3} \right].$$

This formula is due to Lamé.

Note that the divergence of a vector field, as well as Beltrami's differential

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[9] Fr. *champs de tenseurs scalaires*

parameters of the two first orders, involve only the first order partial derivatives of the components $g_{ij}$ of the fundamental tensor.

Finally, note that the second derived tensor $V_{ij}$ of a scalar is *symmetric*; in fact, we have

$$V_{ij} = \frac{\partial^2 V}{\partial u^i \partial u^j} - \Gamma_i{}^k{}_j \frac{\partial V}{\partial u^k};$$

moreover, this property is obvious without calculation, because in rectangular (or Cartesian) coordinates[10] the components $V_{ij}$ reduce to second derivatives $\frac{\partial^2 V}{\partial u^i \partial u^j}$ and that the property of a tensor of being symmetric is independent of the system of coordinates: in fact, it expresses the fact that the invariant sum $V_{ij} X^i Y^j$ does not change when the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are exchanged.

The same property obviously applies to the thrice derived tensor $V_{ijk}$; but if we attempt to verify this by a calculation, we would grind to an halt: we would need to use the conditions that express that the given $ds^2$ is that of Euclidean space which we have not yet encountered and that we must now study.

## VI. – Necessary conditions satisfied by the line element of Euclidean space.

**43.** Given *a priori* a line element

$$ds^2 = g_{ij}\, du^i\, du^j,$$

there is in general no system of coordinates (Cartesian or curvilinear) in Euclidean space which would result in the given $ds^2$. For it to exist, we would need to determine the rectangular coordinates of a variable point $\boldsymbol{M}$ (referred to fixed axes) and the projections of the basis vectors $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ of the natural system of reference attached at $\boldsymbol{M}$ so as to yield the relations

$$dM = du^i \boldsymbol{e}_i, \tag{2.7}$$

$$d\boldsymbol{e}_i = \omega_i{}^k \boldsymbol{e}_k, \tag{2.9}$$

where the $\omega_j{}^i$ are the differential expressions

$$\omega_i{}^j = \Gamma_i{}^j{}_k\, du^k,$$

whose coefficients are given by formulae (2.13).

The conditions of integrability of equations (2.7) are satisfied automatically, given the way in which we determined the coefficients $\Gamma_i{}^j{}_k$.

As for equations (2.9), the conditions of integrability involve the first order partial derivatives of the $\Gamma_i{}^j{}_k$, that is, the second order partial derivatives of the

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[10] Translator's note: Cartan's *Cartesian coordinates* are what in English we call *rectilinear coordinates*, and his *rectangular coordinates* are what we call Cartesian coordinates.

coefficients $g_{ij}$. *We will only solve the problem for the case where the* $g_{ij}$ *have continuous partial derivatives of the two first orders.* This is not to say that the problem does not arise if the $g_{ij}$ only have partial derivatives of the first order. It has not yet been solved for this case.

Equations (2.9) give

$$\frac{\partial \boldsymbol{e}_i}{\partial u^j} = \Gamma_i{}^k{}_j \boldsymbol{e}_k;$$

by writing the second derivative $\dfrac{\partial^2 \boldsymbol{e}_i}{\partial u^r \partial u^s}$ in two different ways, we get

$$\left(\frac{\partial \Gamma_i{}^k{}_r}{\partial u^s} + \Gamma_i{}^h{}_r \Gamma_h{}^k{}_s\right) \boldsymbol{e}_k = \left(\frac{\partial \Gamma_i{}^k{}_s}{\partial u^r} + \Gamma_i{}^h{}_s \Gamma_h{}^k{}_r\right) \boldsymbol{e}_k\,,$$

hence the conditions we seek

$$\frac{\partial \Gamma_i{}^k{}_r}{\partial u^s} - \frac{\partial \Gamma_i{}^k{}_s}{\partial u^r} + \left(\Gamma_i{}^h{}_r \Gamma_h{}^k{}_s - \Gamma_i{}^h{}_s \Gamma_h{}^k{}_r\right) = 0 \qquad (i, k, r, s = 1, 2, ..., n). \quad (2.28)$$

If we replace the $\Gamma_i{}^k{}_j$ in these equations by the values (2.13), we will obtain the necessary conditions that must be satisfied by the functions $g_{ij}$ of the $u^1, ..., u^n$.

**44.** We can arrive differently at these equations. Attach to each point $\boldsymbol{M}$ in space, assumed to be Euclidean, a point $P$ defined by its coordinates $x^1, ..., x^n$ with respect to the natural system of reference attached to the point $\boldsymbol{M}$.

We thus have a *field of points*. The absolute differential of the point $P$ is given (n° 35) by

$$Dx^i = dx^i + du^i + x^k \omega_k{}^i. \qquad (2.17)$$

Put

$$D_r x^i = \frac{\partial x^i}{\partial u^r} + \varepsilon_r{}^i + x^k\, \Gamma_k{}^i{}_r$$
$$(\varepsilon_r{}^i = 0 \;\text{ if }\; i \neq r; \quad \varepsilon_r{}^i = 1 \;\text{ if }\; i = r),$$

so as to have

$$Dx^i = D_r x^i \; du^r.$$

Now let $\alpha$ and $\beta$ be two infinitely small parameters (independent of $u^1, ..., u^n$). Given a point $\boldsymbol{M}(u^1, ..., u^n)$, call $\boldsymbol{M}', \boldsymbol{M}'', \boldsymbol{M}'''$ respectively the points obtained, the first by increasing only the coordinate $u^r$ by $\alpha$, the second by increasing only the coordinate $u^s$ by $\beta$, the third by increasing at the same time the coordinate $u^r$ by $\alpha$ and the coordinate $u^s$ by $\beta$. Denote similarly by $\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}'', \boldsymbol{P}'''$ the points of the field attached respectively to the points $\boldsymbol{M}, \boldsymbol{M}', \boldsymbol{M}'', \boldsymbol{M}'''$.

The infinitely small vector $\overrightarrow{PP'}$ has contravariant components, with respect to the natural system of reference at $\boldsymbol{M}$, the quantities $\alpha D_r x^i$. These quantities define a vector field; when we pass from $\boldsymbol{M}$ to $\boldsymbol{M}''$, the vector of this field undergoes a variation

$$\overrightarrow{P''P'''} - \overrightarrow{PP'}$$

whose contravariant components are $\alpha\beta D_s D_r x^i$.

We see similarly that the infinitely small vector $\overrightarrow{P'P'''} - \overrightarrow{PP''}$ has components $\alpha\beta D_r D_s x^i$. We deduce immediately that

$$D_r D_s x^i - D_s D_r x^i = 0.$$

A calculation gives

$$\begin{aligned} D_s D_r x^i &= \frac{\partial D_r x^i}{\partial u^s} + D_r x^k \, \Gamma_k{}^i{}_s \\ &= \frac{\partial^2 x^i}{\partial u^r \partial u^s} + \frac{\partial x^k}{\partial u^s} \Gamma_k{}^i{}_r + x^k \frac{\partial \Gamma_k{}^i{}_r}{\partial u^s} + \frac{\partial x^k}{\partial u^r} \Gamma_k{}^i{}_s + \Gamma_r{}^i{}_s + x^k \Gamma_k{}^h{}_r \Gamma_h{}^i{}_s \,. \end{aligned}$$

Comparison with $D_r D_s x^i$ leads immediately to equations (2.28).

We can also put things in the following way. The vector $\overrightarrow{PP'''}$ can be calculated in two ways. First, we can first regard it as the sum of the vector $\overrightarrow{PP'}$ and of the vector $\overrightarrow{P'P'''}$; the first has components $\alpha D_r x^i$ and the second, which comes from the vector field $\overrightarrow{PP''}$, has components

$$\beta D_s x^i + \alpha\beta D_r D_s x^i.$$

We thus have

$$\overrightarrow{PP'''} = \alpha D_r x^i + \beta D_s x^i + \alpha\beta D_r D_s x^i.$$

Passing on the other hand through the intermediate point $P''$, we get

$$\overrightarrow{PP'''} = \beta D_s x^i + \alpha D_r x^i + \alpha\beta D_s D_r x^i.$$

**45.** Instead of the two fields of elementary displacements

$$\overrightarrow{MM'} \ \ (\delta u^1 = 0, ..., \delta u^r = \alpha, ..., \delta u^n = 0)$$

and

$$\overrightarrow{MM''} \ \ (\delta u^1 = 0, ..., \delta u^s = \beta, ..., \delta u^n = 0),$$

we can consider two arbitrary displacement fields, which we can define by means of two *commuting* symbols of differentiation $d$ and $\delta$. If we denote by the symbols $D$ and $\Delta$ the corresponding *absolute* differentiations, the required conditions are expressed by the formula

$$D\Delta x^i = \Delta D x^i.$$

Now,

$$\Delta x^i = \delta x^i + \delta u^i + x^k {\omega_k}^i(\delta);$$

$$\begin{aligned} D\Delta x^i &= d(\Delta x^i) + \Delta x^k \, {\omega_k}^i(d) \\ &= d\delta x^i + d\delta u^i + dx^k \, {\omega_k}^i(\delta) + x^k \, d{\omega_k}^i(\delta) \\ &\qquad\qquad + \delta x^k \, {\omega_k}^i(d) + \delta u^k \, {\omega_k}^i(d) + x^k {\omega_k}^h(\delta) \, {\omega_h}^i(d). \end{aligned}$$

Comparing with $\Delta D x^i$ and noting that $d\delta x^i = \delta d x^i, \ d\delta u^i = \delta d u^i$, we get

$$\begin{aligned} x^k \left\{ d{\omega_k}^i(\delta) - \delta{\omega_k}^i(d) + [{\omega_k}^h(\delta) \, {\omega_h}^i(d) - {\omega_k}^h(d) \, {\omega_h}^i(\delta)] \right\} & \\ + ({\Gamma_k}^i{}_h - {\Gamma_h}^i{}_k) \, \delta u^k \, du^h &= 0. \end{aligned}$$

By symmetry of the ${\Gamma_i}^k{}_j$, we get the required conditions in the form

$$d{\omega_k}^i(\delta) - \delta{\omega_k}^i(d) = [{\omega_k}^h(d) \, {\omega_h}^i(\delta) - {\omega_k}^h(\delta) \, {\omega_h}^i(d)] \quad (i, k = 1, ..., n). \tag{2.29}$$

The left hand sides of equations (2.29) are the *bilinear covariants* (n$^\text{o}$ 41) of the ${\omega_k}^i$. Denoting them by $d{\omega_k}^i$, we get the condensed form

$$d{\omega_k}^i = [{\omega_k}^h \, {\omega_h}^i], \tag{2.30}$$

where the symbol $[{\omega_k}^h \, {\omega_h}^i]$ replaces the determinant[11]

$$\begin{vmatrix} {\omega_k}^h(d) & {\omega_h}^i(d) \\ {\omega_k}^h(\delta) & {\omega_h}^i(\delta) \end{vmatrix}.$$

By setting all the $du^i$ to zero, except $du^r = 1$; all the $\delta u^i$ to zero, except $\delta u^s = 1$, we again get formulae (2.28) of n$^\text{o}$ 43.

## VII. – Euclidean line elements.

**46.** We tackle now the problem of knowing whether conditions (2.28), which are necessary for a given $ds^2$ to be that of Euclidean space, are also sufficient.

For the moment, we will narrow down the problem a little. We will consider a $ds^2$, in $n$ variables $u^1, ..., u^n$, whose coefficients $g_{ij}$ are functions with continuous partial derivatives of the first two orders in a certain numerical domain $(\mathcal{D})$, and whose discriminant $g$ is nowhere zero; more briefly, we will say that the metric defined by $ds^2$ is *regular* in the entire domain $(\mathcal{D})$. Finally, we will assume that the domain $(\mathcal{D})$ is *simply connected.* This means that, if we regard $u^1, u^2, ..., u^n$ as the Cartesian coordinates of a point in ordinary $n$-dimensional space, the numerical domain $(\mathcal{D})$ is represented in this space by a domain such that any closed contour in it can, by continuous deformation, be reduced to a point.

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[11] We will come back later to all these notations (Chapter VIII, Section I). The two sides of (2.30) will be interpreted as *exterior* quadratic forms (*cf.* n$^\text{os}$ 12 and 14) constructed from the variables $du^1, du^2, ..., du^n$.

These hypotheses established, we will prove that if relations (2.28) are satisfied, it is possible to represent the numerical domain $(\mathcal{D})$ by a suitably chosen domain $(\Delta)$ in Euclidean space in such a way that the square of the distance between two infinitely close points in $(\Delta)$ is precisely equal to the given $ds^2$.

To do this, we will try to determine a point $P$ and vectors $\boldsymbol{e}_i$ in Euclidean space in such a way that we have identically (n° 33)

$$\left.\begin{array}{rcl} d\boldsymbol{P} & = & du^i \boldsymbol{e}_i, \\ d\boldsymbol{e}_i & = & \omega_i{}^k \boldsymbol{e}_k. \end{array}\right\} \tag{2.31}$$

We include the following initial conditions: for a system $(u^i)_0$ of values of the variables (in the domain $\mathcal{D}$), the point $P$ is at position $P_0$, and the vectors $\boldsymbol{e}_i$ coincide with the given vectors $(\boldsymbol{e}_i)_0$ whose mutual scalar products are equal to the values $g_{ij}$ for the values $(u^i)_0$ of the variables.

We will show that system (2.31) is consistent.

Let

$$(u^1)_1, (u^2)_1, ..., (u^n)_1$$

be any system of values in the domain $(\mathcal{D})$. If the system to be integrated has a solution, we can obtain the point $P$ and the vectors $\boldsymbol{e}_i$ corresponding to this system of values by taking a path in the domain $(\mathcal{D})$ from $(u^i)_0$ to $(u^i)_1$ and integrating equations (2.31) along this path. For the $u^i$ take continuously differentiable functions of an independent variable $t$ which reduce, for example, to $(u^i)_0$ for $t = 0$ and $(u^i)_1$ for $t = 1$, and integrate the differential equations

$$\left.\begin{array}{rcl} \dfrac{d\boldsymbol{P}}{dt} & = & \dfrac{du^i}{dt} \boldsymbol{e}_i, \\ \\ \dfrac{d\boldsymbol{e}_i}{dt} & = & \Gamma_i{}^k{}_h \dfrac{du^h}{dt} \boldsymbol{e}_k, \end{array}\right\} \tag{2.32}$$

by taking as initial values of the unknowns the point $P_0$ and the vectors $(\boldsymbol{e}_i)_0$. In this way, we will thus arrive at a well defined point $P$ and vectors $\boldsymbol{e}_i$ when $t = 1$. Also, we are certain that we can continue the integration over the entire interval $0 \le t \le 1$, because of the linear character of the equations.[12]

**47.** We will first prove that, for each value of $t$, the vectors $\boldsymbol{e}_i$ obtained satisfy the relations

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij}.$$

In fact, we have according to (2.32),

$$\frac{d(\boldsymbol{e}_i \cdot \boldsymbol{e}_j)}{dt} = \left( \Gamma_i{}^k{}_h \, \frac{du^h}{dt} \, \boldsymbol{e}_j \cdot \boldsymbol{e}_k + \Gamma_j{}^k{}_h \, \frac{du^h}{dt} \, \boldsymbol{e}_i \cdot \boldsymbol{e}_k \right).$$

The quantities $\boldsymbol{e}_i \cdot \boldsymbol{e}_j$ thus satisfy a system of differential equations which,

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[12] E. Goursat, *Cours d'Analyse mathématique*, 2nd Edition, Vol. 2 (Paris, Gauthiers-Villars, 1911, p 370-371).

because of how we defined the $\Gamma_i{}^k{}_j$ ($\mathrm{n}^0$ 33), have as solutions the quantities $g_{ij}$. The two solutions $\boldsymbol{e}_i \cdot \boldsymbol{e}_j$ and $g_{ij}$, *which correspond to the same initial conditions for* $t = 0$, are thus identical.

**48.** We will now prove that if, in the domain $(\mathcal{D})$, we move from the system of values $(u^i)_0$ to the system of values $(u^i)_1$ by some other path, we arrive at the same point $P$ and at the same vectors $\boldsymbol{e}_i$ as by the first path.

In fact, since the domain $(\mathcal{D})$ is simply connected, we can pass from the first path to the second by continuous deformation. Consider then a continuous sequence of paths that depends on a parameter $a$ and contains the two given paths. Each path in the sequence can be defined by the formulae

$$u^i = f^i(a, t),$$

and, with no loss of generality, we can assume that, for $t = 0$, we have

$$u^i = (u^i)_0$$

and for $t = 1$,

$$u^i = (u^i)_1$$

irrespective of $a$.

Moving along each path of the family as indicated above, we will get a point $P$ and vectors $\boldsymbol{e}_i$ for each pair of values $a$ and $t$.

By hypothesis we have

$$\left.\begin{aligned} \frac{\partial P}{\partial t} - \frac{\partial u^i}{\partial t}\boldsymbol{e}_i &= 0, \\ \frac{\partial \boldsymbol{e}_i}{\partial t} - \Gamma_i{}^k{}_h \frac{\partial u^h}{\partial t}\boldsymbol{e}_k &= 0. \end{aligned}\right\} \tag{2.32}$$

Now put

$$\left.\begin{aligned} \frac{\partial P}{\partial a} - \frac{\partial u^i}{\partial a}\boldsymbol{e}_i &= \boldsymbol{\varepsilon}, \\ \frac{\partial \boldsymbol{e}_i}{\partial a} - \Gamma_i{}^k{}_h \frac{\partial u^h}{\partial a}\boldsymbol{e}_k &= \boldsymbol{\varepsilon}_i\,. \end{aligned}\right\} \tag{2.33}$$

Differentiate the first equations with respect to $a$, the last with respect to $t$, and subtract. We will get

$$\left.\begin{aligned} \frac{\partial \boldsymbol{\varepsilon}}{\partial t} &= \left(\frac{\partial u^i}{\partial t}\frac{\partial \boldsymbol{e}_i}{\partial a} - \frac{\partial u^i}{\partial a}\frac{\partial \boldsymbol{e}_i}{\partial t}\right), \\ \frac{\partial \boldsymbol{\varepsilon}_i}{\partial t} &= \Gamma_i{}^k{}_h\left(\frac{\partial u^h}{\partial t}\frac{\partial \boldsymbol{e}_k}{\partial a} - \frac{\partial u^h}{\partial a}\frac{\partial \boldsymbol{e}_k}{\partial t}\right) + \frac{\partial \Gamma_i{}^k{}_h}{\partial u^\ell}\left(\frac{\partial u^\ell}{\partial a}\frac{\partial u^h}{\partial t} - \frac{\partial u^\ell}{\partial t}\frac{\partial u^h}{\partial a}\right)\boldsymbol{e}_k\,. \end{aligned}\right\} \tag{2.34}$$

Now replace the $\dfrac{\partial \boldsymbol{e}_\ell}{\partial t}$ and $\dfrac{\partial \boldsymbol{e}_\ell}{\partial a}$ by their values given by (2.32) and (2.33). Relations (2.28), assumed verified, tell us that the right hand sides of equations

(2.34) are zero, if we assume $\varepsilon$ and $\varepsilon_i$ are zero. Equations (2.34) thus reduce to the following:

$$\left.\begin{aligned} \frac{\partial \varepsilon}{\partial t} &= \frac{\partial u^i}{\partial t}\varepsilon_i, \\ \frac{\partial \varepsilon_i}{\partial t} &= \Gamma_i{}^k{}_h \frac{\partial u^h}{\partial t}\varepsilon_k\,. \end{aligned}\right\} \tag{2.35}$$

For $t = 0$, the point $P$, the vectors $\boldsymbol{e}_i$ and the functions $u^i(t, a)$ do not depend on $a$; consequently the values $\varepsilon$ and $\varepsilon_i$ are zero for $t = 0$. Since they satisfy differential equations (2.35) which admit the solution $\varepsilon = \varepsilon_i = 0$, we have identically

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = 0.$$

The $\dfrac{\partial u^i}{\partial a}$ are zero for $t = 1$ by hypothesis; formulas (2.33) show that we also have, for $t = 1$,

$$\frac{\partial P}{\partial a} = 0, \qquad \frac{\partial \boldsymbol{e}_i}{\partial a} = 0.$$

The point $P$ and the vectors $\boldsymbol{e}_i$ thus do not depend, for $t = 1$, on the parameter $a$. This is what we wanted to prove.

**49.** The point $P$ and the vectors $\boldsymbol{e}_i$ are functions of $u^1, ..., u^n$. Given two sets of infinitely close values $(u^i)$ and $(u^i + du^i)$, we can assume that the two corresponding systems $(P, \boldsymbol{e}_i)$ have been determined by means of a path starting from $(u^i)_0$ and passing through $(u^i)$ and $(u^i + du^i)$. This shows that equations (2.31) are identically satisfied. The vectors $\boldsymbol{e}_i$ form the natural system of reference of the Euclidean space referred to the coordinates $(u^i)$ and, since we have (n° 47)

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij},$$

the $ds^2$ of the space is identical to the given $ds^2$.[13]

The rectangular coordinates $x^i$ of the point $P$ which correspond to a set of values $(u^i)$ are determined functions

$$x^i = F^i(u^1, ..., u^n). \tag{2.36}$$

It is easy to see that the functional determinant of the $n$ functions $F^i$ is never zero. The equation

$$(dx^i)^2 = g_{ij} du^i du^j$$

shows in fact that we have

$$\frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} = g_{ij}\,.$$

Calculating the square of the functional determinant of the $x^i$ with respect to

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[13] *See*, in Note V, p 367, another way to prove the consistency of system (2.31), which is what we call a *completely integrable* system.

the $u^i$ gives immediately the value $g$. Since we have assumed the discriminant $g$ to be different from zero in the entire domain $(\mathcal{D})$, this leads to the stated property of the functional determinant.

We cannot conclude from this that to each system of values of the $x^i$ provided by equations (2.36) there corresponds a single system of values of the $u^i$; we can do this only if we we consider a sufficiently small neighbourhood in the domain $(\mathcal{D})$ around the set of values $(u^i)_0$.

*It follows from this that there corresponds to the domain* $(\mathcal{D})$ *a domain* $(\Delta)$ *in Euclidean space that admits the given* $ds^2$*, but this domain* $(\Delta)$ *can overlap itself partially or totally. If we restrict the domain* $(\mathcal{D})$ *to a sufficiently small neighbourhood of the* $(u^i)_0$*, the corresponding domain* $(\Delta)$ *does not overlap itself, and there is a one-to-one correspondence between the considered sets of values of the* $u^i$ *and the points of* $(\Delta)$*. The domain* $(\Delta)$ *is simply connected like the corresponding domain* $(\mathcal{D})$*.*

# 3 Locally Euclidean Riemannian Spaces

## I.– The concept of a manifold.

**50.** The general concept of a manifold is very difficult to define with precision. A surface gives the idea of a two-dimensional manifold. Take for example a sphere, or a torus: we can decompose this surface into a finite number of parts such that there is a one-to-one representation of each of these parts on a simply connected region of the Euclidian plane.

More precisely, given any point $P_0$ of the manifold, we can find a system of coordinates $u, v$ in the neighbourhood of the point $P_0$ such that, if $u_0, v_0$ are the coordinates of $P_0$, there is a positive number $r$ that has the following property. Every set of numbers $u, v$ that satisfies the inequality

$$(u - u_0)^2 + (v - v_0)^2 < r^2 \tag{3.1}$$

are the coordinates of one and only one point near $P_0$ on the manifold, and conversely, in a sufficiently small neighbourhood of $P_0$, any point $P$ has coordinates $u, v$ that satisfy inequality (3.1).

The sphere and the torus are two-dimensional manifolds without boundary. A cylinder of revolution and a hyperbolic paraboloid are *open* two-dimensional manifolds (with boundaries at infinity). A sheet of a cone of revolution, the vertex excluded, is a manifold which has a boundary at infinity and a boundary at finite distance (the vertex).

The volume inside a sphere is an *open* three-dimensional manifold, where the boundary is the surface of the sphere. The volume inside the sphere, including

For the material treated in this Chapter, consult W. Killing, *Einführung in die grundlagen der geometrie*, t. 1, Paderbron, 1893; F. Klein, Conférences sur les Mathématiques faites à l'Exposition de Chicago (Conf. XI); J. Hadamard, *Sur la forme de l'espace* (Proc.-verb. des séances de la Soc. des Sc. phys. et nat. de Bordeaux, 1897-1898, p. 83-85); H. Weyl, *Die idee der Riemannschen fläsche* (Leipzig und Berlin, 1923, and also *Math. Ann.*, t. 77. 1916, p. 349); H. Hopf, *Zum Clifford-Kleinschen Raumproblem* (Math. Ann., t. 95, 1926, p. 313-339). *See* also F. Enriques, *Principes de la géometrié* (Encycl. Sc. math., t. Ill, 1, p. 131-136).

On the general concept of a manifold, we might consult: F. Hausdorff, *Grundzüge der Mengenlehre* (Leipzig, 1914); P. Alexandroff und H. Hopf, *Topologie*, I (Berlin, 1935); N. Bourbaki, Éléments de Mathématique, Llvre III, *Topologie générale* (Paris, Hermann, 1940).

the surface, is a three dimensional manifold with boundary, but the boundary is part of the manifold, which is said to be *closed*.

In the preceding examples, each manifold is defined by a set of points situated in a pre-existent space. But we can also consider manifolds *in abstracto*. In the general case, an $n$-dimensional manifold is characterised by the possibility of representing the neighbourhood of each point $P_0$ by means of a system of $n$ coordinates $u^i$ capable of taking all possible values in the neighbourhood of the system of values $(u_0^i)$ that represent $P_0$.

**51.** The coordinates capable of representing analytically part of a manifold can be chosen in an infinity of ways. By passing from one system of coordinates to another, it is understood that the new coordinates are continuous functions of the old, and conversely. The aim of the *Analysis situs* is the study the properties of manifolds which are invariant under such changes of coordinates.

In differential geometry, we add the condition that the new coordinates, considered as functions of the old, are not only continuous, but furthermore admit continuous partial derivatives up to a certain order. The range of properties that are invariant under such changes of coordinates is already singularly more extensive than in the *Analysis situs*.

Define, for example, a line by giving the coordinates of its points as functions of a parameter $t$. To say that the functions are differentiable with respect to $t$ is to state a property of the line that is conserved by all admitted changes of coordinates: we arrive in this way at the concept of a *line element*. Analytically, a line element is defined by $n$-coordinates $u^1, ..., u^n$ and the mutual ratios of their differentials $du^1, ..., du^n$. Geometrically, it is defined by the set of lines *tangent to each other* at a given point.

We arrive at the concept of a *plane element* by considering the set of line elements issuing from the same point and which satisfy the same system of $n-2$ linear equations in $du^1, ..., du^n$; this is obviously a property of this set of line elements which is conserved by an admissible change of coordinates. We define similarly a plane element of 3, 4, etc. dimensions.

If from a given point there go out 4 line elements tangent to the same plane element, the cross-ratio of these line elements is a number which is conserved by any change of coordinates. These considerations can be generalised in many ways.

In summary, we can say that the study of properties of this nature is the geometry of the manifold from the point of view of the group of continuous and *differentiable* point transformations, while *Analysis situs* is the geometry of the manifold from the point of view of the group of simply continuous point transformations.

If we suppose that the new coordinates admit partial derivatives of the two first orders with respect to the old and conversely, the range of geometric concepts extends still further. We can speak of lines that have between them a contact of second order, etc.

**52.** A *Riemannian manifold or space* is a manifold to which we have attached a metric: this means that, in each part of the manifold, represented analytically by means of a system of coordinates $u^i$, we are given a quadratic differential form

$$ds^2 = g_{ij} du^i du^j.$$

We will suppose that the $g_{ij}$ are continuous functions admitting continuous partial derivatives of the two first orders. Consequently we will admit only changes of coordinates such that the new coordinates admit continuous partial derivatives of the first two orders with respect to the old coordinates, and conversely.

We shall say that a metric is *regular* in a given region of the manifold if, at all points of this region, $ds^2$ is a positive definite form of the $du^i$.

We shall suppose of course that if the manifold is composed of several parts that admit distinct analytic representations, matching the metrics between adjacent parts is possible. We can suppose for example that the analytic representation of each part can be extended a little beyond it into the adjacent parts, and that the two $ds^2$ thus obtained in the common portions are reducible to each other by the change of coordinates which takes one analytic representation into the other.

## II.– Locally Euclidean Riemannian spaces.

**53.** A Riemannian manifold is said to be locally Euclidian if in each of the parts of this manifold, defined analytically by a certain system of coordinates $u^i$, the $ds^2$ satisfies conditions in (28) (n° 43) of the linear element of the Euclidian space.

This means that, according to what was proved at the end of the previous chapter, that the manifold, in a sufficiently small neighbourhood of any of its points $M_0$, can be represented on a small domain of the Euclidian space with conservation of $ds^2$. We shall say that this representation is a *development* of the portion considered of the manifold onto the Euclidian space. Conversely, the domain obtained of the Euclidian space is *developable* on the corresponding small region of the manifold.

If the metric of the Riemannian manifold is everywhere regular, we see that we can develop step by step all the manifold onto the Euclidian space. But it is not certain *a priori*:

1. That every point of the Euclidian space can be obtained in the development.
2. That a point in the Euclidian space obtained in the development of the manifold cannot be obtained more than once.

**54**. Before going further, let us make clear the above by some simple examples borrowed from the case of two dimensions.

A cylinder of revolution immersed in ordinary space has for linear element

$$ds^2 = du^2 + dv^2,$$

where we denote by $u$ the curvilinear abscissa (between 0 and 1) reckoned on a right section and by $v$ the ordinate; this linear element is Euclidian; but the manifold formed by the cylinder is not simply connected, and the development onto the Euclidian plane gives a set of infinite bands of width $\ell$: each point of the cylinder has as correspondents an infinity of points of the plane which are obtained the one from the other by a translation of fixed direction and whose length is an arbitrary multiple of $\ell$. We see that here *the plane is completely covered once*. We have an image of the manifold by taking in the plane a band unlimited in both directions of width $\ell$ and by regarding as identical two points of the two parallels which limit the band when the line that joins them is perpendicular to these parallels.

Another example is provided by a torus; the position of a point on a torus is completely determined by two angles $\theta$ and $\varphi$ both between 0 and $2\pi$; by giving on the torus a linear element

$$ds^2 = a\,d\theta^2 + 2b\,d\theta\,d\varphi + c\,d\varphi^2$$

with constant coefficients, we define on this manifold an Euclidian metric.[2] On the Euclidian plane, the variables $\theta$ and $\varphi$ would be Cartesian coordinates. The torus thus develops onto the Euclidian plane as a parallelogram $0 \leq \theta < 2\pi$, $0 \leq \varphi < 2\pi$, as long as we are compelled, to define the development, to draw on the torus only lines that cross neither the line $\theta = 0$, nor the line $\varphi = 0$. If we lift these restrictions, the torus develops onto all the plane, which is covered once and only once; but the representative domain of the manifold is a parallelogram whose opposite sides are not regarded as distinct.

A last example is provided by an open ended sheet of a cone of revolution whose $ds^2$ is the one that results from the metric of the (ordinary) space in which it is immersed. As we know, this $ds^2$ is also Euclidian. Only here *the vertex of the cone is a singular point for the metric*, since a straight half-line from the vertex and passing successively through all directions (on the cone) *describes an angle less than* $2\pi$. If we want to avoid considering singular points of the metric, we should exclude the vertex of the manifold considered; it will therefore be *open* on the side of the vertex (and on the side of the infinite), thus becoming, from the point of view of the *Analysis situs*, identical to the cylinder of revolution.

[2] According to a remark by W. Killing, we have a concrete realisation of this manifold, for $b = 0$, by taking in the Euclidian space of *four* dimensions, the surface defined by the equations

$$x_1 = \sqrt{a}\,\cos\theta, \quad x_2 = \sqrt{a}\,\sin\theta, \quad x_3 = \sqrt{c}\,\cos\varphi, \quad x_1 = \sqrt{c}\,\sin\varphi.$$

Clifford has given another interpretation in in the elliptic space of three dimensions.

The development of this manifold onto the Euclidian plane will now give all the plane (*with the exception of one point*), *but this plane being covered an infinity of times* (if at least the sine of the half angle at the vertex is an irrational number). The result, as we see, is completely different from that which was obtained in the two preceding cases.

Finally, any developable surface also has an Euclidian $ds^2$, but the line of regression is a locus of singular points for the metric; by taking only one of the sheets of the surface, we obtain a development which covers only one part of the Euclidian plane, and can cover this part many times and even an infinity of times.

**55.** It seems from the above that there is a correlation between the fact that the Euclidean space is covered completely and that it is covered once only. The two last examples cited have this in common that *the manifold considered is open at finite distance*, a circumstance which does not arise in the first two examples (cylinder and torus).

We will exclude from our considerations Riemannian spaces that have a peculiarity like that of the cone. It is necessary for this to consider them *in themselves*, and not in relation to a pre-existent ambient space that contains them.

Define first the distance $[AB]$ between two points $A$ and $B$ in a Riemannian space with metric that is everywhere regular as the lower bound of the length of the arcs of (rectifiable) curves that join the point $A$ to the point $B$. We see easily that given any three points $A$, $B$, $C$, we have the inequality

$$[AC] \leq [AB] + [BC].$$

We will call a *sphere* with centre $A$ and or radius $R$ the set of points $M$ that satisfy the inequality

$$[AM] \leq R.$$

An infinite set of points of the space will be said to be *bounded* if the distance from a fixed point $A$ to the points of the set is bounded; this property is clearly independent of the fixed point $A$ chosen.

We shall say that a point $P$ of the Riemannian space is a *limit point* of an infinite set $(E)$ of points of the space if, in every sphere with centre $P$ and of arbitrarily small radius $r$, there exists at least one point of the set different from $P$; there exist then an infinity.

**56.** With these definitions set down, we will consider only Riemannian spaces with a metric that is regular everywhere and having the property that *any bounded infinite set of points of this space admits at least one limit point.* This property is expressed commonly by saying that the space is *locally compact.* It is clear that an indefinite sheet of a cone, (*vertex excluded*), considered as a Riemannian space of two dimensions endowed with the metric induced on it by the (*ordinary*) Euclidian space in which the cone is immersed, does not have the preceding property. We shall say that a Riemannian space with a metric that is regular everywhere and having the preceding property is *normal.* The cylinder, the torus (equipped with the metric defined above), and the Euclidian space itself are clearly normal.

There are two large classes of normal spaces.

*A normal Riemannian space will be said to be closed, or compact, if every infinite set of points admits at least one limit point.* In such a space, the distance $[AM]$ from a fixed point $A$ to a variable point $M$ is *bounded*; if not, in fact, there would exist an infinite set of points $M_1, M_2, ..., M_n, ...$ such that the distance $[AM_n]$ increases indefinitely; but this is impossible, because one such set would admit at least one limit point $P$ and, in the interior of a sphere with centre at $P$ and radius $r$, there would exist an infinity of points of the set, therefore points $M_n$ at an arbitrarily large distance from $A$, while we have

$$[AM_n] \leq [AP] + r.$$

We can add that the distance $[MN]$ of two variable points is also bounded, by virtue of the inequality

$$[MN] \leq [AM] + [AN].$$

It is clear conversely that if the distance $[MN]$ of two variable points of a normal Riemannian space is bounded, the space is closed, because every infinite set of points of this space is necessarily bounded and consequently admits a limit point.

We can say that a normal Riemannian space which is not closed is *open to the infinite*: it is like this for the cylinder of revolution and the Euclidian space itself. This expression is self explanatory; it expresses the existence of infinite sets of points that extend indefinitely from a given point $A$ without having a limit point.

## III.– Normal locally-Euclidean Riemannian spaces.

**57**. We propose to prove the following fundamental theorem.

*If a Riemannian space with Euclidian metric is normal, its development onto the Euclidian space covers all this space once and only once.*

We have seen (n° 49) that given any point $M_0$ of the Riemannian space, there exists a positive number $r$ such that, if the point $M_0$ corresponds to a point $P_0$ of the Euclidian space, there exists a one-to-one correspondence between the points of the sphere $(S)$ with centre $P_0$ and radius $r$ and the points of the Riemannian space which are in a certain neighbourhood of $M_0$. These points obviously generate a simply connected sphere $(\Sigma)$ with centre $M_0$.

The number $r$ associated with the point $M_0$ is naturally not uniquely determined; we can replace it for example by any smaller positive number. But the following remark is fundamental:

*If the point $M_0'$ is a distance from $M_0$ less than a positive number $\varepsilon < r$, we can* *associate with $M_0'$ the number $r' = r - \varepsilon$.* In fact, since the point $M_0'$ is interior to $(\Sigma)$, there corresponds to it, in the Euclidian space, a point $P_0'$ interior to $(S)$, and the sphere $(S')$ with centre $P_0'$ and radius $r - \varepsilon$ is completely interior to $(S)$, which guarantees the existence of a one-to-one correspondence between the points of (S') and those of a certain neighbourhood of $M_0'$.

**58.** We will deduce from the preceding remark that if the point $M$ remains in the interior or on the boundary of a bounded domain, for example of a sphere of the Riemannian space, the quantity $r$ remains greater than a fixed positive number $\rho$. In fact, if it were not so, we could find an infinite sequence of numbers

$$\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...$$

tending to zero, as well as an infinite sequence of points of the domain

$$M_1, M_2, ..., M_n, ...$$

in such a way that it is impossible to associate with the point $M_n$ a number $r$ greater than $\varepsilon_n$. Now the sequence of points considered will admit at least one limit point $M_0$, with which we can associate a number $r_0$. If $\varepsilon$ denotes a number as small as you want, there would exist an infinity of points of the sequence $M_n$ which would be at a distance from $M_0$ less than $\varepsilon$, and to any of these points we would be able to associate the number $r_0 - \varepsilon$, which obviously would eventually exceed $\varepsilon_n$ for $n$ sufficiently large.

The proposition thus proven, choose in the Riemannian space an origin point $M_0$ and, corresponding to it, a point $P_0$ in the Euclidian space, as well as a Cartesian frame of reference $(R_0)$ defined, up to orientation, by the numerical values at $M_0$ of the coefficients $g_{ij}$ of the fundamental form. We can note by the way that $(R_0)$ is defined up to a rotation and *up to a symmetry.* It is from the point $P_0$, equipped with its frame of reference $(R_0)$, that we shall develop the Riemannian space onto the Euclidian space.

We will prove successively two theorems whose combination will form the fundamental theorem stated above.

**59. Theorem I.** – *The Euclidian space is covered entirely in the development of the Riemannian space.*

In fact, let $P$ be any point in the Euclidian space. Join $P_0$ to $P$ by any path $(C)$ of finite length $\ell$. Consider in the Riemannian space the sphere $(\Sigma)$ with centre $M_0$ and radius $R$, where $R$ is a given number greater than $\ell$. (This sphere can be identical to the Riemannian space itself, if it is closed and if $R$ is very large.) To the sphere $(\Sigma)$ corresponds a positive number $\rho$ less than all the numbers $r$ attached to the different points of $(\Sigma)$.

Imagine, in the Euclidian space, a sequence of spheres of radius $\rho$ where the first has centre $P_0$, the last the point $P$, the intermediate spheres having their centres on the path $(C)$ and overlapping one another in such a way that each point of $(C)$ is *interior* to one of the spheres of the sequence. The path $(C)$ can then be divided into a certain number of partial arcs

$$P_0P_1, \quad P_1P_2, \quad ..., \quad P_{n-1}P,$$

such that each of them is situated inside the same sphere of the sequence.

That said, the arc $P_0P_1$, situated in the interior of the first sphere of radius $\rho$, can be developed on the Riemannian space as an arc $M_0M_1$ *completely interior to* $(\Sigma)$; it will be the same for the arc $P_1P_2$ and the successive arcs which develop onto the Riemannian space without exiting from $(\Sigma)$. The path $(C)$ will thus develop entirely as a certain path $(\gamma)$ which, conversely, will give $(C)$ in the development of the Riemannian space onto the Euclidian space. The theorem is thus proved.

**60.** At the endpoint $M$ of the path $(\gamma)$ the $g_{ij}$ have definite numerical values. If the same system of coordinates is used along all the path $(\gamma)$, there will be, attached to the various points of the path $(C)$, Cartesian frame of references of reference varying continuously, in such a way that each point of $(C)$ will be provided with a frame of reference of reference $(R)$.

If there are successively two different systems of coordinates $(u^i)$ and $(v^i)$ in the regions of the Riemannian space crossed by the path $(\gamma)$, we will have, on a first portion of the path $(C)$, a Cartesian frame of reference $(R_u)$, then, on a second portion of $(C)$, a Cartesian frame of reference $(R_v)$: at the point of separation, there will be a discontinuity in the variation of the frame of reference. It could also happen that the path $(\gamma)$, leaving from $M_0$ in the region with coordinates $u^i$, then crosses the region with coordinates $v^i$, then returns to the region with coordinates $u^i$. We will then have on the path $(C)$ an arc equipped with frames of reference $(R_u)$, a second arc equipped with frames of reference $(R_v)$, and a third arc equipped with frames of reference $(R_u)$.

When the same system of coordinates $u^i$ is used on the entire curve $(\gamma)$, the frame of reference $(R)$ attached to $P$ clearly has the same orientation as the frame of reference $(R_0)$ attached to $P_0$. But if the curve $(\gamma)$, having left a region with coordinates $u^i$, returns to this region after having crossed a region with coordinates $v^i$, the frame of reference $R_u$ attached to $P$ *might not have the same orientation* as the frame of reference $(R_u)_0$ attached to $P_0$. In fact, at the first point of separation of the two regions, the two frames of reference $R_u$ and $R_v$ will have the same orientation or not according to the sign of the functional determinant

$$\frac{D(u^1, u^2, ..., u^n)}{D(v^1, v^2, ..., v^n)};$$

at the second point of separation, the sign of this determinant will intervene again, but there is no reason *a priori* for these signs to be the same at the two points considered.

Finally, let us add the almost self evident remark that a given path coming from $P_0$ in the Euclidian space can only arise from a single path coming from

$M_0$ in the Riemannian space: this follows the same proof that has just been given.

**61. Theorem II.** – *The Euclidian space is covered only once by the development of the Riemannian space.*

It is sufficient to show that two different paths $(C)$ and $(C')$ starting from the point $P_0$ and terminating at the same point $P$ develop onto the Riemannian space following two curves starting from $M_0$ and terminating at the same point $M$. *The proof relies on the property of the Euclidian space of being simply connected.*

Imagine a sequence of paths

$$(C), \quad (C_1), \quad ..., \quad (C_{n-1}), \quad (C')$$

all leaving from $P_0$ and all terminating at point $P$. Denote by $\ell$ and $\ell'$ the lengths of the two paths $(C)$ and $(C')$ and take on each of them a parameter $t$ that varies continuously from 0 to 1 when we describe the path considered. We could take for example

$$t = \frac{s}{l} \qquad \text{on the first path;}$$

$$t = \frac{s}{l'} \qquad \text{on the second path,}$$

where $s$ is the curvilinear abscissa measured from $P_0$.

Let the respective equations of the two curves, in rectangular coordinates, be

$$x_i = f_i(t) \tag{3.1}$$

and

$$x_i = \varphi_i(t). \tag{3.2}$$

Define a curve $(C_\alpha)$ by the equations

$$x_i = \alpha f_i(t) + (1-\alpha)\varphi_i(t) \qquad (0 < \alpha < 1). \tag{3.3}$$

We have, on $(C_\alpha)$,

$$ds^2 = \left[\alpha^2\ell^2 + (1-\alpha)^2\ell'^2 + 2\alpha(1-\alpha)f_i'(t)\varphi_i'(t)\right]\ dt^2$$

Now, the inequality

$$|f_i'(t)\varphi_i'(t)| \leq \sqrt{f_i'(t)}\,\sqrt{\varphi_i'(t)}\ = \ell\ell'$$

gives

$$ds^2 \leq \left[\alpha\ell + (1-\alpha)\ell'\right]^2\ dt^2$$

Let $L$ be the largest of the two lengths $\ell$ and $\ell'$. We see that *the curvilinear abscissa of any point of* $(C_\alpha)$ *is at most equal to* $Lt$, and consequently that *all the curves of the family are of length at most equal to* $L$.

Let $R$ be a number greater than $L$; consider, in the Riemannian space, the sphere $(\Sigma)$ with centre $M_0$ and radius $R$, and let $\rho$ be a positive number less than all the numbers $r$ attached to the different points of $(\Sigma)$.

Finally, evaluate the distance $\delta$ of two points of the two curves $(C_alpha)$ and $(C_{\alpha'})$ corresponding to the same value of $t$. We have

$$\delta^2 = (\alpha - \alpha')^2 \sum_i \; [f_i(t) - \varphi_i(t)]^2 .$$

Let $D^2$ be the maximum of the sum which is on the right hand side, when $t$ varies between 0 and 1. We have

$$\delta \leq (\alpha - \alpha')D.$$

That said, divide the interval (0,1) in which $\alpha$ varies into partial intervals all less than $\frac{2}{3}\frac{\rho}{D}$ and consider the curves $(C), (C_1), ..., (C_n), (C')$ corresponding to values of subdivision. Similarly divide the interval (0,1) of the variation of $t$ into partial intervals each less than $\frac{2}{3}\frac{\rho}{L}$ and let

$$0, \quad t_1, \quad t_2, \quad ..., \quad t_{p-1}, \quad 1$$

be the points of subdivision. To these values of $t$ correspond on each curve $(C_i)$ the point $P_0$, the point $P$ and $p-1$ intermediate points $P_i^1, P_i^2, , ..., P_i^{p-1}$.

*The spheres of radius $\rho$ and with centres*

$$P_0, \quad P_i^1, \quad P_i^2, \quad ..., \quad P_i^{p-1}$$

*have the property that any point, whether of $(C_i)$ or of $(C_{i+1})$, is interior to one of these spheres.* It is clear for $(C_i)$. Then let $Q$ be any point of $(C_{i+1})$ (Figure 1) and let $P_{i+1}^h$ be that point of the $p+1$ points $P_{i+1}^k$ which is the closest to $Q$; we have

$$\text{arc } QP_{i+1}^h < \frac{1}{2}\,\frac{2}{3}\,\frac{\rho}{L}\;L = \frac{1}{3}\;\rho;$$

on the other hand

$$P_i^h P_{i+1}^h < \frac{2}{3}\;\frac{\rho}{D}\;D = \frac{2}{3}\;\rho;$$

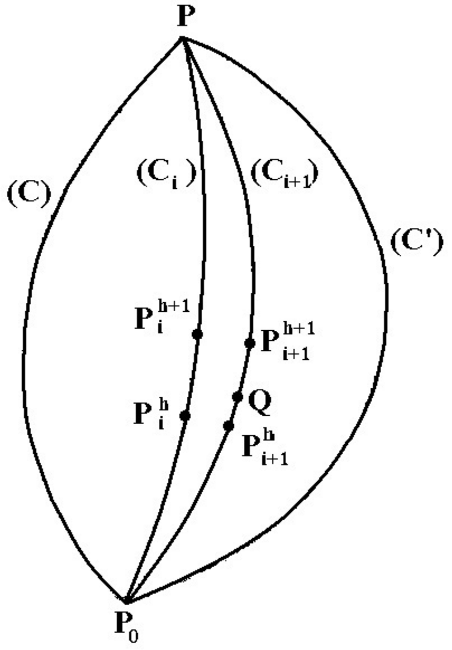
consequently

$$QP_i^h < \frac{2}{3}\;\rho + \frac{1}{3}\;\rho = \rho;$$

in other words the point $Q$ is *interior* to the sphere with centre $P_i^h$.

To $p+1$ spheres of radius $\rho$, having their centres on $(C)$, there correspond, step by step in the Riemannian space, $p+1$ spheres, overlapping one with another and whose centres do not exit $(\Sigma)$; the curve $(C)$ gives in the Riemannian space a curve $(\gamma)$ starting from $M_0$ and terminating a certain point $M$, centre of the last sphere. The curve $C_1$ clearly will also develop following a curve $(\gamma_1)$ leaving from $M_0$ and terminating at $M$. Consideration of the $p+1$ spheres that have their centres on $C_1$ shows similarly that the curve $C_2$ will develop following a curve $\gamma_2$ starting from $M_0$ and terminating at $M$. We thus see step by step that

**Figure 1**



the path $(C')$ will also give in the Riemannian space a path $(\gamma')$ starting from $M_0$ and terminating at $M$, moreover entirely contained in $(\Sigma)$.

*It follows from this that, in the inverse development of the Riemannian space, it is always the same point $M$ that comes to occupy the position $P$.* This is what we wanted to prove.

The fundamental theorem is thus completely proved.

**62.** Suppose in particular that the Riemannian space is simply connected. The development of the Euclidian space onto the Riemannian space covers it entirely once and only once. Consequently *the two spaces are identical*, in this sense, that there exists a one-to-one point correspondence between points of these two spaces, with conservation of distances.

*Every simply connected, normal Riemannian space with a Euclidian metric is identical to the Euclidian space.*

In particular, such a space extends to infinity. It follows that, for $n = 2$, *the surface of a sphere*, which is simply connected, *cannot be defined analytically by means of a single system of everywhere regular coordinates.* Otherwise, in fact, if $u$ and $v$ denote the coordinates, the line element $ds^2 = du^2 + dv^2$ would define on the sphere an everywhere regular Euclidian metric.

## IV.– Holonomy group of a normal locally Euclidean Riemannian space.

**63.** Consider now a locally Euclidian, normal Riemannian space, but not simply connected. The point $M_0$ gives, in the development onto the Euclidian space, many points (or even an infinity)

$$P_0, \quad P_1, \quad P_2, \quad ...$$

each equipped with a Cartesian frame of reference

$$(R_0), \quad (R_1), \quad (R_2), \quad ...$$

corresponding to the same numerical values of the $g_{ij}$ at the point $M_0$. *All these frames of reference are equal or symmetrical.*

It is clear that had we begun the development by making $P_1$ equipped with the frame of reference $(R_1)$ correspond to the point $M_0$, the development of the Riemannian space would not have suffered any essential change; if a certain path $(\gamma)$ going from $M_0$ to $M$ had led in the first development to a point $P$ equipped with a frame of reference $(R)$, the new development would lead to a point $P'$ and to a frame of reference $(R')$ *positioned relative to* $(R_1)$ *in the same way that* $P$ *and* $R$ *were positioned relative to* $(R_0)$.

It follows from this that the displacement $S_1$ (accompanied or not by a symmetry) that, in the Euclidian space, brings $(R_0)$ into coincidence with $(R_1)$, transforms with each other the frames of reference

$$(R_0), \quad (R_1), \quad (R_2), \quad ...$$

In fact, the frame of reference $(R_i)$ is deduced from $(R_0$ by development of a certain closed contour (a cycle) $(\gamma_i)$, starting from $M_0$ and returning to it; the development of the same cycle by starting from $(R_1)$ will give a certain frame of reference $(R_j)$ positioned relative to $(R_1)$ as $(R_i)$ is positioned relative to $(R_0)$; this frame of reference $(R_j)$ is moreover that which we would have obtained by developing first the cycle $(\gamma_1)$, and then the cycle $(\gamma_i)$.

**64.** The preceding considerations show us at the same time that the displacements

$$S_0, \quad S_1, \quad S_2, \quad ...$$

form a group. In fact, by keeping the notation of the preceding paragraph, we see that if we perform successively the operations $S_1$ and $S_i$, the frame of reference $(R_0)$ goes first to coincide with $R_1$, then with $R_j$; the resulting displacement is thus $S_j$:

$$S_i S_1 = S_j.$$

The group $G$ thus obtained is called the *holonomy group* of the Riemannian space. To each frame of reference $R_i$, corresponds a specific operation of this group and only one, namely the one which brings $(R_0)$ into coincide with $R_i$. To the frame of reference $(R_0)$ itself corresponds the *identity operation.*

The operations of the *holonomy group* can be applied to any point $Q$ of the Euclidian space. If the operation $S_1$, for example, transports it to $Q_1$, the point $Q_1$ is positioned relative to $(R_1)$ as $Q$ was positioned relative to $R_0$. So let $\gamma$ be a path traced in the Riemannian space and developing along a given path $C$ leading from $P_0$ to $Q$, and let $N$ be the endpoint of this path. We will obtain obviously the point $Q_1$ by developing the path formed from the cycle $(\gamma_1)$ and from the path $(\gamma)$; this compound path leaves from $M_0$ to end at $N$; *therefore the point $Q_1$ and the point $Q$ correspond to the same point of the Riemannian space.* We shall say that they are *homologous.*

*It follows from this that any operation of the holonomy group takes any point of the Euclidian space to a homologous point*; we can even add that there always exists an operation of this group that takes any point of the space to any homologous point.

**65.** We will now prove two fundamental properties of the holonomy group.

Let $P$ be any point of the Euclidian space; it corresponds to a specific point $M$ of the Riemannian space. To this point we can assign a positive number $r$ such that any point $P'$ different from $P$ and situated at a distance from $P$ less than $r$ comes from a point *$M'$ different from $M$.* Therefore, *the homologues of the point $P$ are all at a distance from $P$ greater than $r$.*

This property expresses the discontinuity of the holonomy group. A group in fact is said to be *discontinuous* if to each point $P$ we can assign a number $r$ such that all the transforms of $P$ are at a distance from $P$ greater than $r$.

It could happen, for a discontinuous group, that there exist certain exceptional points which are their own homologues for a certain number of transformations of the group. That cannot happen here. In other words, *any operation of the holonomy group, other than the identity operation, does not leave invariant any point of the space.*

We thus arrive at the following theorem.

*The holonomy group of a locally Euclidian Riemannian space is discontinuous and none of its operations, other than the identity operation, leaves invariant a point of the space.*

## V.– The fundamental polyhedron.

**66.** When we develop a cylinder of revolution onto a plane, every point of the cylinder is represented, once and only once, on a certain strip of the plane, bounded by two parallel straight lines whose separation is equal to the perimeter of a right-section of the cylinder. We shall show that, in the general case, we can always construct a polyhedron containing one and only one representative of each point of the Riemannian space. We will consider, to clarify the ideas, the case of $n = 3$.

We can form one such a polyhedron by the so-called method of *radiation*. Consider the set of points of the Euclidian space which are closer to the point $P_0$ than to any one of the points $P_1, P_2, ...,$ homologues of $P_0$. This set defines a domain $\mathcal{P}$ (*fundamental domain*) whose boundary will be formed by points equidistant from $P_0$ and one of its homologous points.

The domain $\mathcal{P}$ is *convex*, because if $Q$ and $Q'$ are part of it, the two points are situated on the same side as $P_0$ with respect to the plane perpendicular to the middle of $P_0P_i$ (and that whatever $i$); it is therefore the same for any point $R$ of the segment $QQ'$.

This said, let $Q$ be any point of the Euclidian space. By virtue of the discontinuity of the holonomy group $G$, there exists one or more of the points $P_0, P_1, P_2, ...,$ closer to $Q$ than all the others. Suppose first that there is only one $P_i$. The operation $S_i^{-1}$ of the group $G$ which takes $P_i$ to $P_0$ takes $Q$ to a certain point $R$, which is obviously closer to $P_0$ than any other homologous points, and which consequently is interior to the domain $\mathcal{P}$. On the other hand there does not exist, in this domain, another point $R'$ homologous to $Q$, otherwise the operation of $G$ which takes $R'$ into $Q$ would take $P_0$ into a point $P_j$ closer to $Q$ than any other homologous points and consequently confused with $P_i$; the point $R'$ would therefore be confused with the point $R$. If there are more points such as $P_i$, $P_j$ equally distant from $Q$ and closer to $Q$ than any other homologous points, the operations $S_i^{-1}$, $S_j^{-1}$ would take $Q$ to two points $R$ and $R'$ belonging to the boundary of $\mathcal{P}$ and homologous to each other.

**67.** *The fundamental domain $\mathcal{P}$ is*, as we will show, *a polyhedral volume bounded by a finite number of plane faces.*

Suppose first that the domain $\mathcal{P}$ is situated entirely at a finite distance. Let $R$ be the maximum distance from $P_0$ to the boundary of $\mathcal{P}$. All the points of the sphere with centre $P_0$ and radius $R$, a sphere that contains $\mathcal{P}$ in its interior, are certainly closer to $P_0$ than the points $P_i$ exterior to the sphere with centre $P_0$ and of radius $2R$. It is sufficient therefore for forming $\mathcal{P}$ to consider only the points $P_i$ which are in the interior or on the surface of this last sphere. *These points are finite in number*; consequently the domain $\mathcal{P}$ will be bounded by a finite number of plane faces situated in the planes equidistant from $P_0$ and from certain of the points $P_i$.

Suppose now that the domain $\mathcal{P}$ extends to infinity. That means that there exists at least one half line issuing from $P_0$ and completely interior to $\mathcal{P}$ (and that because of the convexity of $\mathcal{P}$). If $P_0z$ is one such half line, all the points $P_i$ homologous to $P_0$ are obviously in the plane $(\Pi)$ perpendicular to $P_0z$ through $P_0$, or on the side of this plane opposite to $P_0z$.

Suppose first that there is only one half line situated completely in the interior of $\mathcal{P}$. The distance from $P_0$ to the points of $\mathcal{P}$ which are in the plane $(\Pi)$ or on the side of $(\Pi)$ opposite to $P_0z$ has an upper bound $R$; it is sufficient then for forming $\mathcal{P}$ to consider the points $P_i$ situated in the interior or on the surface of a sphere with centre $P_0$ and of radius $2R$, and we arrive at the same conclusion as just now.

Suppose in the second place that there exist two opposite half lines $P_0z$ and $P_0z'$ completely interior to $\mathcal{P}$. The points $P_i$ are necessarily all in the plane $(\Pi)$ the common perpendicular to $P_0z$ and $P_0z'$ through $P_0$. It will be sufficient to consider those which are interior to, or on the surface of, a sphere with centre $P_0$ and radius $2R$, where $R$ denotes the upper bound of the distance from $P_0$ to the points of $\mathcal{P}$ situated on the plane $(\Pi)$. In this case the domain $\mathcal{P}$ is a prism which is indefinite in both directions.

Suppose finally that we are in neither of the two preceding cases. The existence of two half lines (not opposite) $P_0z$ and $P_0z'$ completely interior to $\mathcal{P}$ leads to the same property for all the half lines $P_0z''$ interior to the angle $zP_0z'$ in the plane of this angle. The set of the half lines $P_0z$ interior to $\mathcal{P}$ thus forms a convex conical (or pyramidal) volume, and all the points $P_i$ are interior to or on the surface of the *supplementary* cone. The reasoning ends then as in the preceding cases.

**68.** If we built around each point $P_i$, homologue of $P_0$, the corresponding fundamental domain $\mathcal{P}_i$, we would fill the entire Euclidian space, without the different domains encroaching on each other. All these domains are obviously equal to each other, so that we have a kind of regular *tiling* of the space.

To points $P_i$, which have served effectively for the construction the boundary of $\mathcal{P}$, correspond a finite number $S_i$ of operations of the holonomy group. The operation $S_i^{-1}$ which takes $P_i$ to $P_0$, takes conversely $P_0$ to a point $P_j$, distinct from $P_i$, without which the middle of $P_0P_i$ would be invariant under $S_i$, which is contrary to one of the fundamental properties of the group $G$ (n° 65).

To the points $P_i$ and $P_j$ correspond two plane faces $\mathcal{F}_i$ and $\mathcal{F}_j$ of the polyhedron $\mathcal{P}$, and *the points of these two faces are pairwise homologous*; we pass precisely from a point of $\mathcal{F}_i$ to the homologous point of $\mathcal{F}_j$ by the operation $S_i^{-1} = S_j$.

*The faces of the fundamental domain $\mathcal{P}$ are thus pairwise homologous and the operation which puts into coincidence the points of the one with the homologous points of the other is that which takes the point $P_0$ to the point $P_j$ symmetric to $P_0$ with respect to the second face.*

These operations are called the *generating operations* of the homology group. This term is explained by the fact that *every operation of $G$ results from the composition, in a convenient order, of only the generating operations performed a sufficient number of times.* In fact, we can pass from $P_0$ to $P_i$ by crossing a certain number of fundamental polyhedra; the exit faces of these successive polyhedra are homologous to the faces of $\mathcal{P}$; it is sufficient to perform the generating operations associated with these faces, and that in the inverse order of that in which these faces present themselves, for the resulting operation to take $\mathcal{P}$ to $\mathcal{P}_j$, that is to say $P_0$ to $P_i$.

The holonomy group thus admits, according to the preceding, a finite number of generating operations with the help of which we can generate all the others. Let us add the remark that the holonomy group contains an infinity of operations, otherwise the centre of mean distances from $P_0$ and from its homologues, which would be finite in number, would be invariant under each of the operations of the group.

## VI.– Determination of all normal locally Euclidean Riemannian spaces.

**69.** We are now in a position to reduce the determination of all locally Euclidian, normal Riemannian spaces to a problem in the theory of groups.

In fact, let $G$ be a group of displacements (accompanied or not by a symmetry) which has the two properties stated in n$^{\text{o}}$ 65, namely of being discontinuous, and of not admitting any non-identical operation that leaves a point invariant.

The Euclidian space, *in which we agree to regard as identical two points that are homologous with respect to the group* $G$, and in which we adopt the ordinary metric, is a locally Euclidian normal Riemannian space. To give a clear account of this, let us note that given any point $Q$ of the Euclidian space, all the homologous points, by virtue of the discontinuity of the group $G$, are exterior to a certain sphere with centre $Q$ and radius $R$. Take then take the sphere with centre $Q$ and radius $R/2$; there cannot exist in the interior of this sphere two distinct points $Q_1$ and $Q_2$ homologues of each other; otherwise indeed the operation of $G$ which takes $Q_1$ into $Q_2$, *an operation which does not leave invariant the point* $Q$, would take this point $Q$ to a point $Q'$ with

$$Q_1Q = Q_2Q'\,;$$

we would have then

$$QQ' \leq QQ_2 + Q_2Q'$$

or

$$QQ' \leq QQ_2 + QQ_1 < R,$$

which is contrary to hypothesis. Consequently the whole region of the Euclidian space interior to the sphere considered of radius $R/2$ is formed from *distinct* points of the Riemannian space, and the metric is everywhere regular. The number $r$ which corresponds (n$^{\text{o}}$ 57) to the point $Q$, considered as a point of the Riemannian space, is here equal at least to $R/2$.

We can have a more concrete representation of the Riemannian space by constructing a fundamental polyhedron, which can be done by knowledge alone of the given group $G$, by staring from a point $P_0$ of the space and from its different homologues.

The method that we have adopted for constructing a fundamental polyhedron is not the only one possible. In certain cases there is even advantage in modifying it. Suppose for example that the operations of the holonomy group $G$, which leave invariant the Euclidian metric of the space, also leave invariant another metric defined, in rectangular coordinates, by a quadratic differential form with *constant* coefficients. We can then construct a fundamental polyhedron by applying the method of radiation, but by adopting this new metric. This new polyhedron will not be equal to the first, but that is of no importance.

If the Riemannian space is *closed*, the different fundamental polyhedra that we can construct are all bounded, and *all have the same volume*, namely the total volume of the Riemannian space.

Let us add the remark that the Riemannian spaces formed by a group $G$ of displacements properly so called are *orientable*; the others are *non-orientable*.

## VII.– Two-dimensional locally Euclidean normal spaces.

**70.** Let us apply the general principles above to the case of two dimensions. We have to determine all the discontinuous groups formed by translations (displacements properly so called) or by translations accompanied by a symmetry with respect to a line parallel to the direction of the translation.

An *orientable* space will therefore have a holonomy group formed only of translations. The set of a point $P$ and of its homologues will form therefore either a *linear network* or a *plane network*. The first case corresponds to the cylinder, the second to the torus equipped with a Euclidian metric. The first spaces are distinguished each other only by the amplitude of the generating translation, the last (Clifford spaces) by more essential properties; to each of the two is associated a system of elliptic functions (doubly periodic); it is the *modulus* of these functions which essentially differentiates the corresponding spaces from each other.

The fundamental polygon can here be simply constructed by applying the remark of n° 69. Any group of translations in fact leaves invariant all the metrics with constant coefficients. If we then consider then for example the group,

$$\left.\begin{array}{lll} x' & = & x + pa + qa', \\ y' & = & y + pb + qb', \end{array}\right\} \tag{3.4}$$

where $p$ and $q$ are two arbitrary integers, $a, a', b, b'$ are constants, it will be sufficient to consider the metric in which the two vectors $(a, b)$ and $(a', b')$ are unit and rectangular. The polygon obtained by the method of radiation is a parallelogram having two sides respectively equivalent to these two vectors.

**71.** The theory of analytic functions of a complex variable leads easily to Euclidean metrics, at least in a certain domain. Let $f(z)$ be an analytic function, holomorphic in a neighbourhood of $z_0$, *and which is not zero for* $z = z_0$. The equation

$$ds^2 = |f(z)\ dz|^2$$

defines in the plane of the complex variable $z$ an Euclidean metric that is regular in a neighbourhood of $z_0$; in fact, if we call $\varphi(z)$ a primitive function of $f(z)$, for example

$$\varphi(z) = \int_{z_0}^{z} f(z)\ dz,$$

a function defined in a neighbourhood of $z_0$, and if we put

$$\varphi(z) = P + iQ,$$

we have

$$ds^2 = dP^2 + dQ^2$$

This result is sufficient to show that the metric is Euclidean. It is on the other hand regular, because by putting

$$z = x + iy, \quad f(z) = A(x, y) + iB(x, y)$$

we have

$$ds^2 = (A^2 + B^2)(dx^2 + dy^2)$$

and the coefficient $A^2 + B^2$ is not zero at $z_0$.

It is easy to find locally Euclidean manifolds with a metric that is everywhere regular. First take for $f(z)$ a rational function of $z$ which is not zero for finite $z$:

$$f(z) = \frac{1}{Q(z)}\ ,$$

where $Q(z)$ is an integer polynomial in $z$. For $z$ infinite, the metric

$$ds^2 = \left|\frac{dz}{Q(z)}\right|^2$$

remains regular if, by putting $z = 1/t$, the integrand $dz/Q(z)$ takes the form $\varphi(t)\, dt$ with $\varphi(0)$ different from zero. This is so if $Q(z)$ is a polynomial of the second degree or greater. In this case, the manifold formed by the plane of the complex variable $z$ (including the point at infinity) has a metric that is regular everywhere, except at the two zeros of $Q(z)$; if we remove these two points from the manifold, *points which are, with regard to the metric considered, at infinity*, we obtain a locally Euclidean normal Riemannian space of dimension two with two boundaries at infinity. It is, on the other hand, obviously orientable. Consequently, it develops onto the Euclidean plane as an infinity of bands, in the manner of a cylinder. Then by calling the rectangular coordinates of the Euclidean plane $u$ and $v$, *we see that $z$ is a simply periodic function of $u + iv$.*

The inversion of the integral $\int dz/(az^2+bz+c)$ thus gives a simply periodic, uniform function.

We can also consider a *non-uniform* function $f(z)$; there exists then a *Riemann surface* on which it is uniform. For example, if we take $f(z) = 1/\sqrt{1-z^2}$, the metric

$$ds^2 = \left| \frac{dz}{\sqrt{1-z^2}} \right|^2$$

is, as we easily realise, regular at all points of the Riemann surface, except at the two points $z=\infty$ on the two sheets; if we remove these two points from the manifold, we obtain a manifold with two boundaries, *but situated at infinity with respect to the metric considered.* We thus obtain again, by inversion, a simply periodic function.

Take finally an elliptic integral $\int dz/\sqrt{R(z)}$ where $R(z)$ is a polynomial of fourth degree. The metric

$$ds^2 = \left| \frac{dz}{\sqrt{R(z)}} \right|^2$$

is everywhere regular on the Riemann surface of the function $\sqrt{R(z)}$, and this *without exception.* We thus define a *closed* manifold with Euclidean metric that is regular everywhere. Consequently, this manifold develops onto the Euclidean plane as a net of parallelograms and $z$, a uniform function of a point of the manifold, is a doubly periodic, uniform, analytic function of the complex variable $u+iv$, where $u$ and $v$ are the rectangular coordinates of the Euclidean place on which the development is made. *We have therefore thus proved the fundamental property on the inversion of the elliptic integral.*

The same reasoning will apply without modification to the integral of the first kind associated with any algebraic curve of rank 1 and of degree $p$,

$$f(z,t) = 0.$$

The metric

$$ds^2 = \left| \frac{Q(z,t)\,dz}{f'_t} \right|^2,$$

where $Q$ is the adjoint polynomial of degree $p-3$, is everywhere regular on the Riemann surface of the curve considered.

This reasoning will no longer apply to a hyper-elliptic integral

$$\int \frac{dz}{\sqrt{R(z)}},$$

because the metric will cease to be regular at two points $z=\infty$, *points which are not at infinity for this metric.*

**72.** Move on to non-orientable spaces. Any operation of the group $G$ involving a symmetry can be defined, in rectangular co-ordinates, by equations of the form

$$\left.\begin{array}{rcl} x' &=& x+\alpha, \\ y' &=& -y\,; \end{array}\right\} \tag{3.5}$$

this operation, repeated twice in a row, gives a translation parallel to $Ox$ and of amplitude $2\alpha$. There exist then in $G$ translations parallel to $Ox$; let $a$ be the smallest (positive) amplitude of these translations: we can always reduce $\alpha$ to have a value less than $a$, and since $2\alpha$ must be a multiple of $a$, we can suppose that $\alpha = a/2$. We will then have in the group G the operations

$$\left.\begin{array}{rcl} x' &=& x+na, \\ y' &=& y\,; \end{array}\right\} \tag{3.6}$$

$$\left.\begin{array}{rcl} x' &=& x+\left(n+\frac{1}{2}\right)a, \\ y' &=& -y\,; \end{array}\right\} \tag{3.7}$$

Suppose that we have in the group $G$ translations that are not parallel to $Ox$; let

$$x' = x+\lambda,$$
$$y' = y+\mu,$$

be one of these translations. The group $G$ will then contain the translation

$$x' = x+\lambda,$$
$$y' = y-\mu,$$

which is the result of three successive translations

$$x' = x+\frac{a}{2}, \qquad x' = x+\lambda \qquad x' = x+\frac{a}{2},$$
$$y' = -y; \qquad y' = y+\mu; \qquad y' = -y.$$

Consequently we will have also the translation parallel to $Ox$ of amplitude $2\lambda$; thus $\lambda$ is an integer multiple of $a/2$, a multiple that we can always reduce to 0 or $a/2$. Now, it is not reducible to $a/2$, otherwise the group $G$ will contain the operation

$$x' = x,$$
$$y' = -y+\mu,$$

which leaves invariant the point $(0, \mu/2)$. Thus the group $G$ contains one translation parallel to $Oy$. The general form of the operations of the group follows immediately:

$$\left.\begin{array}{rcl} x' &=& x+pa, \\ y' &=& y+qb\,; \end{array}\right\} \tag{3.8}$$

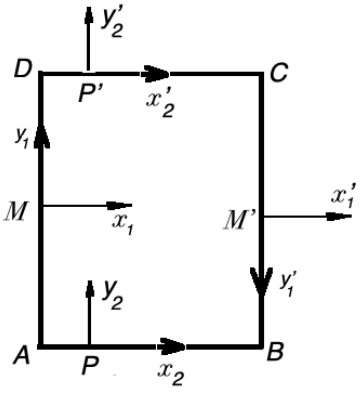
$$\left.\begin{aligned} x' &= x + \left(p + \tfrac{1}{2}\right) a, \\ y' &= -y + qb\,, \end{aligned}\right\} \tag{3.9}$$

where $p$ and $q$ denote arbitrary integers.

**73.** There exist therefore two classes of non-orientable, locally Euclidean Riemannian spaces, which correspond to two classes of orientable spaces.

We can take for the fundamental domain of spaces of the second class a rectangle having as centre the origin, whose sides are parallel to the axes and of lengths respectively $a/2$ and $b$ (Figure 2).

**Figure 2**



The two generating operations of the group are

$$\left.\begin{aligned} x' &= x + \tfrac{a}{2}\,, \\ y' &= -y, \end{aligned}\right\} \qquad \left.\begin{aligned} x' &= x, \\ y' &= y + b\,; \end{aligned}\right\}$$

the first takes the side $AD$ onto the side $CB$, where the frame at a point $M$ of $AD$ changes orientation at the homologous point $M'$ of $CB$; the second takes by translation the side $AB$ onto the side $DC$, where the frame at a point $P$ of $AB$ keeps its orientation at the homologous point $P'$ of $DC$.

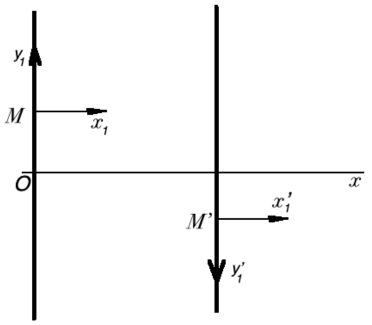
**74.** Let us emphasise a little more the first class of non-orientable spaces whose holonomy group is defined by the formulae

$$\left.\begin{aligned} x' &= x + na, \\ y' &= y\,; \end{aligned}\right\}$$

$$\left.\begin{aligned} x' &= x + \left(n + \tfrac{1}{2}\right) a, \\ y' &= -y\,. \end{aligned}\right\}$$

We can take as fundamental domain (Figure 3) the strip on the plane included

**Figure 3**

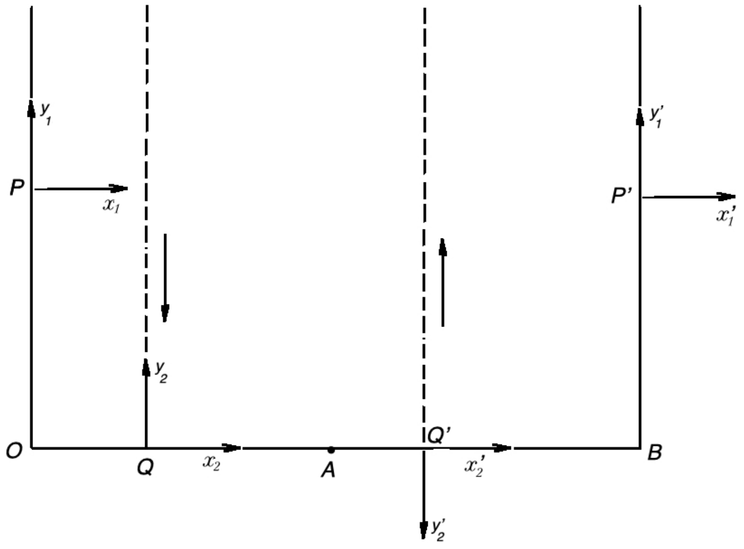


between the straight lines $x = 0$ and $x = a/2$, where the generating operation which takes the first line onto the second is

$$x' = x + \frac{a}{2},$$
$$y' = -y\,.$$

But we can take another more convenient fundamental domain, by replacing the part of the first which is above $Ox$ by its homologue with respect to the generating operation. We obtain in this way (Figure 4) the region of the plane situated above $Ox$ between the lines $x = 0$ and $x = a$. To this domain correspond

**Figure 4**



*two* generating operations. The one, defined by

$$x' = x + a,$$
$$y' = y\,,$$

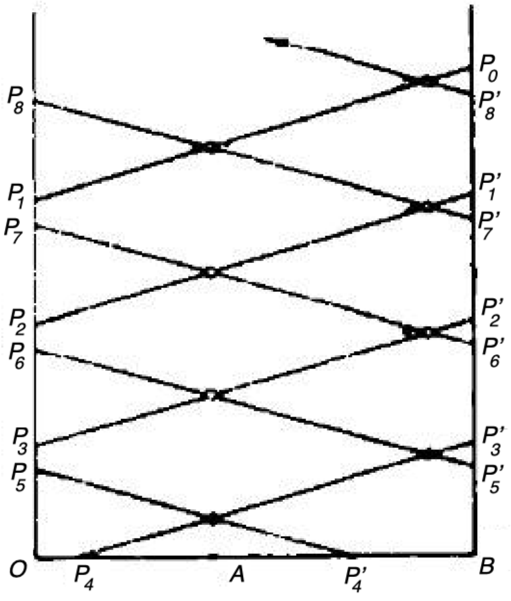
puts the unlimited side $x = 0$ into coincidence with the unlimited side $x = a$. The second, defined by

$$x' = x + \frac{a}{2},$$
$$y' = -y\,,$$

puts into coincidence the side $OA$ with its extension $AB$. The correspondence between the points situated on the boundary and the frames of reference that we can attach to them is shown in the Figure.

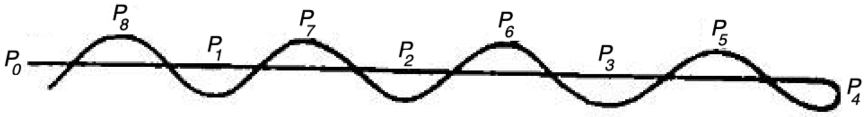
The space admits a closed straight line of length $a/2$, namely $OA = AB$; the straight lines parallel to $Ox$ are also closed but of double the length $a$; the straight lines perpendicular to $Ox$ are unlimited in the two directions and represented in the interior of the fundamental domain by two half-lines travelled in opposite direction; the first leads for example to the point $Q$ and the second leaves from the homologous point $Q'$ (Fig. 4). As regards the other straight lines, they are also unlimited In the two directions; Figure 5 represents one, brought back entirely into the interior of the fundamental polygon. We see *that it intersects itself* in an infinity of points; if we follow it starting from the point $P_0$, the first point where

**Figure 5**



it intersects itself is between $P'_4$ and $P_5$; they succeed each other regularly, so that from the point of view of the *Analysis situs* the straight line considered has the form represented by Figure 6.

**Figure 6**



We see from this example that, in certain locally Euclidean Riemannian spaces, *direction is not absolute*; a vector whose origin moves and which remains step by step parallel to itself can return with a direction different from its initial directlon when the origin has described a cycle. In the space considered, there are two absolute or *stable* directions, the direction parallel to the $x$-axis and the perpendicular direction.

**75.** The two classes of locally Euclidean closed Riemannian spaces both have a parallelogram as fundamental polygon. In the two cases we verify easily that there are *two* sides (not homologous to each other) and *only one* vertex (since the four vertices are all homologous to each other). Denoting by $A$ the number of sides and $S$ the number of essentially distinct vertices, we therefore have

$$A = S + 1.$$

We can link this relation to a much more general property. Decompose in any way a locally Euclidean space of two dimensions and *closed* (and hence of finite total area) into a certain number of sufficiently small polygons so that we can apply to them the theorems of plane geometry. Let $F$ be the number of these polygons, $A$ the number of their distinct sides and $S$ that of their vertices. The sum of the angles of a polygon with $n$ sides is

$$\pi(n - 2);$$

consequently the total sum of all the angles of all the polygons is

$$\pi(2A - 2F);$$

on the other hand, around each vertex are distributed angles whose sum is equal to $2\pi$. We thus have the relation

$$\pi(2A - 2F) = 2\pi S$$

or

$$F + S = A.$$

According to a theorem from the *Analysis situs*, if this relation Is verified for one particular decomposition, then it is verified for any other decomposition, in particular when we regard the total space as forming a single polygon: we have then $F = 1$ and $A = S + 1$.

We recover in this way the theorem previously proved that the surface of a sphere cannot be given an Euclidean metric that is everywhere regular; in fact, the decomposition of the surface of the sphere into polygons gives rise to a classical formula of Euler

$$F + S = A + 2.$$

**76.** The determination of the discontinuous groups $G$ of the space of three dimensions can be made without too many difficulties. If the Riemannian space is orientable, its group $G$ can contain only translations and helical displacements. Apart from the case where $G$ is generated by a single helical displacement

$$\begin{aligned} x' &= x \cos n\alpha - y \sin n\alpha, \\ y' &= x \sin n\alpha + y \cos n\alpha, \\ z' &= z + nh, \end{aligned}$$

where $n$ Is an arbitrary Integer, $h$ a length and $\alpha$ a constant angle, the helical displacements which feature in the group $G$ correspond to an angle of rotation equal to $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$ or $\pi$. The fundamental domain, in the first case indicated, where the angle $\alpha$ can be incommensurable with $\pi$, is, for example, the volume of the space contained between the two planes $z = 0$ and $z = h$.

## VIII.– Locally Euclidean normal Riemannian spaces and elementary geometry.

**77.** In any sufficiently small region of a locally Euclidean, normal Riemannian space, all the theorems of elementary geometry are valid, and it is impossible, if we do not exit this region, to establish that we are not In an Euclidean space. Certain axioms of elementary geometry that use properties of the space, taken as a whole, remain true, but the others cease to be so.

Amongst the former, we point out this one: *Through any two points of the space, there passes one straight line.* In representing a locally Euclidean Riemannian space by means of Its fundamental polyhedron, the axiom becomes obvious.

On the contrary, the axiom according to which through any two points, there passes only one straight line is true only in Euclidean space. In fact, let $P$ and $Q$ be two points of the fundamental polyhedron of a locally Euclidean, normal Riemannian space; let us figure out the homologues, infinite in number, of $Q$; let them be $Q_1, Q_2, ..., Q_n, ...$. The lines $PQ, PQ_1, PQ_2, ..., PQ_n, ...$ of the Euclidean space, brought back if we want to the interior of the fundamental polyhedron, provide an infinity of lines joining the two points of the Riemannian space that correspond to $P$ and to $Q$.

We see that the axiom according to which *through any two points there pass only a finite number of lines* would be sufficient to characterise the Euclidean space among all locally Euclidean Riemannian spaces.

**78.** We could take, with W. Killing, another point of view. In the Euclidean space, a motion impressed on a solid body transmits itself to all solid bodies invariably tied to the first or, if you want, to the whole space. In a locally Euclidean Riemannian space, solid bodies of sufficiently small dimensions admit the same degree of mobility (without deformation) as in Euclidean space. It is no longer the same if the solid body is too extended, more precisely, if it is impossible to keep its *development* onto the Euclidean space inside a single fundamental polyhedron. The Euclidean motion impressed on a neighbourhood of a point $P_0$ gives, in the Euclidean space, to the neighbourhood of a point $P_1$ homologous to $P_0$ a motion different in general from the motion considered; the velocity of $P_1$ for example will not be placed with respect to the frame of reference $(R_1)$ at $P_1$ as the velocity of $P_0$ with respect to the frame of reference $(R_0)$. We can say that if we consider a *closed* chain of solid bodies each of which is secured to the preceding one, a motion impressed on the first body and which transmits itself from one to the other could return to impress on the first body a motion completely different from the initial motion. In other words, the locally Euclidean Riemannian space does not admit the complete mobility of the Euclidean space.

If we want that an overall motion $T$ of the whole space to be possible, it is necessary and sufficient that the displacements $P_iQ_i$ undergone by the different points $P_i$ homologous to a point $P_0$ all be placed in the same way with respect to the corresponding frames of reference $(R_i)$. The succession of displacements $S_i$ and $T$, which take $P_0$ to $Q_i$ via $P_i$, must thus be equivalent to the succession of the same displacements, but taken in inverse order, $T$ and $S_i$, which likewise take $P_0$ to $Q_i$ via $Q_0$.

*The search for the overall motions that the Riemannian space is likely to take thus comes back to the search for the motions of the Euclidean space that commute with each of the operations of the holonomy group.*

We can show easily that if a displacement, accompanied or not by a symmetry, commutes with all the translations, it is itself a translation. From this it follows that there does not exist any displacement (accompanied or not by a symmetry) which commutes with all the displacements properly so called. We arrive therefore at the following theorem:

*If a locally Euclidean, normal Riemannian space has the perfect mobility of the Euclidean space, it merges with Euclidean space.*

Clifford's space of two dimensions admits only translations, while the cylinder admits translations and also symmetries with respect to the closed lines that it contains.

# 4 Riemannian Spaces. Tangent and Osculating Euclidean Spaces.

## I. — Tangent Euclidean space at a point.[1,2]

**79.** Consider a Riemannian space defined as a manifold with any line element

$$ds^2 = g_{ij}\ du^i\ du^j; \tag{4.1}$$

we assume that the right hand side is a positive definite differential form whose coefficients are continuous and have continuous first order partial derivatives. Some considerations in this section do not even require the existence of derivatives.

The simplest way to investigate the geometric properties of this space is to identify it *insofar as possible* with Euclidean space. A first step in this direction is the introduction of the concept of the *Euclidean metric tangent to the given metric at a point*, or of *the Euclidean space tangent to the given space at a point.*

The metric defined by an Euclidean line element

$$d\sigma^2 = \gamma_{ij}\ du^i\ du^j, \tag{4.2}$$

constructed with the variables $u^1, ..., u^n$, and such that for $u^i = u^i_0$ we have $\gamma_{ij} = g_{ij}$, is called the *Euclidean metric* tangent to the given metric (4.1) at the point $A(u^1_0, ..., u^n_0)$. There are clearly an infinite number of tangent Euclidean metrics at a given point: for example, it is sufficient to take $\gamma_{ij} = (g_{ij})_0$. On the other hand, the set of these metrics is independent of the choice of coordinates which define the Riemannian space analytically, because if we replace the variables $u^i$ by other given variables $v^i$, the new values of the coefficients of the line element at $A$ depend only on their old values at $A$. The equality $\gamma_{ij} = g_{ij}$ thus remains valid at $A$ with the new variables.

Instead of saying that we have given the manifold considered a new (Euclidean) metric, we can say that we have represented the Riemannian space on an Euclidean space in a such a way that, in this representation, the line element of the Euclidean space becomes $d\sigma^2$. This Euclidean space will be called the *Euclidean space tangent to the given Riemannian space at A.* This is only a linguistic convention, helpful because of the image it creates. We could say that there are

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[1] Fr. *l'espace euclidien tangent en un point.*

[2] TRANSLATOR'S NOTE: This phrase could be translated as either *the tangent Euclidean space at a point* or *the Euclidean space tangent at a point.* I have opted for the former except when the context favours the latter.

an infinite number of tangent Euclidean spaces at $A$, in the sense that the line element $d\sigma^2$ depends on an infinite number of arbitrary elements. But since in what follows we will consider only the geometric properties common to all these Euclidean spaces, we can speak not inappropriately[3] of *the tangent Euclidean space at* $A$.

**80.** A first geometric concept that follows from consideration of the tangent Euclidian space is that of the distance of a point $A$ to an infinitely close point, which distance is equal to $d\sigma$ or also to $ds$: after all, it is this very notion which is the basis for the definition of a Riemannian space.

*The angle between two directions* $d$ *and* $\delta$ issuing from $A$ is given in the tangent Euclidian space (n° 29) by the formula

$$\cos\varphi = \frac{\gamma_{ij}\ du^i\ \delta u^j}{\sqrt{\gamma_{ij}\ du^i\ du^j}\ \sqrt{\gamma_{ij}\ \delta u^i\ \delta u^j}};$$

in any Riemannian space, therefore, we can define the cosine of the angle between two directions by the formula

$$\cos\varphi = \frac{\gamma_{ij}\ du^i\ \delta u^j}{\sqrt{\gamma_{ij}\ du^i\ du^j}\ \sqrt{\gamma_{ij}\ \delta u^i\ \delta u^j}}; \tag{4.3}$$

*we can be certain in advance that the right-hand side is independent of the choice of coordinates.*

Similarly, we can define the angle between a plane element of $p$ dimensions and of a plane element of $q$ dimensions, and all the theorems of elementary geometry relating to angles, *all with the same vertex*, between lines, surfaces, etc., that pass through this common vertex, remain valid in any Riemann space.

**81.** At point $A$ of the tangent Euclidean space, we can define a *vector* with contravariant components $X^i$ (or covariant components $X_i$) relative to the natural system of reference attached at $A$ (which frame is well defined in size and shape). We can thus define contravariant or covariant vectors at each point of a Riemannian space. A very simple way to represent a vector is to imagine a moving point and the *velocity* of this point: this is the vector whose contravariant components are $\dfrac{du^i}{dt}$; it has an intrinsic existence in the tangent Euclidean space independent of the choice of coordinates.

We can define similarly bivectors, multivectors, and generally any tensor, at a point $A$ of the Riemannian space.

But the concept of the tangent Euclidean space allows us to introduce into the geometry of Riemannian spaces concepts involving an entire line, an entire surface, an entire volume of these spaces.

First, since we know the elementary distance between two points, we can

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[3] Fr. *sans inconvénient.*

deduce by addition (integration) the length $s$ of an arc of any curve[4]

$$s = \int \sqrt{g_{ij} du^i du^j}\ ;$$

from this we could immediately deduce, as Riemann does, the concept of a *straight line*, or *geodesic*, as a line that realises the extremum of the distance. Here, we only point out this possibility; we will return to it later.

Taking into account the expression for the elementary volume in the tangent Euclidean space, the elementary volume of the space is given by

$$d\tau = \sqrt{g}\ du^1 du^2 \dots du^n; \tag{4.4}$$

it follows that a finite volume is defined by the integral

$$\int \sqrt{g}\ du^1 du^2 \dots du^n.$$

We can also measure the area of a portion of surface. In three dimensional space, if the coordinates of a point on a surface are defined as functions of two parameters $\alpha$ and $\beta$, an element of surface at $A$ is a bivector whose contravariant components are

$$\begin{aligned} du^2 du^3 &= \frac{D(u^2, u^3)}{D(\alpha, \beta)}\ d\alpha\, d\beta = a^{23}\ d\alpha\, d\beta, \\ du^3 du^1 &= \frac{D(u^3, u^1)}{D(\alpha, \beta)}\ d\alpha\, d\beta = a^{31}\ d\alpha\, d\beta, \\ du^1 du^2 &= \frac{D(u^1, u^2)}{D(\alpha, \beta)}\ d\alpha\, d\beta = a^{12}\ d\alpha\, d\beta; \end{aligned}$$

the measure of this element of surface is thus

$$\sqrt{a^{23} a_{23} + a^{31} a_{31} + a^{12} a_{12}}\ d\alpha\, d\beta.$$

For a finite portion of surface, we deduce the expression for the area

$$\iint \sqrt{a^{23} a_{23} + a^{31} a_{31} + a^{12} a_{12}}\ d\alpha\, d\beta.$$

**82.** Given a vector field in Riemannian space, we do not yet know how to define the tensor *derived* from this field. Nevertheless, we can easily define its *curl* by means of the Pfaffian expression

$$X_1\ du^1 + X_2\ du^2 + \cdots + X_n\ du^n,$$

which, in the tangent Euclidean space at each point, has an intrinsic meaning. The bilinear covariant

$$\frac{1}{2}\left(\frac{\partial X_j}{\partial u^i} - \frac{\partial X_i}{\partial u^j}\right)(du^i \delta u^j - du^j \delta u^i)$$

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[4] We have already implicitly used this concept when defining in n° 55 the distance $[AB]$ between two points $A$ and $B$ of a Riemannian space.

leads immediately to the tensor

$$X_{ji} = \frac{\partial X_j}{\partial u^i} - \frac{\partial X_i}{\partial u^j},$$

which is the curl.

Consideration of the *elementary flow of a vector*, that is

$$\sqrt{g}\left(X^1\ du^2 du^3 + X^2\ du^3 du^1 + X^3\ du^1 du^2\right),$$

which also involves only the tangent Euclidean space, leads by application of Ostrogradsky's formula to the concept of divergence (n° 41):

$$\operatorname{div} \mathbf{X} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u^1}(\sqrt{g}\ X^1) + \frac{\partial}{\partial u^2}(\sqrt{g}\ X^2) + \frac{\partial}{\partial u^3}(\sqrt{g}\ X^3)\right]. \tag{4.5}$$

Similarly, given a scalar field $V(u^1, \ldots, u^n)$, we easily define the two Beltrami differential parameters (n° 42)

$$\Delta_1 V = g^{ij}\ \frac{\partial V}{\partial u^i}\frac{\partial V}{\partial u^j}, \tag{4.6}$$

$$\Delta_2 V = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial u^i}\left(\sqrt{g}\ g^{ik}\ \frac{\partial V}{\partial u^k}\right). \tag{4.7}$$

**83.** In spite of the large number of concepts of Euclidean origin that we have been able to generalise to any Riemannian space, there are fundamental elementary concepts that are still missing, for example that of the angle between two directions issuing from two different points. Generally, any geometric concept which involves a *scalar* at each point is easily generalised; the same is true of a concept which involves one or more vectors, *provided they all have the same origin.* The divergence of a vector field seems to be an exception, but only because we were able to deduce it from the concept of the elementary flow of a vector, *which in fact only involves the field at a single point.* In short, up to now the Riemannian space has been for us a collection of small morsels of Euclidian space, but to a certain extent it remains amorphous because we have not yet related these various morsels to one another by defining their mutual orientation. This is what we will now accomplish with the concept of the *osculating Euclidian space.*

## II. — The osculating Euclidean space.

**84.** An osculating Euclidean metric at $\mathrm{A}(u_0^i)$ to the given metric is defined by a line element

$$d\sigma^2 = \gamma_{ij}\ du^i du^j$$

such that its coefficients, *as well as their partial derivatives of first order*, have the same numerical values at $A$ as the given line element.

If there are osculating Euclidean metrics, the set of these metrics is independent of the choice of coordinates, because, *for a given change of variables*, the new numerical values of the $g_{ij}$, and of their partial derivatives of first order, are known as soon as we know the old numerical values of these same quantities.

Instead of speaking of an osculating Euclidean metric, we can speak about *an osculating Euclidean space.*

First, we must prove the existence of osculating Euclidean spaces at a given point $A$ of the Riemannian space. Now, if such a space exists, the coefficients ${\Gamma_i}^k{}_j$ which allow us to position the natural system of reference attached to the point $(u^i + du^i)$ with respect to the natural system of reference at the point $u^i$ are given by the formulae established in Chapter II (n° 33). It is thus sufficient to show that we can always find coordinates $(u^i)$ in the Euclidean space such that, at a point $O$ of this space, the $g_{ij}$ and the ${\Gamma_i}^k{}_j$ have given numerical values (knowledge of the $g_{ij}$ and the $\dfrac{\partial g_{ij}}{\partial x^k}$ lead to those of the ${\Gamma_i}^k{}_j$ and conversely). Now the formulae of Euclidean geometry

$$\begin{aligned} dM &= du^k e_k, \\ de_i &= {\Gamma_i}^k{}_r \, du^r \; e_k, \end{aligned}$$

give

$$\frac{\partial M}{\partial u^i} = e_i, \quad \frac{\partial^2 M}{\partial u^i \partial u^j} = {\Gamma_i}^k{}_j \; e_k.$$

That said, take at point $O$, in the Euclidean space, a Cartesian system of reference $(e_1, ..., e_n)$ defined by the numerical values at $A$ of the coefficients $g_{ij}$, and determine the coordinates $x^i$ (with respect to this system of reference for example) of a variable point $M$ in this space, by the condition that for $u^i = u^i_0$, we have

$$\frac{\partial M}{\partial u^i} = e_i, \quad \frac{\partial^2 M}{\partial u^i \partial u^j} = \left({\Gamma_i}^k{}_j\right)_0 e_k.$$

*This is possible* in an infinite number of ways; *for example* it is sufficient to take

$$x^i = u^i - u^i_0 + \frac{1}{2}\left({\Gamma_r}^i{}_s\right)_0 (u^r - u^r_0)(u^s - u^s_0) \qquad (i = 1, 2, ..., n);$$

the line element of the Euclidean space, referred to this system of curvilinear coordinates, will obviously have the property that, for $u^i = u^i_0$, its coefficients and their partial derivatives of first order will have the same numerical values as the line element of the given Riemannian space. This is what we needed to prove.

**85.** All geometric properties common to the various osculating Euclidean spaces at a point will obviously be intrinsic geometric properties of the Riemannian space. This will be so for all those which involve only the numerical values at $A$ of the $g_{ij}$ and the ${\Gamma_i}^k{}_j$.

Imagine a fixed point $A$ in the osculating Euclidean space, and also the natural system of reference $R_0$ which is attached to it. Any point $M$ of the Riemannian space is represented by a point $\overline{M}$ whose coordinates with respect to $R_0$ are well defined, up to infinitesimals of order *greater than the second.* The natural system of reference attached to the point $\overline{M}$ in the Euclidean space is, as regards size and orientation, well defined up to infinitesimals of order *greater than the first*; furthermore, it is equal to the natural system of reference attached to the point $M$ in the Riemannian space, also up to infinitesimals of order higher than the first. Any vector of the Riemannian space with origin $M$ is represented to the same degree of approximation by an equal vector. In general, if $M$ and $N$ are any two points (close to $A$) of the Riemannian space, and if their representative points $\overline{M}$ and $\overline{N}$ in the osculating Euclidean space at $A$ are in the interior of a sphere with centre $A$ and radius $r$, the scalar product of the representative vectors of the two vectors with origin $M$ and $N$, is well defined up to a quantity that is infinitely small with respect to $r$.

A vector $\boldsymbol{x}$ with origin $A$ and a vector $\boldsymbol{x}'$ with infinitely close origin $A'$ in the Riemannian space are represented in the osculating Euclidean space by two vectors whose *geometric difference* is, according to what precedes, well defined up to an infinitesimal of order greater than the first. This leads us to the extension of the concept of an *absolute* or *covariant differential* of a vector or, more generally, of a tensor, to any Riemannian space; we will continue to denote this differential by the symbol $D$.

We will thus have, by definition,

$$DX^i = dX^i + \omega_k{}^i X^k, \tag{4.8}$$

$$DX_i = dX_i - \omega_i{}^k X_k. \tag{4.9}$$

However, it is important to note here that the absolute differential in Euclidean geometry was a *true* differential, whereas it is no longer certain that it is so in Riemannian geometry.

Denoting the coordinate vectors by $e_i$, we have in particular

$$De_i = e_k \omega_i{}^k$$

Two vectors with infinitely close origins $A$ and $A'$ will be said to be *equal*[5,6] if they are equal in the osculating Euclidean space at $A$ or, which comes down to the same thing, if the absolute differential of the first vector (when we pass from this first vector to the second) is zero. The conditions for equality are thus

$$dX^i + \omega_k{}^i X^k = 0, \tag{4.10}$$

$$dX_i - \omega_i{}^k X_k = 0. \tag{4.11}$$

To *parallel-transport*[7,8] a vector $\boldsymbol{x}$ with origin $A$ to an infinitely close point $A'$

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[5] French: *équipollent.*

[6] Translator's note: *équipollent* means *have the same value*, or *equi-valent.* Thus "equipollent vectors" are equal vectors, or vectors that have the same length and the same direction.

[7] French: *transporter par équipollence.*

[8] Translator's note: The phrase *transporter par équipollence* expresses a stronger concept than the English phrase *parallel transport.* The latter term only reflects the fact that the

is to construct the vector $\boldsymbol{x}'$ with origin $A'$ equal to $\boldsymbol{x}$. It is clear that this vector is defined only up to infinitesimals of order greater than one. In the osculating Euclidean space at $A$, this operation becomes a simple translation from $A$ to $A'$.

Parallel transport has some important and obvious properties: *if two vectors with common origin $A$ are parallel-transported from $A$ to $A'$, their scalar product does not change.*

Should we want to verify this theorem analytically, it would be sufficient to note that the calculation of the differential of the scalar product involves only the numerical values at $A$ of the $g_{ij}$ and of their first order partial derivatives; the proof would then be the same as if the space were Euclidean, in which case the theorem is obvious.

The absolute derivative applies to any tensor field. In particular, the absolute differential of the fundamental tensor $g_{ij}$ is identically zero.

Finally, we define derived tensors as in n° 41.

**86.** The definition of the acceleration of a moving point in a Riemannian space now encounters no difficulties; if $\boldsymbol{v}$ is the velocity vector, the acceleration vector is $\dfrac{D\boldsymbol{v}}{dt}$. We define accelerations of different orders in the same way. The contravariant components of the acceleration are, as in Euclidean geometry,

$$\gamma^i = \frac{d^2u^i}{dt^2} + \Gamma_{k\ h}^{\ i} \frac{du^k}{dt}\frac{du^h}{dt},$$

and its covariant components can again be calculated by Lagrange's algorithm.

A moving point whose acceleration is constantly zero has its velocity constantly equal to itself. Generally, a *straight line* of the Riemannian space will be a line whose tangent remains parallel to itself from point to point; if such a line is traversed by a moving point with speed equal to 1, the acceleration of this moving point will be zero. The equations

$$\frac{d^2u^i}{dt^2} + \Gamma_{k\ h}^{\ i} \frac{du^k}{dt}\frac{du^h}{dt} = 0 \tag{4.12}$$

thus define the *straight lines* of Riemannian spaces.

We will see later (n° 95) that the straight lines of Riemannian space are the lines of shortest distance; they are thus identical with Riemann's geodesics.

The equations of the dynamics of a point generalise immediately from Euclidean space to any Riemannian space; they can be written in one of the two forms,

$$\gamma^i = F^i,$$
$$\gamma_i = F_i,$$

direction of the vector does not change from point to point, whereas the term *Transporter par équipollence* stresses the fact that vectors change neither in length nor in direction as they are moved from point to point of the manifold. However, these two terms are clearly synonymous, so I have chosen to translate *Transporter par équipollence* as *parallel transport.*

depending on whether we use the contravariant or covariant components of the force.

It is remarkable that the general equations of the dynamics of systems (holonomic and with ideal constraints) reduce to equations of the dynamics of a point by regarding the set of positions of a material system as a manifold in which each element (or point) is one of the positions of the system. By adopting the product of the *vis viva* of the system by the square of $dt$ as the line element in this manifold, the covariant components of the acceleration of a point of the manifold are (n° 37)

$$\gamma_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{u}^i} - \frac{\partial T}{\partial u^i}. \tag{4.13}$$

According to the equations of Lagrange, the most general motions of the system are given by the equations

$$\gamma_i = Q_i\,,$$

where $Q_i\,\delta u^i$ denotes the sum of the elementary work of the given forces. *There is thus a perfect correspondence between the motions of the given system, assumed to have $n$ degrees of freedom, and the motions of a point in a Riemannian space of $n$ dimensions*; the correspondence conserves the *vis viva* as well as the sum of the elementary work of the forces.

**87.** The theory of the curvature of curves generalises without modification from the Euclidean space to any Riemannian space[9]. In fact, let $M$ be a point on the curve, $\boldsymbol{t}$ the unit vector tangent to the curve at $M$, and $ds$ the element of arc of the curve. The vector $\frac{D\boldsymbol{t}}{ds}$ is normal to $\boldsymbol{t}$ (we see this immediately by using the osculating Euclidean space at $M$); let

$$\frac{D\boldsymbol{t}}{ds} = \frac{1}{\rho}\,\boldsymbol{n},$$

where $\boldsymbol{n}$ denotes an unit vector (principal normal). Let $\boldsymbol{b}$ be the unit vector perpendicular to $\boldsymbol{t}$ and $\boldsymbol{n}$, and forming with $\boldsymbol{t}$ and $\boldsymbol{n}$ a right-handed trihedron, and let

$$\begin{aligned}\frac{D\boldsymbol{n}}{ds} &= \alpha\boldsymbol{t} + \beta\boldsymbol{n} + \gamma\boldsymbol{b},\\ \frac{D\boldsymbol{b}}{ds} &= \alpha'\boldsymbol{t} + \beta'\boldsymbol{n} + \gamma'\boldsymbol{b}.\end{aligned}$$

The relations

$$\boldsymbol{t}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{t}\cdot\boldsymbol{b} = 0, \quad \boldsymbol{n}\cdot\boldsymbol{b} = 0, \quad \boldsymbol{n}^2 = 1, \quad \boldsymbol{b}^2 = 1$$

---

[9] For the properties of curves and surfaces in ordinary space, see *Elèments de Gèométrie infinitèsimale* of G. JULIA (Paris, Gauthiers-Villars, 1927).

give, by absolute differentiation,

$$\frac{1}{\rho} + \alpha = 0, \quad \alpha' = 0, \quad \gamma + \beta' = 0, \quad \beta = 0, \quad \gamma' = 0.$$

Then, by putting

$$\gamma = -\beta' = \frac{1}{\tau}$$

we obtain the generalised Frenet formulae,

$$\left.\begin{aligned} \frac{D\boldsymbol{t}}{ds} &= \frac{1}{\rho}\,\boldsymbol{n}, \\ \frac{D\boldsymbol{n}}{ds} &= -\frac{1}{\rho}\,\boldsymbol{t} + \frac{1}{\tau}\,\boldsymbol{b}, \\ \frac{D\boldsymbol{b}}{ds} &= -\frac{1}{\tau}. \end{aligned}\right\} \tag{4.14}$$

The quantities $\frac{1}{\rho}$ and $\frac{1}{\tau}$ are the *curvature* and the *torsion* of the curve.

The straight lines of the Riemannian space are the lines of zero curvature. As for the lines of zero torsion, we can characterise them by the property that the osculating plane element at a point $M$ of the curve is *parallel* to the osculating plane element at an infinitely close point $M'$; the vectors $\boldsymbol{t} + D\boldsymbol{t}, \boldsymbol{n} + D\boldsymbol{n}$, which are the unit vectors on the tangent and the principal normal at $M'$, are in fact parallel transported from $M'$ to $M$,

$$\boldsymbol{t} + \frac{ds}{\rho}\,\boldsymbol{n}, \quad \boldsymbol{n} - \frac{ds}{\rho}\,\boldsymbol{t} + \frac{ds}{\tau}\boldsymbol{b}\,;$$

they are in the osculating plane element at $M$ if the torsion is zero, and in this case only.

If we represent the Riemannian space on the osculating Euclidean space at $M$, the given curve $(C)$ has as image a certain curve $(\Gamma)$ *which has the same curvature at $M$ as $(C)$.* Moreover, the vectors $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ of $(\Gamma)$ are the images of the analogous vectors of $(C)$.

Things are even clearer if we represent them in the following way. *Whether we adopt in Riemannian space the given metric or any osculating Euclidean metric at $M$, the curve $(C)$ has at $M$ the same tangent, the same principal normal, the same binormal and the same curvature; but it does not necessarily have the same torsion with the two metrics.*

Specifically, any straight line in Riemannian space has a point of inflection at $M$ for any observer who adopts the Euclidean metric osculating at $M$.

**88.** The classical theory of the curvature of surfaces generalises, according to the foregoing, with the greatest of ease to Riemannian spaces. The various curves drawn on a surface $(S)$ and passing through a given point $M$ on this surface have the same principal normal and the same curvature, whether we adopt the metric of the Riemannian space or the osculating Euclidean metric at $M$. It follows that the laws which govern the variation of the curvature of these curves, as we vary their tangent around $M$, are the same as in Euclidean space. All curves tangent to each other have the same *normal curvature*; this normal curvature is equal to $\dfrac{\cos V}{\rho}$, where $V$ denotes the angle of the principal normal of the curve with the normal to the surface, and $\rho$ denotes the radius of curvature of the curve; *this is Meusnier's theorem.*

There are two orthogonal directions tangent to the surface that correspond to the maximum and the minimum of the normal curvature: these are the principal directions, where the corresponding normal curvatures are the principal curvatures. *Lines of curvature* are lines which, at each of their points, are tangent to one of the principal directions at that point. If $\theta$ denotes the angle that a curve makes with the line of curvature of the first family, we have for this curve

$$\frac{\cos V}{\rho} = \frac{\cos^2 \theta}{R_1} + \frac{\sin^2 V}{R_2}, \tag{4.15}$$

where we denote by $\dfrac{1}{R_1}$ and $\dfrac{1}{R_2}$ the two principal curvatures.

Asymptotic lines are those whose normal curvature is zero; they are characterised by the property that their osculating plane element is tangent to the surface. We can also characterise them by the condition that the straight line in Riemannian space tangent at $M$ to an asymptotic line has a second order contact with the surface at this point; this is the same as saying that, in Euclidean space, a line tangent to a surface at a point $M$ and having a point of inflexion at $M$ has a second order contact with this surface when it is tangent at $M$ to one of the asymptotic tangents of the surface.

We can also characterise the lines of curvature by the condition that, given any point $M$ of the line, the normal to the surface at a point $M'$ on the line infinitely close to $M$, once parallel-transported from $M'$ to $M$, is in the same plane element as the normal to the surface at $M$ and the tangent to the line at $M$.

If we denote by $\boldsymbol{v}$ the unit vector normal to the surface, by $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ the unit vectors tangent to the surface in the direction of the lines of curvature, we have, by moving in the direction of the vector $\boldsymbol{t}_1$ a distance $ds_1$,

$$D\boldsymbol{v} = -\frac{ds_1}{R_1}\, \boldsymbol{t}_1,$$

and by moving in the direction of the vector $\boldsymbol{t}_2$ by the distance $ds_2$,

$$D\boldsymbol{v} = -\frac{ds_2}{R_2}\, \boldsymbol{t}_2.$$

These formulae are the classic formulae of Olinde Rodrigues. If we move by distance $ds$, in a direction making angle $\theta$ with $\boldsymbol{t}_1$, we have in general

$$\frac{D\boldsymbol{v}}{ds} = -\frac{\cos\theta}{R_1}\, \boldsymbol{t}_1 - \frac{\sin\theta}{R_2}\, \boldsymbol{t}_2\,. \tag{4.16}$$

Finally, the total curvature $\dfrac{1}{R_1R_2}$ of the surface at a point $M$ can be defined by the method used by Gauss in Euclidean space. Consider a surface element $d\sigma$ surrounding a point $M$ and, in the Euclidean space tangent at $M$, the sphere with centre $M$ and radius 1. Parallel-transport in $M$ the normals to the surface at various points of the element $d\sigma$, and consider the small region of the sphere determined by the tracks of the normals thus transported; the ratio $\dfrac{d\omega}{ds}$ of the area of this portion of the sphere to the given area is equal, in the limit, to the total curvature $\dfrac{1}{R_1R_2}$. We can also regard $d\omega$ as the solid angle of the cone with vertex $M$ obtained by transporting to $M$ by parallelism the normals at different points of the element $d\sigma$.

**89.** We can generalise to the most general Riemannian spaces the concept of *geodesic torsion* of a curve drawn on a surface. In fact, denote by $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ the unit vectors on the tangent, the principal normal and the binormal to the curve, and continue to denote by $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and $\boldsymbol{v}$ the unit vectors on the principal tangents and the normal to the surface. Finally, let $\theta$ be the angle between the tangent to the curve and the first principal tangent, and $V$ the angle between the principal normal to the curve and the normal to the surface. The direction cosines of the vectors $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ with respect to the vectors $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{v}$ are given in the following table

| | $\boldsymbol{t}_1$ | $\boldsymbol{t}_2$ | $\boldsymbol{v}$ |
|---|---|---|---|
| $\boldsymbol{t}$ | $\cos\theta$ | $\sin\theta$ | $0$ |
| $\boldsymbol{n}$ | $-\sin\theta\sin V$ | $\cos\theta\sin V$ | $\cos V$ |
| $\boldsymbol{b}$ | $\sin\theta\cos V$ | $-\cos\theta\cos V$ | $\sin V$ |

Start with the generalised Frenet formulae,

$$\left.\begin{aligned} \frac{D\boldsymbol{t}}{ds} &= \frac{1}{\rho}\,\boldsymbol{n}, \\ \frac{D\boldsymbol{n}}{ds} &= -\frac{1}{\rho}\,\boldsymbol{t} + \frac{1}{\tau}\,\boldsymbol{b} \\ \frac{D\boldsymbol{b}}{ds} &= -\frac{1}{\tau} \end{aligned}\right\}$$

and the formula

$$\frac{Dv}{ds} = -\frac{\cos\theta}{R_1}\,t_1 - \frac{\sin\theta}{R_2}\,t_2$$

If we differentiate the relations

$$\boldsymbol{t}\cdot\boldsymbol{v} = 0, \qquad \boldsymbol{n}\cdot\boldsymbol{v} = \cos V, \qquad \boldsymbol{b}\cdot\boldsymbol{v} = \sin V,$$

we obtain two distinct equations, namely

$$\frac{\cos V}{\rho} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2},$$
$$\frac{dV}{ds} + \frac{1}{\tau} = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\sin\theta\cos\theta. \tag{4.17}$$

The second of these equations contains on the left hand side that which we call the *geodesic torsion* of the curve; the form of the right hand side shows that it is the same for all curves that have the same tangent. It is zero for lines of curvature.

Applied to the case of an asymptotic line different from a straight line ($V = \pi/2$), formula (4.17) gives

$$\frac{1}{\tau} = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\sin\theta\cos\theta;$$

now, for an asymptotic line, we have

$$\frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2} = 0;$$

we deduce

$$\frac{1}{\tau} = \pm\sqrt{\frac{-1}{R_1R_2}}\ ; \tag{4.18}$$

in the case of any Riemannian space, this is the *Beltrami-Enneper theorem.*

*The asymptotic lines issuing from a point $M$ on a surface have equal and opposite torsions at this point, whose absolute value is equal to the square root of the total curvature of the surface with sign changed.*

In particular, the torsion of a *double* asymptotic line is constantly zero.

We know that, in Euclidean space, if the two families of asymptotics of a surface coincide, the asymptotic lines are straight lines. This theorem is no longer true in any Riemannian space;[10] nevertheless we can affirm from Enneper's theorem, that these are curves of zero torsion.

**90.** The theory of *conjugate tangents* generalises to any Riemannian space. If $M$ and $M'$ are two infinitely close points on a curve $(C)$ drawn on a surface $(S)$, the plane element tangent to the surface at $M'$, transported by parallelism to $M$, has in common with the plane element tangent at $M$ a certain direction *conjugate* to the direction of the curve $(C)$ at $M$. Two conjugate directions are harmonic conjugates with respect to the asymptotic directions. Finally the absolute differential of the unit vector normal to the surface, when we move from $M$ to $M'$, is perpendicular to the conjugate direction of the curve. All these properties become obvious when representated on the osculating Euclidean space at $M$.

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[10] E. Cartan, *Sur le courbes de torsion nulle et les surfaces dèveloppable dans les espaces de Riemann* (*Comptes rendus*, t. 184, 1927, p. 138-141).

**91.** We finish by generalising a famous theorem of Dupin. We know that the system formed by three one-parameter families of surfaces which intersect at a right angles is called a triply orthogonal system. The theorem of Dupin states that the curve of intersection of two surfaces belonging to two different families is a line of curvature for each of these surfaces.

The analytic proof that we will give applies equally well to Riemannian spaces and to Euclidean space. Take as coordinates the parameters $u^1, u^2, u^3$ of the surfaces of the three families. The line element of the space will then be of the form

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2,$$

since the cosine of the angle at which two different coordinate curves intersect is zero. The condition that expresses for example that the coordinate line $(C_1)$ ($u^1$ variable) is a line of curvature for the surface $(S_3)$ ($u^3$ constant), is that the vector $\boldsymbol{e}_3 + D_1\boldsymbol{e}_3\, du^1$, normal to this surface at the point $M'$ infinitely close to $M$ on the line $(C_1)$, is in the same plane element with the two vectors $\boldsymbol{e}_3$ and $\boldsymbol{e}_1$, in other words, that the coefficient of $\boldsymbol{e}_2$ in the expression for $D_1\boldsymbol{e}_3$ is zero. Now, we have

$$\begin{aligned} D\boldsymbol{e}_3 &= \omega_3^1\boldsymbol{e}_1 + \omega_3^2\boldsymbol{e}_2 + \omega_3^3\boldsymbol{e}_3, \\ D_1\boldsymbol{e}_3 &= {\Gamma_3}^1{}_1\boldsymbol{e}_1 + {\Gamma_3}^2{}_1\boldsymbol{e}_2 + {\Gamma_3}^3{}_1\boldsymbol{e}_3. \end{aligned}$$

It is thus necessary to prove that we have ${\Gamma_3}^2{}_1 = 0$; or, since

$$\Gamma_{321} = g_{22}\,{\Gamma_3}^2{}_1,$$

it is necessary to prove that the Christoffel symbol $\begin{bmatrix} 3\,1 \\ 2 \end{bmatrix}$ is zero. Now, this is obvious since $g_{23} = g_{13} = g_{12} = 0$.

### III. — Euclidean space of connection along a curve.[11]

**92.** We can obtain new geometric properties and new theorems by *developing*a line of the Riemannian space onto Euclidean space.

Start from a point $A_0$ of the line; suppose the coordinates of a point on this line are expressed as a function of a parameter $t$ that is zero at $A_0$. In Euclidean space, take a point of departure $O$ to which we attach a Cartesian system of reference $(R_0)$ determined in size and in shape by the numerical values of the $g_{ij}$ at the To each point $M$ of parameter $t$ on the line assign a point $M'$ in Euclidean space and a Cartesian system of reference $(e'_1, ..., e'_n)$ attached to this point. Start, as we have already done (n° 46), from the differential equations

$$\left.\begin{aligned} d\boldsymbol{M}' &= du^i\,\boldsymbol{e}'_i, \\ d\boldsymbol{e}'_i &= \omega_i^k\,\boldsymbol{e}'_k, \end{aligned}\right\} \tag{4.19}$$

---

[11] French: *Espace Euclidien de raccordement le long d'une ligne.*

where $t$ is the independent variable. These unknown functions $M'$, $e'_i$ are determined by the following initial conditions: for $t = 0$, the point $M'$ is at $O$, and the vector $\boldsymbol{e}'_i$ is equal to the vector $(e_i)_0$ of the system of reference $(R_0)$. We will prove, as we did before (n° 47), that the frame $(R)$ attached to the variable point $t$ has the scalar products of the vectors that determine it equal to the corresponding coefficients $g_{ij}$. Furthermore, if $M$ and $M_1$ are any two infinitely close points on the given line, $M'$ and $M'_1$ the corresponding points in Euclidean space, two equivalent vectors with origins $M$ and $M_1$ have as images two equivalent vectors (in the ordinary sense of the word) with origins $M'$ and $M'_1$.

Thanks to the determination at each point $M'$ of a Cartesian system of reference $(R')$, we have in fact developed onto Euclidean space not only the given curve, but also the whole infinitesimal region of Riemannian space which surrounds this curve. To obtain the absolute geometric variation of a vector whose origin describes an arc of the given line in the Riemannian space, *it is sufficient to construct the ordinary geometric difference of the two vectors obtained from the preceding development.* We thus get an exact and rigorous evaluation of this absolute geometric variation. We know also in a precise and rigorous way what it is to transport a vector by equivalence along the given path.

**93.** We can make the preceding operation still more precise by showing that there exists an osculating Euclidean metric for the given metric all along the given line, or again that there exists *an Euclidean metric of connection* along the given line, or finally that there exists *a Euclidean space of connection* along the given line. This means that we can determine an Euclidean line element constructed with variables $u^i$ and such that the coefficients $g_{ij}$ and their partial derivatives of first order have the same numerical values, all along the line, as the given line element.[12]

To prove this proposition, consider the development which we have just made of the given line onto Euclidean space. Suppose, for simplicity, that $n = 3$ and also, with no loss of generality, that the line is defined by the equations $u^1 = 0$, $u^2 = 0$. The development has given us, as a function of $u^3$, a point $M'$ and vectors $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$, and we have, according to equations (4.19),

$$\left.\begin{aligned} \frac{d\boldsymbol{M}'}{du^3} &= \boldsymbol{e}'_3, \\ \frac{d\boldsymbol{e}'_i}{du^3} &= \Gamma_i{}^k{}_3\, \boldsymbol{e}'_k. \end{aligned}\right\} \tag{4.20}$$

That said, let us determine a point $P$ of the Euclidean space as a function of $u^1, u^2, u^3$ from the following initial conditions:

---

[12] This theorem was stated for the first time by Fermi (Rend. Acc. Lincei, I, $31^2$, 1922, p. 21-23, 51-52).

For $u^1 = u^2 = 0$, we have

$$
\left.\begin{aligned}
&P = M' \\
&\frac{\partial P}{\partial u^1} = \boldsymbol{e}'_1, \qquad \frac{\partial P}{\partial u^2} = \boldsymbol{e}'_2, \\
&\frac{\partial^2 P}{\partial (u^1)^2} = \Gamma_1{}^k{}_1\, \boldsymbol{e}'_k, \quad \frac{\partial^2 P}{\partial u^1 \partial u^2} = \Gamma_1{}^k{}_2\, \boldsymbol{e}'_k, \quad \frac{\partial^2 P}{\partial (u^2)^2} = \Gamma_2{}^k{}_2\, \boldsymbol{e}'_k,
\end{aligned}\right\} \tag{4.21}
$$

where the $\Gamma_i{}^h{}_j$ are replaced by the numerical values that they take when we put $u^1 = u^2 = 0$.

These conditions are clearly compatible; it is sufficient, for example, to take

$$
P = M' + u^1 \boldsymbol{e}'_1 + u^2 \boldsymbol{e}'_2 + \frac{1}{2}(u^1)^2\, \Gamma_1{}^k{}_1 \boldsymbol{e}'_k + u^1 u^2\, \Gamma_1{}^k{}_2 \boldsymbol{e}'_k + \frac{1}{2}(u^2)^2\, \Gamma_2{}^k{}_2 \boldsymbol{e}'_k.
$$

*By this formula, the Euclidean space is referred to a system of curvilinear coordinates* $u^1, u^2, u^3$. For $u^1 = u^2 = 0$, the *natural* system of reference attached to point $P$ (which here reduces to $M'$) is defined by the vectors $\dfrac{\partial P}{\partial u^i}$, which here reduce to $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$; the coefficients of $ds^2$ of the Euclidean space are thus the $g_{ij}$ of the given $ds^2$ of the Riemannian space. As regards the coefficients $\Gamma_i{}^k{}_j$ of the Euclidean space, we obtain them by taking the coefficient of the $\boldsymbol{e}'_k$ in $\dfrac{\partial^2 P}{\partial u^i \partial u^j}$. If the indices $i$ and $j$ are both different from 3, we find, according to (4.21), the coefficients themselves of the Riemannian space; if the index $i$ is different from 3 and the index $j$ is equal to 3, we have, for $u^1 = u^2 = 0$, according to (4.20),

$$
\left(\frac{\partial^2 P}{\partial u^i \partial u^3}\right)_0 = \frac{d}{du^3}\left(\frac{\partial P}{\partial u^i}\right)_0 = \frac{d\boldsymbol{e}'_i}{du^3} = \Gamma_i{}^k{}_3\, \boldsymbol{e}'_k,
$$

and the conclusion is the same. Finally, if $i = j = 3$, we have

$$
\left[\frac{\partial^2 P}{\partial (u^3)^2}\right]_0 = \frac{d\boldsymbol{e}'_3}{du^3} = \Gamma_3{}^k{}_3\, \boldsymbol{e}'_k;
$$

and the conclusion still holds. The theorem is therefore proved.

We see that *the determination of an Euclidean space of connection along a given line is made without integration, once we have developed the line onto Euclidean space.*

**94**. The consequences of the preceding theorem are many and important. First, we see that *an observer who moves only along the given line and who is content to make measurements in the immediate neighbourhood of this line, would no way of perceiving that he is not in Euclidean space, as long as he neglects infinitesimal quantities of order greater than the first.*

Another important consequence is the following. If we consider in the Riemannian space an arc of a line infinitely close to the given line $C$, *the length of this arc of line, whether we measure it according to the metric of the Riemannian* *space, or whether we measure it according to the Euclidean metric of connection along $C$, is the same up to infinitesimals of second order*[13].

In fact, at a point infinitely close to the line $C$, the coefficients of the given line element and the Euclidean line element of connection are equal up to infinitesimals of second order. *The representation of the Riemannian space on the Euclidean space of connection thus preserves, up to infinitesimals of second order, the distances measured in the neighbourhood of the given line.*

**95.** When we develop a curve on the Euclidean space, the developed curve obviously has the same curvature and the same torsion at each point as the given curve, since in the development the absolute differential of a vector whose origin describes the curve becomes the ordinary geometric variation of this vector. The generalised Frenet formulae thus become, in the Euclidean space of connection, the *ordinary* Frenet formulae

$$\frac{d\boldsymbol{t}}{ds} = \frac{1}{\rho}\,\boldsymbol{n}, \qquad \frac{d\boldsymbol{n}}{ds} = -\frac{1}{\rho}\,\boldsymbol{t} + \frac{1}{\tau}\,\boldsymbol{b}, \qquad \frac{d\boldsymbol{b}}{ds} = -\frac{1}{\tau}\,\boldsymbol{n}.$$

A curve of zero torsion thus develops as a plane curve, and a curve of zero curvature as a straight line. *The straight lines of the Riemannian space are thus those curves that develop as straight lines.* From this there follows immediately the equivalence of the definition of straight lines given in n° 75 (and which we have adopted in the preceding) and of the definition of geodesics given by Riemann. In fact, if a curve $(C)$ develops as a straight line and we draw in the Riemannian space an infinitely close curve $(C')$ starting from a given point $A$ and ending at a given point $B$ of $(C)$, the image curve in the Euclidean space of connection will have the same length as $(C')$, up to infinitesimals of the second order. Now, in this Euclidean space, the image curve of $(C')$ is the result of the variation of a segment of a straight line; its length is thus, up to infinitesimals of second order, *the same* as that of a straight line segment. Consequently, in the Riemannian space, the first variation of a segment of a straight line is identically zero and the "straight line" thus indeed realises the extremal of distance, in accordance with Riemann's classic definition.

We deduce from this another consequence; if we consider an arc $AB$ of a geodesic, and an arc $A'B'$ of an infinitely close geodesic, we have, *as in Euclidean space*, up to infinitesimals of second order,

$$\text{arc } A'B' - \text{ arc } AB = -AA'\cos(AB, AA') - BB'\cos(BA, BB').$$

In particular, put on the various geodesics issuing from $A$, a constant length $R$ starting from $A$; we obtain a surface analogous to the sphere as the locus of the endpoints: *the plane element tangent to this surface at a point $M$ is normal to the geodesic from $A$ that ends at $M$.* The same is true if we have issuing from $A$ only a family of geodesics that generate a surface: if we place on each of them a

---

[13] It is necessary to suppose that the two arcs of line correspond point by point in such a way that at two corresponding points the directions of the tangent elements are infinitely close.

constant length, the line element tangent to the locus of endpoints at any point $M$ is normal to the geodesic issuing from $A$ that terminates at this point $M$.

Another consequence is the generalisation of the properties of *parallel surfaces.* If at the various points of a surface we construct the normal geodesic, and if we put a constant length on this geodesic, the locus of points thus obtained is a surface which is normal to the corresponding geodesic at each of its points. It follows that if a two parameter family of geodesics is normal to a surface, it is normal to an infinity of surfaces.

**96.** By considering the Euclidean space of connection, we can easily generalise certain classical theorems, such as *Joachimstal's theorem.* Consider a line $(C)$ and two surfaces $(S_1)$ and $(S_2)$ passing through $(C)$; if the line $(C)$ is a line of curvature for each of these surfaces, this property is conserved in the development onto the Euclidean space of connection; consequently, in this space of connection, the two surfaces intersect at a constant angle: *it will thus be the same in the Riemannian space.* The converse is equally obvious.

**97.** What we have done for a line cannot in general be done for a surface: *in general there is no Euclidean space of connection along a surface.* The reason is simple: we cannot in general develop a surface onto an Euclidean space. Suppose, for simplicity, that the surface is defined by $u^3 = 0$, which can be done with no loss of generality. The development would require the integration of total differential equations in two independent variables $u^1$ and $u^2$

$$\left.\begin{array}{rcl} d\boldsymbol{M}' &=& du^1\boldsymbol{e}'_1 + du^2\boldsymbol{e}'_2, \\ d\boldsymbol{e}'_1 &=& \left(\Gamma_i{}^k{}_1\, du^1 + \Gamma_i{}^k{}_2\, du^2\right) \boldsymbol{e}'_k; \end{array}\right\} \tag{4.22}$$

now, the integrability conditions in general are not satisfied.

Nevertheless there is an important result to remember. *If development onto the Euclidean space is possible, there exists an Euclidean space of connection.* In fact, to say that the development is possible, is to say that there exists in Euclidean space a point $M'$ and vectors $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$ functions of $u^1, u^2$, that satisfy equations (4.22) and the relations

$$\boldsymbol{e}'_i \cdot \boldsymbol{e}'_j = g_{ij}(u^1, u^2, 0).$$

That said, let us determine in the Euclidean space a point $P$ function of $u^1, u^2, u^3$ by the condition that we have, for $u^3 = 0$,

$$\left.\begin{array}{rcl} P &=& M', \\ \dfrac{\partial P}{\partial u^3} &=& \boldsymbol{e}'_3, \\ \dfrac{\partial^2 P}{\partial (u^3)^2} &=& \Gamma_3{}^k{}_3\, \boldsymbol{e}'_k. \end{array}\right\} \tag{4.23}$$

This is possible in an infinity of ways; it is sufficient for example to take

$$P = M' + u^3 \boldsymbol{e}'_3 + \frac{1}{2}\,(u^3)^2\,\Gamma_3{}^k{}_3\,\boldsymbol{e}'_k.$$

We thus refer the Euclidean space to curvilinear coordinates $u^1, u^2, u^3$. For $u^3 = 0$, the natural system of reference defined by these curvilinear coordinates is precisely

$$\frac{\partial P}{\partial u^1} = \frac{\partial M'}{\partial u^1} = \boldsymbol{e}'_1, \qquad \frac{\partial P}{\partial u^2} = \frac{\partial M'}{\partial u^2} = \boldsymbol{e}'_2, \qquad \frac{\partial P}{\partial u^3} = \boldsymbol{e}'_3;$$

the coefficients of the line element of the space are therefore, for $u^3 = 0$, the same as those of the Riemannian space. As for the coefficients $\Gamma_i{}^k{}_j$, they are determined, for $u^3 = 0$, by the vectors $\partial^2 P/\partial u^i \partial u^j$. Now we have, for $i, j \neq 3$,

$$\frac{\partial^2 P}{\partial u^i \partial u^j} = \frac{\partial^2 M'}{\partial u^i \partial u^j} = \Gamma_i{}^k{}_j\,\boldsymbol{e}'_k\,;$$

for $i \neq 3, j = 3$,

$$\frac{\partial^2 P}{\partial u^i \partial u^3} = \frac{\partial}{\partial u^i}\left(\frac{\partial P}{\partial u^3}\right) = \frac{\partial \boldsymbol{e}'_3}{\partial u^i} = \Gamma_3{}^k{}_i\,\boldsymbol{e}'_k\,;$$

finally, for $i = j = 3$,

$$\frac{\partial^2 P}{\partial (u^3)^2} = \Gamma_3{}^k{}_3\,\boldsymbol{e}'_k\,.$$

This clearly shows the equality, for $u^3 = 0$, of the partial derivatives of the coefficients of $ds^2$ of the Riemannian space and of $ds^2$ of the Euclidean space.

Note that the possibility of developing a surface onto Euclidean space is guaranteed if the parallel transport of any vector along the surface is *holonomic*, that is to say, it gives a result that is independent of the path followed on the surface to go from the origin of the initial vector to the origin of the final vector.

**98.** Let us point out a last application of the concept of the Euclidean space of connection. In a three dimensional Riemannian space take a surface $(S)$ and a *geodesic* $(C)$ on this surface, that is, a line of minimum of distance *on the surface.* In the Euclidean space of connection along $(C)$, the line $(C)$ still has a stationary length with respect to all the lines drawn on the surface that have their endpoints at two given points of $(C)$; thus, according to the classic property of geodesics of a surface in Euclidean space, the osculating plane at a point of $(C)$ is normal to the surface. *This property,* which is true in the Euclidean space of connection, *is still true in the Riemannian space.*

## IV. — Application to the theory of surfaces in ordinary space.

**99.** A surface immersed in an ordinary Euclidean space can be regarded as a two dimensional Riemannian space whose line element is the ordinary (Gaussian) line element for the surface. We have here a *concrete interpretation of the osculating Euclidean space* (here a plane) at a point $M$. In fact, project orthogonally the points of the surface onto the tangent plane at $M$; we thus obtain a particular representation of the surface on the Euclidean plane, and *the line element of the plane osculates that of the surface.* This becomes obvious if we place the point $M$ at the origin of coordinates and if we take rectangular axes such that the $xy$ plane is the tangent plane. Keeping the classic notation of Monge, we have, for the $ds^2$ of the surface,

$$ds^2 = dx^2 + dy^2 + (p\,dx + q\,dy)^2,$$

while that of the plane is

$$ds^2 = dx^2 + dy^2;$$

we see that at the origin of coordinates the coefficients of the two line elements are equal, *as are their first order partial derivatives.*

More generally, we could map onto a point $M'$ $(x, y, z)$ of the surface the point $(x_1, y_1)$ of the tangent plane obtained by drawing a line of variable direction through $M'$, but making with the normal to the surface at $M$ an angle that tends to zero as $M'$ tends to $M$; more precisely, we will take

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = z,$$

where $\ell$ and $m$ are continuous functions and that have continuous first order derivatives, with the condition that they are zero when $x = y = 0$. We have in fact

$$x_1 = x - \ell z, \qquad y_1 = y - mz,$$
$$dx_1^2 + dx_2^2 = (dx - \ell\,dz - z\,d\ell)^2 + (dy - m\,dz - z\,dm)^2;$$

by neglecting terms of the second order in the coefficients of the expansion side in $dx$ and $dy$ on the right hand, we find

$$dx_1^2 + dy_1^2 = dx^2 + dy^2.$$

It follows from the preceding that the $ds^2$ of two surfaces tangent at a point $A$ are osculating at this point if we establish a point correspondence between the two surfaces either through the perpendiculars to the common tangent plane at $A$, or through perpendiculars dropped from the points of one of the surfaces onto the other.

**100.** Gauss was the first to develop the *intrinsic* theory of surfaces by studying those properties which depend only on their line element; in this respect he is a precursor of Riemann. In particular he introduced the concept of *geodesic curvature* of a curve drawn on the surface. This is what we call the *curvature* of the curve, considered as drawn inside the two dimensional Riemannian space formed by the surface. To avoid any confusion, we adopt Gauss's notation and denote the geodesic curvature of a curve by $\dfrac{1}{\rho_g}$; we reserve the term *curvature* for the ordinary curvature of the curve considered as immersed in the Euclidean space, and denote it by $\dfrac{1}{\rho}$.

According to the general theory and according to n° 99, the geodesic curvature at a point $M$ of a curve $(C)$ is equal to the ordinary curvature of the projection $(C')$ of the curve $(C)$ onto the tangent plane at $M$. If we now consider the cylinder projecting orthogonally the curve $(C)$ onto the tangent plane, the two curves $(C)$ and $(C')$ have, on this cylinder, the same normal curvature; since the angle made by the principal normal to the curve $(C)$ with the normal to the cylinder is the complement of the angle $V$ made by the principal normal with the normal to the surface $(S)$, we have the formula

$$\frac{1}{\rho_g} = \frac{\cos\left(\frac{\pi}{2} - V\right)}{\rho} = \frac{\sin V}{\rho}.$$

The *geodesics* of the surface (curves of zero geodesic curvature) are, according to this, either straight lines of the space $(1/\rho = 0)$, or lines whose osculating plane is normal to the surface $(\sin V = 0)$.

We can get another expression for the geodesic curvature that does not involve the ambient space. First note that, in a two-dimensional Riemannian space, two vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ with infinitely close origins $M$ and $M'$ are *parallel* if they make the same angle with the geodesic $MM'$: this is due to the fact that the direction of the geodesic remains the same from point to point when we move along this geodesic. That said, start from the formula of Frenet

$$\frac{D\boldsymbol{t}}{ds} = \frac{1}{\rho_g}\, \boldsymbol{n}_g,$$

where $\boldsymbol{n}_g$ denotes the unit vector normal to the curve, *but tangent to the surface.* The number that measures the vector $D\boldsymbol{t}$ is equal to the angle $\varepsilon$ made by the tangent at $M$ with the tangent at $M'$, transported parallel to itself from $M'$ to $M$ (angle of contingency). Consequently *the geodesic curvature is equal to the ratio* $\dfrac{\varepsilon}{ds}$ *of the angle of contingency to the arc* $MM'$. To calculate $\varepsilon$, consider (Figure 1) the two geodesics tangent at $M$ and $M'$ to the given curve and let $P$

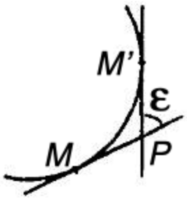

Figure 1

be the point (infinitely close to $M$ and to $M'$) where they intersect. To transport the direction of the tangent at $M'$ parallel to itself from $M'$ to $M$, we can, up to infinitesimals of the second order, first transport this direction from $M'$ to $P$: we thus obtain the direction at $P$ of the *geodesic* $PM'$; by then transporting it from $P$ to $M$, the angle made by this direction with the geodesic $MP$ does not change; consequently *the angle of contingency is simply the angle at which the two geodesics considered intersect at* $P$. We thus find the *same* definition of geodesic curvature as on the Euclidean plane.

**101.** We also have a concrete interpretation of the *Euclidean space* (here a plane) *of connection* along a curve $(C)$ by projecting orthogonally the points of the surface onto the developable $(\Sigma)$ circumscribed on the surface along the curve $(C)$. In fact, the metrics of the two surfaces osculate at each point of $(C)$ (n° 99) and, on the other hand, the metric of $(\Sigma)$ is Euclidean. Consequently the surface $(\Sigma)$, unrolled on a plane, indeed provides an Euclidean space of connection along $(C)$.

This provides a mechanism, as it were, for parallel-transporting a vector along a curve $(C)$ of the given surface. If, for example, we consider a sphere of radius $R$ and, on this sphere, an arc of a line of latitude with colatitude $\alpha$,[14] we can propose to parallel-transport from $A$ to $B$ the tangent $AT$ to the meridian $PAP'$ that passes through $A$ (Figure 2). The developable circumscribing the sphere is

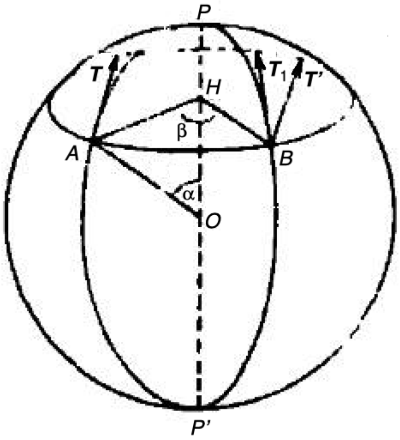

**Figure 2**

here a cone whose generators, confined to the given parallel on the sphere, have length $R\tan\alpha$; if $\beta$ is the angle at the centre corresponding to the arc $AB$, the development will give (Figure 3) an arc of a circle of radius $R\tan\alpha$ and length $R\beta\sin\alpha$. The geodesic curvature of the line of latitude is thus $\dfrac{\cot\alpha}{R}$; the angle at

[14] Translator's note: Colatitude is the angle made by a radial line with the radial line

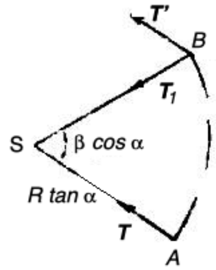

**Figure 3**

the centre of the sector provided by the development of the cone will be $\beta\cos\alpha$. It follows immediately that this is the angle that the vector $BT'$ (resulting from the parallel transport of $AT$) will make on the sphere with the tangent $BT_1$ to the meridian passing through $B$. This angle $\beta\cos\alpha$ is zero if the circle considered is the great circle of the equator: this is due to the fact that the equator is a geodesic; the tangent to this geodesic remains constantly parallel to itself, so the vector $AT$ constantly makes a right angle with this geodesic during its transport.

We can find the preceding results analytically by starting from the classic $ds^2$ of the sphere, which we will assume to be of radius 1:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2.$$

Calculating the Christoffel symbols gives

$$\Gamma_2{}^1{}_2 = \left\{ \begin{matrix} 2\,2 \\ 1 \end{matrix} \right\} = -\sin\theta\cos\theta, \qquad \Gamma_1{}^2{}_2 = \left\{ \begin{matrix} 1\,2 \\ 2 \end{matrix} \right\} = \cot\theta,$$

where the other quantities $\Gamma_i{}^k{}_j$ are zero. The equations that express that a vector is parallel-transported along a line of latitude are

$$\begin{aligned} dX^1 + X^2\omega_2^1 &= dX^1 - X^2\sin\theta\cos\theta\ d\varphi = 0,\\ dX^2 + X^1\omega_1^2 &= dX^2 + X^1\cot\theta\ d\varphi = 0. \end{aligned}$$

Here we have

$$\theta = \alpha, \qquad (X^1)_0 = 1, \qquad (X^2)_0 = 0;$$

the integration, performed from $\varphi_0$ to $\varphi_0 + \beta$, gives at point $B$

$$X^1 = \cos(\beta\cos\alpha), \qquad X^2 = -\frac{1}{\sin\alpha}\ \sin(\beta\cos\alpha),$$

which is in full agreement with the result obtained geometrically.

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through the north pole of the sphere. In spherical coordinates, this is the angle commonly denoted by $\theta$. It is the angle complementary to the angle of latitude.

**102.** The procedure described in n$^o$ 99 for obtaining an osculating Euclidean metric at a point of a surface can be generalised to any $n$-dimensional Riemannian space. It is sufficient to imagine, in an Euclidean space of a sufficiently large number $N$ of dimensions, an $n$-dimensional manifold that has precisely the given line element. We obtain such a manifold if we can find $N$ functions $x_1, ..., x_N$ of $u^1, ..., u^n$ that satisfy the identity

$$dx_1^2 + dx_2^2 + \cdots + dx_N^2 = g_{ij} du^i \, du^j;$$

this identity amounts to a system of $\dfrac{n(n+1)}{2}$ first order partial differential equations for $N$ unknown functions. This system will certainly be compatible (or so a closer examination shows) if we do not have more equations than unknowns, in particular if we take $N = \frac{n(n+1)}{2}$.[15]

That said, let there be a manifold $V_n$, with the given line element, in the $N$ dimensional Euclidean space. We will obtain an osculating Euclidean metric at a point $M$ of the manifold by projecting orthogonally the points of the manifold onto the plane manifold of dimension $n$ tangent at $M$: the line element thus obtained for this plane manifold satisfies the required conditions. From this follows a procedure for obtaining two vectors tangent to $V_n$ that have as their origins two infinitely close points $M$ and $M'$ and that satisfy the condition of parallelism: it is sufficient that the orthogonal projection of the second onto the plane manifold tangent to $V_n$ at $M$ be parallel to the first vector in the ordinary sense of the word. It is in this way that Leví-Civitá first introduced the concept of parallelism;[16] but if we approach matters *a priori* from this point of view, it is necessary to prove that parallel transport thus defined involves only the line element of the manifold: this is what Leví-Civitá proved.

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[15] See M. Janet, *Annales Soc. Pol. Math.,* Vol. 5, 1926, p. 38-43, and E. Cartan, same journal, Vol. 6, 1927, p. 1-7.

[16] T. Leví-Civitá, *Nozione di parallelismo in una varietà qualunque* (*Rend. Circ. matem. Palermo,* Vol. 42, 1917, p. 173-205).

# 5 Geodesic Surfaces; The Axiom of the Plane and the Axiom of Free Mobility

## 5.1 Surfaces Geodesic at a Point; Severi's Theorem

**103.** If we construct the various geodesics[1] through a given point $A$ of the Riemannian space tangent to a given plane element at this point, we obtain a surface which is said to be *geodesic at* $A$. To determine whether a given surface is geodesic at one of its points, it is thus sufficient to consider the geodesics tangent to the surface at this point; they must all be entirely contained in it.

A surface that is geodesic at $A$ has, at this point, its two principal curvatures equal to zero, since the normal curvature is zero when we pass from point $A$ to an infinitesimally close point $A'$ on the surface. Thus if we parallel-transport a vector tangent to the surface at $A$, from $A$ to $A'$, it remains tangent to the surface at $A'$ (up to an infinitesimal of order greater than the first).

We deduce from this a geometric construction due to F. Severi[2] on the parallel transport of a vector.

*To parallel-transport a vector* $\boldsymbol{x}$ *with origin* $A$ *to an infinitely close point* $A'$, *construct the geodesic joining* $AA'$, *as well as the geodesic surface at* $A$ *tangent at* $A$ *to this geodesic and to the vector* $\boldsymbol{x}$; *the required vector* $\boldsymbol{x}'$ *is tangent at* $A'$ *to this surface, and makes the same angle with the geodesic* $AA'$ *extended beyond* $A'$, *as the given vector* $\boldsymbol{x}$ *makes at* $A$ *with the geodesic* $AA'$.

The last part of the preceding statement follows from the fact that the direction at $A'$ of the geodesic $AA'$ is *parallel* to its direction at $A$ and that, in parallel transport, the angle between two directions is constant.

It is important to note that if we were to parallel-transport the vector $\boldsymbol{x}$ along a *finite* arc of the geodesic $AA'$ (sufficiently extended), the vector obtained would in general not be tangent to the geodesic surface considered, because there is no *a priori* reason why this surface, geodesic at $A$, should be geodesic at other points of the geodesic $AA'$.

If two surfaces which intersect in a line $(C)$ are geodesic at all points on $(C)$,

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[1] We are dealing here, and in what follows, with the *geodesics* or the *straight lines of the space.*

[2] F. SEVERI, *Sulla curvatura delle superficie e varietà* (Rend. Circ. matem. Palermo, t. 42, 1917, p. 227-259).

they intersect at constant angle, because the line $(C)$ can be regarded as a line of curvature of each of the surfaces (n° 96).

## 5.2 Totally Geodesic Surfaces; Planes

**104.** A surface which is geodesic at each of its points is said to be *totally geodesic.*[3] It thus has the characteristic property that any geodesic tangent to it is entirely contained in it.

An equally characteristic property is that any geodesic that contains two of its (sufficiently close) points is entirely contained in it. In fact, let $A$ and $A'$ be these two points; from point $A$ go out an infinity of geodesics on the surface which fill the entire surface in a sufficiently small neighbourhood; one of them thus passes through $A'$, and since only one geodesic passes through two sufficiently close points of the space, this is precisely the geodesic considered. The converse is proved in the same way.

Totally geodesic surfaces thus have the characteristic properties of a *plane* in Euclidian space. We can still call them *planes.* But we will see that *the existence of planes in a Riemannian space is unusual.*

**105.** A totally geodesic surface has the following three properties, which are equivalent:

First, *it has zero principal curvatures at each of its points.*

Second, *an unit vector normal to the surface remains normal if we parallel-transport it in any way along the surface.*

Third, *any vector tangent to the surface remains tangent if we parallel-transport in any way along the surface.*

The second property is an immediate consequence of the first, and the first is in turn a consequence of the second. As for the second and the third, these are clearly equivalent to one another.

We shall show that the third property is *characteristic* of totally geodesic surfaces; it follows that each of the first two is also characteristic of these surfaces.

Suppose that any vector tangent to a given surface $S$ remains tangent to it if it is parallel-transported with its origin describing any curve on the surface. It follows that the acceleration of a point that describes any such curve on the surface is always tangent to the surface. If the coordinates of a point on the surface are expressed as a function of two parameters $\alpha$, $\beta$, *two* second order differential equations in $\alpha$ and $\beta$ are sufficient to express the fact that the acceleration of a moving point is zero: *there is then on the surface a geodesic passing through an arbitrary point on this surface and having at this point an*

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[3] This concept is due to J. Hadamard (*Bull. Soc. Math.* 2° series, Vol. 25, 1901, p. 37-40).

*arbitrary direction tangent to the surface*; consequently the surface is totally geodesic.

Verification by calculation is easy. Suppose, with no loss of generality, that the surface is defined by the equation $u^3 = 0$. If we look for the geodesics of the space that lie on the surface, we must satisfy three equations:

$$\begin{aligned} \frac{d^2(u^1)}{ds^2} + \Gamma_{i}{}^{1}{}_{j} \frac{du^i}{ds}\frac{du^j}{ds} &= 0, \\ \frac{d^2(u^2)}{ds^2} + \Gamma_{i}{}^{2}{}_{j} \frac{du^i}{ds}\frac{du^j}{ds} &= 0, \\ \Gamma_{i}{}^{3}{}_{j} \frac{du^i}{ds}\frac{du^j}{ds} &= 0 \end{aligned}$$

For the surface to be totally geodesic, it is therefore necessary and sufficient that we have

$$\Gamma_{1}{}^{3}{}_{1} = \Gamma_{1}{}^{3}{}_{2} = \Gamma_{2}{}^{3}{}_{2} = 0 \tag{5.1}$$

These equations express either that the principal curvatures are zero, or that the absolute differential of each of the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ tangent to the surface is a vector tangent to the surface

$$D\boldsymbol{e}_1 = \left(\Gamma_{1}{}^{1}{}_{1}\, du^1 + \Gamma_{1}{}^{1}{}_{2}\, du^2\right) \boldsymbol{e}_1 + \left(\Gamma_{1}{}^{2}{}_{1}\, du^1 + \Gamma_{1}{}^{2}{}_{2}\, du^2\right) \boldsymbol{e}_2;$$

or that the absolute differential of the normal vector, with covariant components $X_1 = X_2 = 0, X_3 = 1$, is normal to the surface

$$\begin{aligned} DX_1 &= dX_1 - X_i\omega_1^i = -\omega_1^3 = -\Gamma_{1}{}^{3}{}_{1}\, du^1 - \Gamma_{1}{}^{3}{}_{2}\, du^2 = 0, \\ DX_2 &= dX_2 - X_i\omega_2^i = -\omega_2^3 = -\Gamma_{2}{}^{3}{}_{1}\, du^1 - \Gamma_{2}{}^{3}{}_{2}\, du^2 = 0. \end{aligned}$$

**106.** Another important property of totally geodesic surfaces is the following: *If we develop on the Euclidian space any of the curves that lie on a totally geodesic surface, we obtain a plane curve.* In fact, since the absolute differential of the unit tangent vector is tangent to the surface, the principal normal is tangent to the surface; the unit vector on the binormal is thus normal to the surface and its absolute differential is zero; the torsion of the curve is thus zero. Since the torsion is zero, in the development the curve becomes plane.

*The converse is also true.* Suppose that the torsion of any curve on the surface is zero, or, which comes to the same thing, that the velocity, the firdst order acceleration and and the second order acceleration of a point that moves in any way on the surface are constantly coplanar. Let the surface be defined by $u^3 = 0$. We can always imagine a second moving point travelling across the surface and such that, at a given instant, the quantities $\frac{du^i}{dt}$ and $\frac{d^2u^i}{dt^2}$ have the same numerical values as for the first moving point, whereas the quantities $\frac{d^3u^1}{dt^2}$ and $\frac{d^3u^2}{dt^3}$, on the other hand, vary from the first moving point to the second, by arbitrary quantities $\alpha^1$ and $\alpha^2$ respectively. It follows that the arbitrary vector $(\alpha^1, \alpha^2)$, tangent to the surface, is coplanar with the velocity and the acceleration of the first moving point; consequently this acceleration is in the plane tangent to the surface and all the curves drawn on the surface are asymptotic. The surface is thus totally geodesic.

**107.** We now deduce from the above a remarkable theorem due to G. Ricci.[4]

*If there is a one-parameter family of planes in Riemannian space, their orthogonal trajectories establish an isometric point correspondence between the various planes of the family.*

In fact, let $(P)$ be a plane of the family, and let $(P')$ be an infinitely close plane. Let $(C)$ be any curve on the plane $(P)$, and let $(C')$ be the locus of the marks[5] on $(P')$ of the orthogonal trajectories passing through the various points of $(C)$. Represent this in the connecting Euclidian space along $(C)$. In this representation the curve $(C)$ is *plane* and the curve $(C')$ is deduced by running an infinitesimal length, normal to the plane of $(C)$, through each point of $(C)$; consequently the *first variation* of the length of any arc of (C) is zero when we pass go $(C)$ to $(C')$. On the other hand, since the length of $(C')$ is conserved in the representation up to infinitesimals of the second order, we see that, in the Riemannian space, the first variation of the arc-length of $(C)$ is zero when we pass from the plane $(P)$ to the infinitely close plane $(P')$. This is what needed to be proved.

This leads to an interesting consequence. In one of the planes $(P_0)$ of the family, take any system of coordinates $u, v$, and denote by $w$ a variable parameter that distinguishes the various planes of the family. The line element of the space is then of the form

$$ds^2 = d\sigma^2 + H(u, v, w)\ dw^2, \tag{5.2}$$

where we denote by $d\sigma^2$ the line element of the plane $P_0$:

$$d\sigma^2 = E(u, v)\ du^2 + 2F(u, v)\ du\, dv + G(u, v)\ dv^2. \tag{5.3}$$

We can also arrive at this result by calculation if we take the trajectories orthogonal to the planes of the family as the third family of coordinate curves ($u$ = const., $v$ = const.), and the planes themselves as the third coordinate surfaces ($w$ = const.). The line element of the space satisfies the conditions

$$g_{13} = g_{23} = 0. \tag{5.4}$$

For the surface $w$ = constant to be totally geodesic, it is necessary and sufficient

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[4] G. Ricci, *Formole fondamentali nella theoria generale delle varieta e della loro curvatura* (Rend. Acc. Lincei, Vol. 12, 1903, p. 409-420).

[5] Fr. *traces*

that we have

$$\Gamma_1{}^3{}_1 = \Gamma_1{}^3{}_2 = \Gamma_2{}^3{}_2 = 0,$$

or, taking (5.4) into account,

$$\Gamma_{131} = \Gamma_{132} = \Gamma_{232} = 0,$$

or again, in Christoffel's notation,

$$\begin{bmatrix} 1\,1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\,2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\,2 \\ 3 \end{bmatrix} = 0.$$

These equations reduce to

$$\frac{\partial g_{11}}{\partial w} = \frac{\partial g_{12}}{\partial w} = \frac{\partial g_{22}}{\partial w} = 0,$$

and we recover the form of $ds^2$ indicated above.

Since the line element obtained contains only *one* arbitrary function of three arguments (and some functions of two arguments), it follows that Riemannian spaces that admit one-parameter families of planes are exceptions since, according to a remark due to Riemann, the most general $ds^2$ in three variables depends on *three* arbitrary functions of three variables (there are six arbitrary coefficients, but the possibility of an arbitrary transformation of coordinates reduces the number of arbitrary functions to three).

## 5.3 The Axiom of the Plane and the Axiom of Free Mobility in Space

**108.** In ordinary space, a plane always passes through three arbitrarily given points or, equivalently, a plane always passes through two intersecting lines. We will say that a Riemannian space satisfies the *axiom of the plane* if it has the following property:

> *Through any point in the space and tangentially to any plane that has this point as origin, there passes a totally geodesic surface, that is, such that any geodesic that contains two of its points is completely contained in it.*

A little further on we will determine all the Riemannian spaces that satisfy the Axiom of the Plane. Before doing so, we will show that they have another very important property in common with Euclidean space.

Euclidian space admits an infinite number of isometric point transformations; in other words, a figure can always be moved, *without it ceasing to be equal to itself*, in such a way as to make any point $A$ of the figure coincide with an arbitrarily given point $B$, and any two half lines issuing from $A$ with any two half lines issuing from $B$, provided that they have between them the same angle as the first two.

Elementary geometry is largely based on the existence of these displacements or, which amounts to the same thing, on the concept of equality.

We will say that a Riemannian space satisfies the Axiom of Free Mobility if it also admits displacements that do not deform the figures and have the same degree of generality as those of Euclidian space.

We will show that, in Riemannian space, *the Axiom of the Plane leads to the Axiom of Free Mobility and conversely.*

**109.** First, we will prove that if all geodesic surfaces at a particular point $A$ are totally geodesic, then the space has *free mobility around* $A$, that is, it always admits an isometric point transformation that leaves point $A$ invariant and transforms any two directions at $A$ into any two other directions that have the same angle between them as the two given directions.

It is clear that if such an isometric point transformation exists, it is well defined, because in the Euclidian space tangent at $A$ it necessarily becomes a rotation around an axis through $A$; since the transformed direction of another is well defined, any point $M$ on the geodesic through $A$ in the first direction corresponds to a point $M'$ on the geodesic through $A$ in the transformed direction and at the same distance from $A$ as the point $M$. The preceding assumes that the isometric transformation is *direct*, that is, it conserves the orientation of the trihedron with origin $A$.

These preliminaries being set down, suppose that all geodesic surfaces at $A$ are totally geodesic. Consider a *sphere* with centre $A$, that Is, the locus of points obtained by marking out a constant length $R$, starting from $A$, on the various geodesics from $A$. Call the section of this sphere by a *plane* passing through $A$ a *great circle* of the sphere, and consider on the sphere two arcs of a great circle, $MN$ and $M'N'$, corresponding to two equal angles $\widehat{MAN}, \widehat{M'AN'}$ at the centre.

It is possible to draw on the sphere an arc of a curve $MM'$ which is normal to the great circle $MN$ at $M$, and to the great circle $M'N'$ at $M'$. Consider the family of planes obtained by taking an arbitrary point $P$ on the curve $MM'$ and constructing the geodesic surface containing the geodesic $AP$ and normal at $P$ to the curve $MM'$; this is possible since the radius $AP$ is normal to the sphere at $P$ (n° 95).

The orthogonal trajectories of the planes of the family define an isometric point correspondence on these planes; now the point $M$ on the plane $AMN$ (which belongs to the family) corresponds to the point $M'$ on the plane $AM'N'$ (which also belongs to the family); the geodesic $AN$ corresponds to the geodesic $AN'$ which makes the same angle with $AM'$ that $AN$ makes with $AM$, and consequently the point $N$ corresponds the point $N'$. The isometric correspondence thus takes the arc of the great circle $MN$ to the arc of the great circle $M'N'$.

It follows from this that *on any sphere with centre A, equal angles at the centre correspond to equal arcs of great circles* (that is, of equal length).

Consider now the point transformation of the Riemannian space which, as explained above, results from a rotation around $A$ in the tangent Euclidean space at $A$. Let $M$ and $N$ be two neighbouring points on the same geodesic from $A$, and let $M'$ and $N'$ be the transformed points; they too are on the same geodesic from $A$ and clearly we have $M'N' = MN$. Now let $M$ and $N$ be two neighbouring points situated on the same sphere with centre $A$; let $M'$ and $N'$ be their transformed points also situated on the same sphere; we again have $M'N' = MN$, since the corresponding angles at the centre are obviously equal. Since any elementary displacement in the Riemannian space can be seen as the result of two elementary orthogonal displacements, one radial, the other normal to the radius vector from $A$, this proves that *the point transformation considered is indeed isometric.*

We can add something more. The reasoning just given can be applied without modification to the point transformation resulting from a *symmetry* in the tangent space at $A$.

*Therefore, if the geodesic surfaces at $A$ are all totally geodesic, the space admits $\infty^3$ direct point transformations and $\infty^3$ inverse point transformations, all isometric, which leave the point $A$ fixed.*

**110.** We now prove the converse. Suppose that the space has the property of Free Mobility around $A$. It immediately follows that, on any sphere with centre $A$, two elementary arcs corresponding to equal angles at the centre are equal. Consider then a plane element at $A$ and, in the tangent Euclidean space at $A$, the symmetry with respect to this plane element. It leads to a point transformation of the Riemannian space, with a point $M$ transforming into a point $M'$ such that the two geodesics $AM$ and $AM'$ have their directions at $A$ symmetric with respect to the given plane element and such that the two lengths $AM$ and $AM'$ are equal. This point transformation, which conserves the lengths of the elementary arcs drawn on a sphere with centre $A$ and the lengths of the elementary arcs normal to this sphere, is isometric. In other words, the existence of $\infty^3$ *direct* isometric transformations about $A$ implies the existence of as many *inverse* isometric transformations.

That said, consider the geodesic surface at $A$, tangent at this point to a given plane element. It is clearly invariant under the *symmetry* with respect to the given plane element, whose existence we have just proved. If $M$ and $N$ are any two points on the geodesic surface, the geodesic $MN$ is clearly invariant under this symmetry; it is thus the same for the geodesic $AP$ that joins the point $A$ to any point $P$ of $MN$; now the only geodesics from $A$ that are invariant under the symmetry are found in the geodesic surface considered; thus any point $P$ of $MN$ is in the geodesic surface. In other words, *all surfaces geodesic at $A$ are totally geodesic.*

**111.** The converse that we have just proved shows immediately that, if a Riemannian space satisfies the axiom of free mobility, it satisfies the axiom of the plane, since all surfaces that are geodesic at *any* point of the space are totally geodesic. Conversely, if a Riemannian space satisfies the axiom of the plane, it satisfies the axiom of free mobility. In fact, let $A$ and $A'$ be any two (sufficiently close) points, and let $\widehat{BAC}$ and $\widehat{B'A'C'}$ be two equal angles; a suitably chosen (isometric) rotation around the middle of the geodesic $AA'$ will take $A$ into $A'$, and a second rotation around $A'$ will bring the two given angles into coincidence.

**112.** We will prove a very remarkable theorem of F. Schur[6], namely that *if there are two specific (sufficiently close) points $A$ and $B$ in a Riemannian space such that any geodesic surface at one of these points is totally geodesic, the space satisfies the axiom of the plane; moreover, we can represent it in ordinary space in such a way that any geodesic of the space is represented by a straight line.*

The first part of this theorem can also be stated as follows:

*If a Riemannian space has free mobility about two particular points, it satisfies the axiom of free mobility.*

From this point of view the theorem will be easy to prove. In fact, if $M$ is any point (sufficiently close to A and to B), the space can be rotated by any angle around the geodesic $MA$ and rotated by any angle around $MB$. Now in the tangent Euclidean space at $M$, any rotation about $M$ can be obtained by composing in a suitable order a sufficient number of rotations about two given lines emanating from $M$. These compositions, made in the Riemannian space itself, give rise to isometric transformations that prove the free mobility of the space around the point $M$.

**113.** We will prove Schur's theorem in a purely projective way. Any geodesic from $A$ can be defined analytically by the direction parameters $x, y, z$ of this geodesic at $A$, parameters which are referred to a Cartesian system of reference attached to point $A$, and whose mutual ratios alone are relevant. Similarly any geodesic from $B$ can be defined analytically by its direction parameters $x', y', z'$ at $B$. To any point $M$ of the space (sufficiently close to $A$ and $B$) are thus attached six numbers $x, y, z; x', y', z'$, of which the first three, as well as the last three, are involved only by their mutual ratios. Therefore in reality this means *four* coordinates, between which there is necessarily one relation.

To obtain this relation, note first that any plane (totally geodesic surface) that passes through $A$ is represented by a linear and homogeneous equation in $x, y, z$; similarly any plane passing through $B$ is represented by a linear and homogeneous equation in $x', y', z'$. Consequently any plane containing the geodesic $AB$ will be represented in two different ways; on one hand, by an equation of the form

$$a_1x + b_1y + c_1z + m(a_2x + b_2y + c_2z) = 0, \tag{5.5}$$

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[6] F. Schur, *Ueber den Zusammenhang der Räume oonstanten Krümungsmasses mit den projectiven Räumen* (Math. Ann., Vol. 27, 1886, p. 537-567).

where $m$ is a parameter that varies with the plane considered; on the other hand, by an equation of the form

$$a'_1x' + b'_1y' + c'_1z' + m'(a'_2x' + b'_2y' + c'_2z') = 0, \tag{5.6}$$

where $m'$ is parameter that also varies with the plane considered. The equations

$$a_1x + b_1y + c_1z = 0, \qquad a_2x + b_2y + c_2z = 0$$

define two particular planes (*basis planes*) of the bundle formed by the planes containing $AB$; it is the same for the equations

$$a'_1x' + b'_1y' + c'_1z' = 0, \qquad a'_2x' + b'_2y' + c'_2z' = 0.$$

Nothing stops us from supposing that we have chosen the same two basis planes at $A$ and at $B$; we can even suppose that a third particular plane corresponds at the same time to the value 1 of the parameter $m$, and to the value 1 of the parameter $m'$.

That said, since the quantity $\dfrac{a'_1x' + b'_1y' + c'_1z'}{a'_2x' + b'_2y' + c'_2z'}$ has the same value for all points that correspond to the same numerical value of $\dfrac{a_1x + b_1y + c_1z}{a_2x + b_2y + c_2z}$, these two quantities are functions of each other, and this is how we get the relation that we seek between the coordinates assigned to any point. Now It Is easy to see *that the cross-ratio of four planes passing through $AB$ is the same at $A$ and at $B$*. In fact, consider a surface $(S)$ intersecting the geodesic $AB$ at a point $C$. Any point on this surface can be defined by the homogeneous coordinates $x, y, z$ of the geodesic which joins it to the point $A$. If we intersect the surface $(S)$ with four specific planes that pass through $AB$, we will get four curves emanating from $C$ and defined by the equations

$$a_1x + b_1y + c_1z + m_i(a_2x + b_2y + c_2z) = 0;$$

consequently *the cross-ratio of these four curves at $C$ is equal to the cross-ratio of the four values of $m$*. In fact, if we suppose that at $C$ we have $z \neq 0$, we can assume that $z = 1$ and, putting $\dfrac{x}{z} = X$, $\dfrac{y}{z} = Y$, the *directions* of the four curves at $C$ are defined by the equations

$$a_1\,dX + b_1\,dY + m_i(a_2\,dX + b_2\,dY) = 0.$$

The same reasoning shows that this cross-ratio is equal to the cross-ratio of the four values $m'_i$ of $m'$ which correspond to four planes. *The relation which exists between the parameters $m$ and $m'$ of the same plane containing $AB$ is thus homographic.* Since the values $m = 0, \infty, 1$ correspond to the same values $m' = 0, \infty, 1$, the homographic relation reduces to $m = m'$. In other words, between the six quantities $x, y, z$; $x', y', z'$ we have the relation

$$\frac{a_1x + b_1y + c_1z}{a_2x + b_2y + c_2z}, = \frac{a'_1x' + b'_1y' + c'_1z'}{a'_2x' + b'_2y' + c'_2z'}\,.$$

We can thus write

$$\begin{aligned} a_1x + b_1y + c_1z &= \rho(a_1'x' + b_1'y' + c_1'z'), \\ a_2x + b_2y + c_2z &= \rho(a_2'x' + b_2'y' + c_2'z'). \end{aligned}$$

Take then a linear form $a_3x + b_3y + c_3z$ that is linearly independent of the two forms

$$a_1x + b_1y + c_1z \quad \text{and} \quad a_2x + b_2y + c_2z.$$

Similarly, introduce a form $a_3'x + b_3'y + c_3'z$ that is linearly independent of the two forms

$$a_1'x' + b_1'y' + c_1'z' \quad \text{and} \quad a_2'x' + b_2'y' + c_2'z'.$$

Finally, put

$$\left.\begin{array}{rcl} X &=& a_1x + b_1y + c_1z = \rho(a_1'x' + b_1'y' + c_1'z'), \\ Y &=& a_2x + b_2y + c_2z = \rho(a_2'x' + b_2'y' + c_2'z'), \\ Z &=& a_3x + b_3y + c_3z, \\ T &=& \rho(a_3'x + b_3'y + c_3'z). \end{array}\right\} \tag{5.8}$$

Any point of the space is completely determined by the four quantities $X, Y, Z, T$ (or rather by their mutual ratios), since knowledge of $X, Y, Z$ leads to that of $x, y, z$, and that of $X, Y, T$ leads to that of $x', y', z'$ up to a factor.

**114.** With the coordinate system thus obtained, *any plane passing through* $A$ *is defined by a linear equation in* $X, Y, Z$, *and any plane through* $B$ *by an equation linear in* $X, Y, T$. Take now any geodesic of the space; this geodesic and the point $A$ determines a totally geodesic surface, therefore defined by a linear equation in $X, Y, Z$; the geodesic also determines with the point $B$ a totally geodesic surface defined by a linear equation in $X, Y, T$. Thus *any geodesic is defined by two equations of the first degree in* $X, Y, Z, T$.

The converse is true, because a system of two equations of first degree is equivalent to a system formed by an equation in $X, Y, Z$ and an equation in $X, Y, T$; it thus defines the curve of intersection of two totally geodesic surfaces, which is necessarily *the geodesic* joining two points on this curve.

If we regard $X, Y, Z, T$ as the homogeneous coordinates of a point in ordinary space, we see then that *the portion of the Riemannian space close to the points* $A$ *and* $B$ *has a representation in ordinary space, in which the geodesics are represented by straight lines* (geodesic representation).

The axiom of the plane can be deduced immediately from the previous result; the surface which can be represented by any plane in ordinary space is in fact a totally geodesic surface, since a infinite number of geodesics situated on the surface start from it and these geodesics are tangent at this point to the same plane element (see Note I at the end of this book).

In particular, we see that *the axiom of the plane leads to the possibility of a* *geodesic representation of the Riemannian space on ordinary space; the converse is obvious*, as we have just seen.

So, *if we consider the following three properties of a Riemannian space:*

***I.*** *It satisfies the axiom of free mobility;*
***II.*** *It satisfies the axiom of the plane;*
***III.*** *It admits a geodesic representation on ordinary space;*

*each of these properties implies the other two.*

**115.** The preceding theorems generalise to the case of any number of dimensions $n > 3$. We will confine ourselves to indicating the general definition of geodesic and totally geodesic manifolds and to proving a theorem related to these manifolds.

A $p$-dimensional manifold $V_p$ through a point $A$ is said to be *geodesic at* $A$ if it contains all the geodesics emanating from $A$ and tangent at this point to the same $p$-dimensional plane element $E_p$. A manifold $V_p$ will be said to be *totally geodesic* if it is geodesic at each of its points, or if any geodesic that has two of its points in it is completely contained in it.

Suppose that all surfaces (two-dimensional manifolds) that are geodesic at a point $A$ are totally geodesic. We shall prove that all manifolds $V_p$ $(p > 2)$ that are geodesic at $A$ are also totally geodesic. In fact, let $M$ and $N$ be any two points of one of these manifolds; the geodesics $AM$ and $AN$ determine at $A$ a plane element $E_2$, contained in the plane element $E_p$ which defines the manifold $V_p$; there exists a surface $S$, geodesic at $A$ and tangent to $E_2$; this surface contains the points $M$ and $N$; since it is totally geodesic, it completely contains the geodesic $MN$; the manifold $V_p$, which contains the surface $S$, thus contains also the geodesic $MN$; it is therefore totally geodesic.

Conversely, suppose that all manifolds $V_p$ $(p > 2)$ that are geodesic at $A$ are totally geodesic. Consider a surface $S$, geodesic at $A$ and tangent at $A$ to the plane element $E_2$. This plane element can be regarded as the common intersection of an infinite number of plane elements $E_p$ of dimension $p$, and consequently the surface $S$ can be regarded as the common intersection of an infinite number of manifolds $V_p$ that are geodesic at $A$. So let $M$ and $N$ be two points in $S$; they belong to each of the preceding manifolds $V_p$; therefore the geodesic $MN$ belongs in its entirety to each of these manifolds, assumed by hypothesis to be totally geodesic. Therefore it belongs in its entirety to their common intersection, that Is, to the surface $S$, which is thus totally geodesic.

**116.** In the case $n = 2$, the three properties stated in n° 114, the second makes no sense. *The equivalence of properties I and III is still true,* but for reasons that are much more difficult to prove geometrically than in the case $n = 3$. Moreover, in the theory of Weyl spaces, which generalise those of Riemann, the equivalence of properties I, II, III is still exact for $n = 3$, but the equivalence of properties I and III ceases to be true for $n = 2$.

It was Beltrami who, while trying to produce a geodesic representation of a given surface on a plane, discovered the impossibility of such a representation for an arbitrary surface[7]. The only surfaces for which the representation is possible are the surfaces of constant total curvature, applicable onto a sphere (with real or pure imaginary radius); these are also the only ones that satisfy the axiom of free mobility.

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[7] E. Beltrami, *Risoluzione del problema: riportare i punti di una superficie sopra un piano in modo che le linie geodetiche veggano rappresentate su linee rette* (*Ann. di Matem.,* Series 1, 7, 1865, p. 185-204; *Opere Matem.,* I, Milan, 1902, p. 262-280).

# 6 Non-Euclidean Geometries. Spherical Space. Elliptic Space. Hyperbolic Space.

**117.** In this chapter we will study briefly a class of Riemannian spaces that has the property of admitting a geodesic representation on ordinary space; these are the so called spaces of *constant curvature*. These spaces are also said to be *non Euclidean*; the geometry of these spaces is the geometry called *non Euclidean.*[1] We begin with the simple case $n = 2$.

## I.– Spherical geometry of two dimensions

**118.** Consider in ordinary space a sphere of radius $R$. Every point of the sphere can be defined analytically by its coordinates $x, y, z$ referred to three rectangular axes that have as origin the centre of the sphere; these quantities are related by the equation

$$x^2 + y^2 + z^2 = R^2; \tag{6.1}$$

we then have

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{6.2}$$

The lines that play here the role of straight lines (geodesics) are the great circles of the sphere. It is clear that the sphere admits a geodesic representation on a plane: it is sufficient to perform a central projection onto a plane (where the viewing point is the centre of the sphere). It is also clear that the axiom of free mobility is satisfied, because we can always, by a rotation, take any point of the sphere into any other point in such a way that any direction at the first point comes into coincidence with any direction at the second point. All the axioms on equality, stated at the beginning of plane Euclidean geometry are satisfied In spherical geometry.

The fundamental difference between Euclidean plane geometry and spherical geometry is the following: through two points on a sphere can pass more than one line (great circle), and therefore there passes an infinity: this circumstance presents itself when the two points are diametrically opposite.

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[1] The bibliography on non Euclidean geometries is considerable. We could consult especially the beautiful Memoirs by F. Klein, entitled: *Ueber die sogenannte nicht-euklidische Geometrie* (*Math. Ann.*, t. 4, 1871, and t.6, 1873). — *See* also P. Barbarln, *Le Gèométrie non euclidienne*, followed by notes by A. Buhl (collection *Scientia*; Paris, Gauthler-Villars, 1928).

There exist other essential differences; for example, the space of two dimensions formed by the surface of the sphere is finite (and of area $4\pi R^2$); the straight lines (great circles) are closed curves of finite length $2\pi R$.

These differences are nevertheless more superficial than essential, because we have seen (Chapter III) that certain *locally Euclidean* spaces present even more pronounced qualitative differences with Euclidean space properly so called.

A *local* difference between the sphere and the Euclidean plane is given by the sum of the angles of a triangle. On the sphere it is greater than $\pi$, and the excess of this sum over $\pi$ is equal to the quotient by $R^2$ of the area of the triangle.

We point out moreover that a circumference has two centres and consequently two radii $r$ and $\pi R - r$, that the radius of curvature of such a circumference is $R \tan r/R$, and finally that a circumference of radius $\pi R/2$ has zero curvature, in other words it is a straight line: all radii are then perpendicular to the line.

## II.– Elliptic geometry in two dimensions

**119.** Perform the geodesic representation of the sphere on a plane $(P)$ by projecting from the centre $O$ of the sphere. Any point of the sphere gives one and only one point on the plane (at finite or infinite distance), but conversely a point of the plane arises from two diametrically opposite points of the sphere. Define in the plane the elementary (*non Euclidean*) distance of two infinitely close points $M$ and $N$ by the ordinary distance (calculated on the sphere) of the two corresponding points $M'$ and $N'$ of the sphere, a distance which does not change if we replace the two points $M'$ and $N'$ by the two diametrically opposite points, which give rise in the plane $(P)$ to the same points $M$ and $N$). The two dimensional Riemannian space thus defined is called the *elliptic plane*, and the geometry on this plane Is *elliptic geometry*.

It Is necessary to note that the elliptic plane Is a closed manifold, since the points which, in the ordinary sense of the word, are at infinity are ordinary points that are, from the point of view of elliptic geometry, at finite distance (and they form a *straight line*, that which corresponds to the great circle of the sphere parallel to the plane $P$). From the point of view of the *Analysis situs*, the elliptic plane is thus identical to the projective plane.

The topological differences that exist between the sphere and the elliptic plane can be highlighted if we note that to any point of the sphere there corresponds one and only one *ray* from $O$, whereas to any point of the elliptic plane there corresponds one and only one *straight line* passing through $O$. Consider then a plane $(\Pi)$ passing through $O$; the manifold of rays issuing from $O$ is partitioned by this plane into two separate regions; we cannot pass by continuity from a ray situated on a certain side of the plane to a ray situated on the other without crossing the plane. In contrast, the plane $(\Pi)$ *does not separate* the manifold of straight lines passing through $O$ into two distinct parts; we can pass by continuity from any straight line to any other straight line passing through $O$ without ever crossing the plane (Π). In other words, a straight line (great circle) of the sphere partitions the sphere Into two distinct parts, while a straight line of the elliptic plane does not partition the plane Into two distinct parts: anyhow we see this very well in the case where the straight line Is the ordinary straight line at infinity: we can in fact pass from a point of the plane to any other point without crossing the line at infinity.

**120.** If the elliptic plane is *locally* identical to the sphere, there are however, as we have just seen, essential differences between the two manifolds. Here are others:

Through any two points of the elliptic plane there always passes one and only one straight line, and conversely any two straight lines always intersect at one and only one point. *The first axioms of plane Euclidean geometry are thus true for the elliptic plane.* One might wonder at what point in the sequence of theorems that are usually proved in the treatises on geometry does elliptic geometry breaks away from Euclidean geometry. The divorce is surely complete at the moment when we usually introduce the postulate of Euclid. Indeed it is proved immediately before that through any point taken outside of a straight line there definitely passes a parallel to this straight line; now this theorem is false in elliptic geometry, since two straight lines always have a common point. Now the proof of the theorem in question rests on a preceding theorem, according to which we can drop from a given point only one perpendicular onto a straight line.

Let us recall the proof of this last theorem. Let $D$ be a given straight line, $A$ an exterior point (Figure 1). Join $A$ to any point $M$ of the straight line $D$ and form

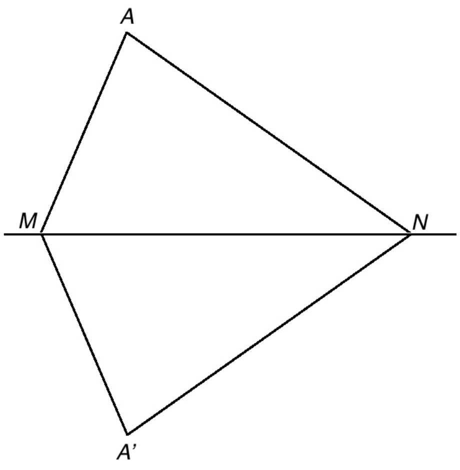

**Figure 1** .

at $M$, on the other side of the straight line $D$, an angle $DMA'$ equal to the angle $DMA$; finally take $MA' = MA$. If $N$ is any other point of the straight line $D$, we see immediately that the two triangles $AMN$, $A'MN$ are equal, since they have an equal angle between two sides equal to each other; consequently, the angles $MNA'$, $MNA$ are also equal. That said, if the line $AN$ is perpendicular to the straight line $D$, the two angles $MNA$ and $MNA'$ are supplementary and the points $A$, $N$, $A'$ are on a straight line; the converse is true. Consequently there exists only one perpendicular dropped from $A$ onto the straight line $D$, and it is the line $AA'$.

The preceding conclusion is valid *as long as the point $A'$ is different from the point $A$*; now this last circumstance might not arise In the elliptic plane, *which is not divided into two distinct parts by a straight line.* If $A'$ coincides with $A$, *all straight lines passing through $A$ are perpendicular to the straight line $D$.* This in fact is what occurs when we take in the elliptic plane a point and a straight line resulting from a point on the sphere and from its polar great circle. In the elliptic plane, any straight line can be regarded as a circumference of radius $\frac{\pi R}{2}$ whose centre is the *pole* of the straight line. Using the sphere as intermediary, we see easily that a circumference of radius $r < \frac{\pi R}{2}$ has length $2\pi R \sin \frac{r}{R}$; when $r$ tends towards $\frac{\pi R}{2}$, this length tends to $2\pi R$, *but the length of the limit line is only half* $\pi R$. Similarly the area bounded by a circle of radius $r$ is $4\pi R^2 \sin^2 \frac{r}{2R}$; it tends toward $2\pi R^2$, the total area of the elliptic plane, when $r$ tends toward $\frac{\pi R}{2}$.

**121.** If we take the plane $(P)$ tangent to the sphere at a point $A$, projection from the centre $O$ of the sphere (situated on the normal to the sphere at $A$) produces [in the plane $(P)$ supposed endowed with the ordinary Euclidean metric] an Euclidean metric that *osculates* the metric of the sphere (see n° 99), and consequently that of the elliptic plane. We thus have in this way in the plane $(P)$ a representation of the elliptic geometry that conserves the straight lines, and a metric that osculates the given metric at the point $A$ (therefore *with conservation of the curvature of curves passing through $A$*). We see something more. A small circle of centre $A$ is represented by a circumference of centre $A$; now the radius of geodesic curvature of the small circle $B'C'$ of the sphere (Figure 2) is equal to the radius $AB = AC$ of the circumference $BC$; consequently, *the representation considered conserves the curvature of circumferences with centre $A$.* We see clearly how the radius of curvature of these circumferences increases from zero to $+\infty$ when the (non Euclidean) radius increases from zero to $\pi R/2$.

**122.** We propose now to define directly the (non Euclidean) distance between two points in the elliptic plane of curvature[2] $1/R^2$ by *purely projective*

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[2] We recall, by this manner of speech, the total curvature of the sphere from which the elliptic plane was derived.

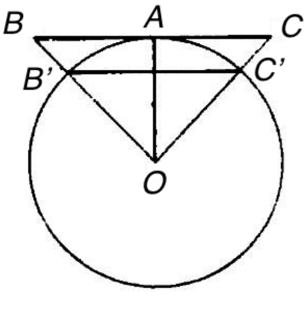


**Figure 2**

concepts. For this we can define analytically a point of the plane by the rectangular coordinates $x, y, z$ of one of the corresponding points of the sphere (*normal coordinates*); these are basically projective coordinates in the plane, but subject to satisfying the relation

$$x^2 + y^2 + z^2 = R^2.$$

This said, consider two points $M, M'$ of the plane with respective coordinates $x, y, z$; $x', y', z'$. We have clearly, by denoting their non Euclidean distance by $d$,

$$R^2 \cos \frac{d}{R} = xx' + yy' + zz'.$$

We interpret geometrically the right hand side of this formula.

The isotropic cone having as vertex the centre of the sphere cuts the plane $(P)$ in a conic $\Gamma$ (called the *absolute*) with equation

$$x^2 + y^2 + z^2 = 0.$$

Let $N_1$ and $N_2$ be two points (Imaginary conjugates) where the line $MM'$ meets the absolute. The coordinates of every point of the line $MM'$ can be put in the form

$$x + \lambda x, \quad y + \lambda y', \quad z + \lambda z',$$

with a variable parameter $\lambda$. To the point $M$ corresponds the value zero of the parameter, to the point $M'$ the value $\infty$; let $\lambda_1$ and $\lambda_2$ be the values corresponding to the points $N_1$ and $N_2$. They are given by the equation of second degree

$$\lambda^2(x'^2 + y'^2 + z'^2) + 2\lambda(xx' + yy' + zz') + x^2 + y^2 + z^2 = 0,$$

or

$$\lambda^2 + 2\lambda \cos \frac{d}{R} + 1 = 0.$$

The cross ratio $(MM', N_1N_2)$ is equal, as we know, to the cross ratio of the four numbers $0, \infty, \lambda_1, \lambda_2$, that is, to $\lambda_1/\lambda_2$; an easy calculation gives

$$\frac{\lambda_1}{\lambda_2} = e^{2id/R}.$$

Consequently, the distance $d$ can be defined by the formula, due to Cayley,

$$d = \frac{R}{2i}\ \ln(MM', N_1N_2), \tag{6.3}$$

which involves the Naperian logarithm of the cross ratio of the four points $M, M', N_1, N_2$. *The length $d$ thus defined is the Cayley distance of the two points, with respect to the absolute* $\Gamma$.

We can deduce from this formula another immediate consequence. If through a point $O$ (that we can suppose to be the centre of the sphere) we take any two lines, the angle of these two lines is equal to the quotient by $2i$ of the logarithm of the cross ratio of the two given lines and the two isotropic lines through $O$ and situated in the plane of the two given lines. This theorem is due to Laguerre end provides in ordinary geometry a *projective* definition of the angle.

Let us return to the elliptic plane. We can see that two points harmonic conjugates with respect to the absolute are at a distance $\pi R/2$ from each another: the cross ratio $(MM', N_1N_2)$ is then in fact equal to $-1$, whose logarithm is $i\pi$. The pole of a straight line is thus its pole (in the ordinary sense) with respect to the absolute.

We can similarly define projectively, in the elliptic plane, the angle of two straight lines through a point $A$. This angle, which depends only on the numerical values at $A$ of the coefficients of the linear element of the elliptic plane, could be defined as the quotient by $2i$ of the logarithm of the cross-ratio of the two given lines and of the two isotropic lines (of zero length) through $A$; now a straight line $AA'$ is isotropic if the two points of intersection $B_1, B_2$ with the absolute $\Gamma$ are the same [because then the cross-ratio $(AA', B_1B_2)$ will be equal to 1 and the Cayley distance $AA'$ will be zero]. Consequently, *the angle of two straight lines through $A$ is the quotient by $2i$ of the logarithm of the cross-ratio formed by these two straight lines and the two tangents from $A$ to the absolute.*

We will note that *if the curvature $1/R^2$ of the elliptic plane is equal to 1, there is perfect duality between the concept of distance and the concept of angle*; the distance of two points transforms by duality into the angle of two straight lines.

**123.** We can offer to look for the equation of a circle. Let $a, b, c$ be the normal coordinates of a point $A$ and $x, y, z$ the normal coordinates of a point $M$ situated at the distance $r$ from $A$; we have

$$ax + by + cz = R^2 \cos\frac{r}{R};$$

this is the equation we sought. We can make It homogeneous by squaring it and taking into account that the coordinates used are normal; we thus obtain

$$(ax+by+cz)^2 = \cos\frac{r}{R}\,(x^2+y^2+z^2)(a^2+b^2+c^2).$$

We see that *any circle is represented on the elliptic plane by a conic bitangent to the absolute*; the line of contact is the line $ax+by+ax=0$, that is to say *the polar of the centre of the circle*, both with respect to the absolute and with respect to the circle Itself.

**124.** Finally let us look for the analytic expression of the $ds^2$ of the elliptic plane. We will perform the central projection of the sphere onto the plane $(P)$ tangent at one of Its points $A$ and we will define analytically a point $M$ of the plane $(P)$ by Its rectangular coordinates $X, Y$ referred to two axes with origin $A$. Thus let $d\sigma = \sqrt{dX^2+dY^2}$ be the ordinary distance of two infinitely close points $M$ and $N$ of the plane $(P)$ (Figure 3), let $\alpha = ds/R$ be the angle at the centre $MON$. Finally denote by $\varphi$ the angle $ONM$. We have

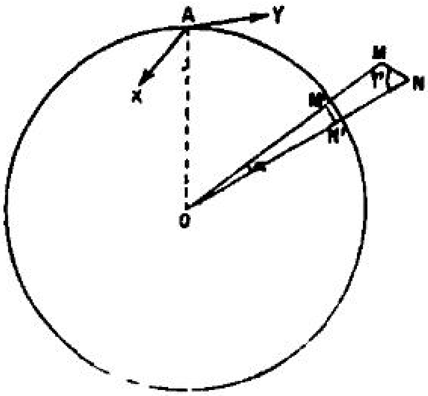


Figure 3

$$\frac{MN}{\alpha} = \frac{OM}{\sin\varphi},$$

from which

$$ON.MN\ \sin\varphi = \alpha.OM.ON = \frac{ds}{R}(X^2+Y^2+R^2).$$

The product $ON.MN\ \sin\varphi$ Is equal to double the area of the triangle $OMN$, or again it is the measure of the bivector determined by the two vectors $OM$ and $MN$, the projections of these two vectors being respectively

$$\begin{array}{ccc} X, & Y, & R \\ dX, & dY\,, & 0; \end{array}$$

consequently

$$ON.MN \ \sin\varphi = \sqrt{(X\,dY - Y\,dX)^2 + R^2(X^2+Y^2)}\ .$$

We thus have finally

$$ds^2 = R^2\ \frac{R^2(X^2+Y^2) + (X\,dY - Y\,dX)^2}{(X^2+Y^2+R^2)^2}.$$

Put $1/R^2 = K$; the preceding formula can be written as

$$ds^2 = \frac{X^2+Y^2+K(X\,dY - Y\,dX)^2}{[1+K(X^2+Y^2)]^2}. \tag{6.4}$$

It highlights the property already pointed out that the Euclidean metric of the plane $(P)$ $(d\sigma^2 = dX^2 + dY^2)$ is the osculator at $A$ to the metric of the elliptic plane. The linear element obtained Is thus defined by the double property that the straight lines are represented by equations linear in $X$, $Y$ and that the Euclidean plane with rectangular coordinates $(X, Y)$ is the osculator to the elliptic plane at the origin of coordinates.

It Is good to note that the linear element found applies to the entire elliptic plane, *with the exception of the points situated on the polar line of the point* $X = Y = 0$ (which corresponds to infinite values of $X$ and $Y$).

## III.– Hyperbolic geometry in two dimensions

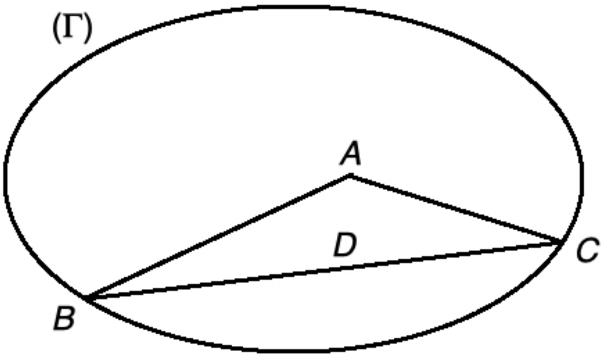
**125.** The formulae that translate the geometric properties of figures drawn on the elliptic plane of curvature $1/R^2$ contain a positive parameter $K = 1/R^2$. If we give this parameter a negative value $K = -1/R^2$, we obtain the *hyperbolic* geometry. We can define it directly by starting, in the projective plane, from a *real* conic $\Gamma$ (the absolute) and calling (Cayley, or non Euclidean) *distance* between two points $M$ and $M'$ of the plane the product by $R/2$ of the logarithm of the cross ratio formed by the two given points and the two points where the straight line that joins them cuts the absolute. If we want this distance to be real for all line segments issuing from a point $M$, it is necessary (and sufficient) that this point be situated in the *interior* of the absolute. *The hyperbolic plane is thus the manifold formed by the points interior to the absolute* $\Gamma$*.* When, with the point $M$ remaining fixed, the point $M'$ tends towards a point of the absolute, we see immediately that the (Cayley) distance between the two points increases indefinitely. The absolute is thus the locus of points at infinity.

The angle of two straight lines issuing from a point $A$ is, as in the elliptic plane, the quotient by $2i$ of the logarithm of the cross ratio formed by the two given straight lines and the two tangents drawn from the point $A$ to the absolute. In particular, two straight lines are perpendicular (in the sense of Cayley) if they are conjugates with respect to the absolute, that is to say if one of them passes through the pole of the other (a pole which in reality does not exist in the hyperbolic plane properly so called).

The geodesics of the hyperbolic plane are clearly the straight lines, because the calculation which leads to these geodesics is the same as that which we would perform in the elliptic plane (with the only difference that the positive parameter $K$ is negative here).

The postulate of Euclid is not true in hyperbolic geometry; through a point $A$ situated outside of a straight line $D$ we can draw an infinity of straight lines that do not intersect the line $D$; among these, two are limits; these are the two *parallels of Lobachevsky*, which join the point $A$ to two points $B$ and $C$ of intersection of the straight line $D$ with the absolute (Figure 4).

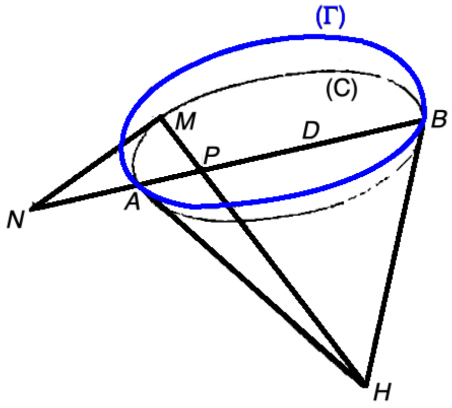
**Figure 4**



**126.** We saw above that the perpendiculars raised at different points of a straight line $D$ form, *in the complete projective plane*, a pencil of lines whose vertex is the pole $H$ of the straight line $D$ with respect to the absolute. The straight line $D$ is thus an orthogonal trajectory (in the non Euclidean sense) of this pencil. It is easy to have others. Consider in fact (Figure 5) a conic $(C)$ bi-tangent to the absolute at points $A$ and $B$ where the straight line $D$ intersects $(\Gamma)$; let $M$ be a point of this conic; the tangent at $M$ passes through the pole $N$ of the straight line $HM$ with respect to the conic $(C)$, a pole which is on the straight line $AB$ and which is also the pole of $HM$ with respect to $\Gamma$. The tangent $MN$ to $(C)$ is thus (from the non Euclidean point of view) normal to the ray $HM$ of the pencil; the conic $(C)$ is thus indeed an orthogonal trajectory of the straight lines perpendicular to $AB$.

On the other hand, we know (n° 95) that if in any Riemannian space of two dimensions, we place onto the geodesics normal to a fixed geodesic a constant length, the locus of points thus obtained is normal to all these geodesics. Consequently *the conic $(C)$ is the locus of points obtained by placing onto the perpendiculars to the straight line $D$ a constant length*: we call it *the line of equal distance* or *equidistant* or again *hypercycle*.

**Figure 5**



If, instead of taking in the projective plane a pencil of straight lines with summit exterior to the absolute, we take a pencil of straight lines issuing from a point $A$ interior to the absolute, the orthogonal trajectories are clearly *non Euclidean* circumferences with centre $A$. Consequently, *non Euclidean circumferences are, in the hyperbolic plane, conics bi-tangent to the absolute, where the straight line of contact is exterior to the absolute, and the pole of this straight line of contact is the centre of this circumference.*

An intermediate case is that of a pencil of straight lines having as vertex a point $\Pi$ on the absolute. The orthogonal trajectories are conics that admit at $\Pi$ a contact of third order with the absolute: we call them the *horocycles*: *these are thus basically orthogonal trajectories of a family of Lobachevsky parallels.*

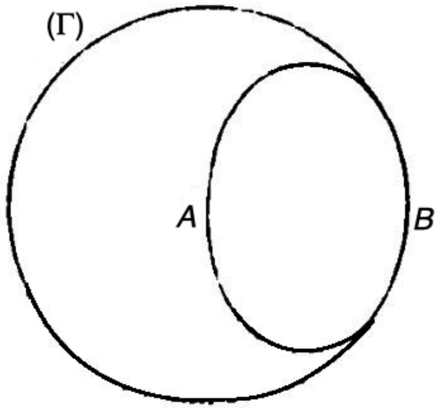
If we imagine a point $A$, and a straight line $D$ passing through $A$, the conics bi-tangent to the absolute that pass through $A$ and are tangent at $A$ to the straight line $D$ contain three categories of distinct curves: circumferences, two horocycles and hypercycles.

To appreciate how the curvature of these curves varies, let us represent the hyperbolic geometry on the plane in such a way that in *ordinary rectangular coordinates*, the equation of the absolute is

$$X^2 + Y^2 - R^2 = 0;$$

at the origin $A$ of coordinates, the Euclidean metric of the plane osculates its (hyperbolic) Cayley metric. The absolute is a circle of radius $R$ (Figure 6). If we picture a horocycle that passes through $A$ and touches $\Gamma$ at $B$, the radius of curvature (in the ordinary sense) of this conic at $B$ is the same as at $A$; consequently, the radius of curvature at $A$ is equal to $R$. So finally *any horocycle has radius of curvature* $R = 1/\sqrt{-K}$. We see then *that any circumference has a radius of curvature less than* $R$ *and any hypercycle a radius of curvature greater than* $R$.

**Figure 6**



Moreover we know, by analogy with what we have seen in elliptic geometry, that the ordinary curvature of circles with centre $A$ is equal to their non Euclidean curvature; it is thus always greater than $1/R$.

We arrive at the same result by starting from the formula

$$\rho = R \tan \frac{r}{R} \ ,$$

which gives the radius of geodesic curvature $\rho$ of a circle drawn on the sphere and of radius (reckoned on the surface of the sphere) equal to $r$. If we pass from elliptic geometry to hyperbolic geometry, the formula becomes

$$\rho = R \tanh \frac{r}{R} \ ;$$

this shows that $\rho$ increases from zero to $R$ as $r$ increases from zero to $+\infty$. We obtain similarly the curvature of equidistants by noting that, on the sphere, the circle which is the locus of points situated at distance $a$ from a great circle has radius $\frac{\pi R}{2} - a$ and as geodesic radius of curvature

$$\rho = R \cot \frac{a}{R} \ ;$$

in hyperbolic geometry, the radius of curvature of the locus of points situated at distance $a$ from a fixed straight line is thus

$$\rho = R \coth \frac{a}{R} \ ;$$

it is always greater than $R$.

**127.** The formula which gives the area of a spherical triangle

$$A + B + C - \pi = \frac{S}{R^2} = KS$$

remains valid in hyperbolic geometry; the sum of the angles of a triangle is smaller than two right angles and the difference $\pi - (A + B + C)$ is equal to $S/R^2$. The area of a triangle thus cannot exceed the value $\pi R^2$.

This limit is attained by a triangle whose three vertices lie on the absolute; the three sides of this triangle are pairwise (Lobachevsky) parallels.

**128.** Calculation of the linear element of the hyperbolic plane leads to a result analogous to that which was obtained for the elliptic plane. If we adopt in the plane (homogeneous) projective coordinates such that the equation of the absolute is

$$F(x, y, z) = 0,$$

and if we suppose, which is always allowed, that the points interior to the absolute make $F$ negative, we will have

$$ds^2 = F(dx, dy, dz),$$

provided that $x, y, z$ are subject to the condition

$$F(x, y, z) = -R^2 = \frac{1}{K} \ .$$

We can also point out the form already reported (n$^{\text{o}}$ 124)

$$ds^2 = \frac{dX^2 + dY^2 + K(X\,dY - Y\,dX)^2}{[1 + K(X^2 + Y^2)]^2} \ .$$

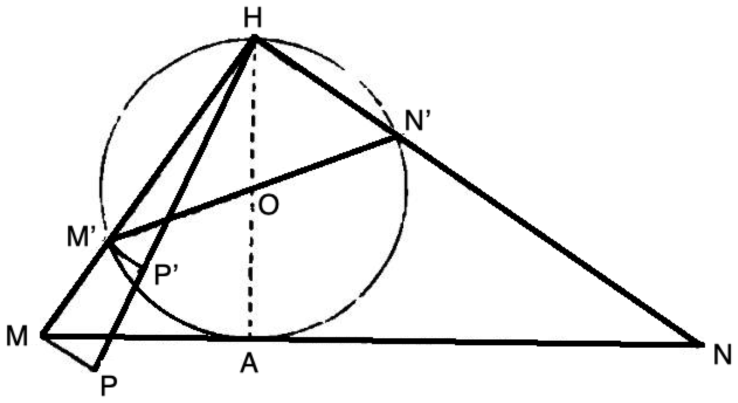
## IV.– Conformal representation of spherical and hyperbolic geometries

**129.** In the above we used the geodesic representation of spherical, elliptical and hyperbolic geometry, a representation in which lines have lines as images. Another very important representation is that which conserves angles.

We get there easily, as far as spherical geometry is concerned, by performing a stereographic projection of the sphere onto the plane; it is not necessary then to consider this plane as a projective plane, but as the plane of the theory of functions, which has only one point at infinity (that which corresponds on the sphere to the pole $H$ of the stereographic projection). Preferably take as the plane of projection the plane tangent to the sphere at the point diametrically opposite the point $H$: we will thus have in fact an Euclidean plane at $A$ osculating the Riemannian space of two dimensions formed by the surface of the sphere.

In this representation of the spherical geometry, a circle is represented by a circle, a line (a great circle) also by a circle, but having a characteristic property, namely that *the power of the point $A$ with respect to this circle is constant and equal to* $-4R^2$. We have, in fact, according to Figure 7,

**Figure 7**



$$AM = 2\,R\,\tan\widehat{MHA},$$
$$AN = 2\,R\,\tan\widehat{NHA},$$

from which

$$AM.AN = -4R^2.$$

We can say furthermore that lines are represented by circles that are orthogonal to the (imaginary) circle with centre $A$ and radius $2iR$; this circle is called the *absolute*; its equation is, in rectangular coordinates,

$$X^2 + Y^2 + 4R^2 = 0.$$

It is the intersection of the plane with the isotropic cone with vertex $H$.

The linear element of the sphere, with the rectangular coordinates $X, Y$ of the stereographic projection, is easy to determine. If, in fact, $M$ and $P$ are two infinitely close points of the plane, arising from two points $M', P'$ of the sphere (Figure 7), we have

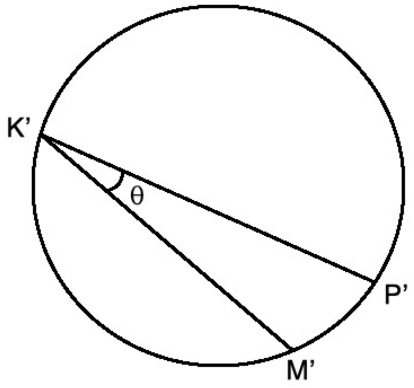
$$\frac{MP}{M'P'} = \frac{HM.HP}{4R^2}\ ,$$
$$\frac{\sqrt{dX^2 + dY^2}}{ds} = \frac{X^2 + Y^2 + 4R^2}{4R^2} = 1 + \frac{K}{4}\ (X^2 + Y^2),$$

$$ds^2 = \frac{dX^2 + dY^2}{\left[1 + \dfrac{K}{4}\ (X^2 + Y^2)\right]^2}\ ; \tag{6.5}$$

we verify that the Euclidean $ds^2$, $dX^2 + dY^2$, osculates at $A$ the spherical $ds^2$.

**130.** To define directly, in the *conformal* plane, the distance between two points ? and ? by means of the absolute, let us consider the circle orthogonal to the absolute that passes through $M$ and $P$, and let $Q_1$ and $Q_2$ be the two (conjugate imaginary) points where it meets the absolute. Since the four points $M, P, Q_1, Q_2$ are situated on the same circle, they admit on this circle a certain cross ratio. To evaluate it, note that these four points arise from the projection of four points $M', P', Q_1', Q_2'$ of a great circle of the sphere; now the absolute Is the trace, on the plane of projection, of the isotropic cone with vertex $H$, a cone which meets the sphere following the umbilical;[3] the points $Q_1'$ and $Q_2'$ are thus the two cyclic points at infinity $I$ and $J$ of the great circle $M'P'$ of the sphere. If then we take any point $K'$ on this great circle (Figure 8), the cross ratio of the four lines $K'M', K'P', K'I, K'J$ is equal to $e^{2i\theta}$, where we denote by $\theta$ the angle $M'K'P'$, or also to $e^{id/R}$, where we denote by $d$ the distance, calculated on the sphere, between the two points $M', P'$. We thus have, on the plane of projection, for the (spherical) distance between two points $M, P$, the formula

**Figure 8**



$$d = \frac{R}{i} \ \ln(MPQ_1Q_2). \tag{6.6}$$

**131.** To obtain a conformal representation of hyperbolic geometry, let the absolute $(\Gamma)$ on a plane $(\Pi)$ be given. There exist an infinity of points $H$ such that the cone with vertex $H$ and base $(\Gamma)$ is one of revolution (in the ordinary sense of the word). Take one of these points and inscribe in the cone a sphere $(\Sigma)$ (Figure 9); the cone touches it in a circle $(\Gamma')$ which partitions the sphere into two caps. This established, we shall let correspond to each point $M$ (interior to the absolute) in the plane $(\Pi)$ the point $M'$ where the line $HM$ meets one of the two caps, chosen once and for all; we obtain in this way a representation of the

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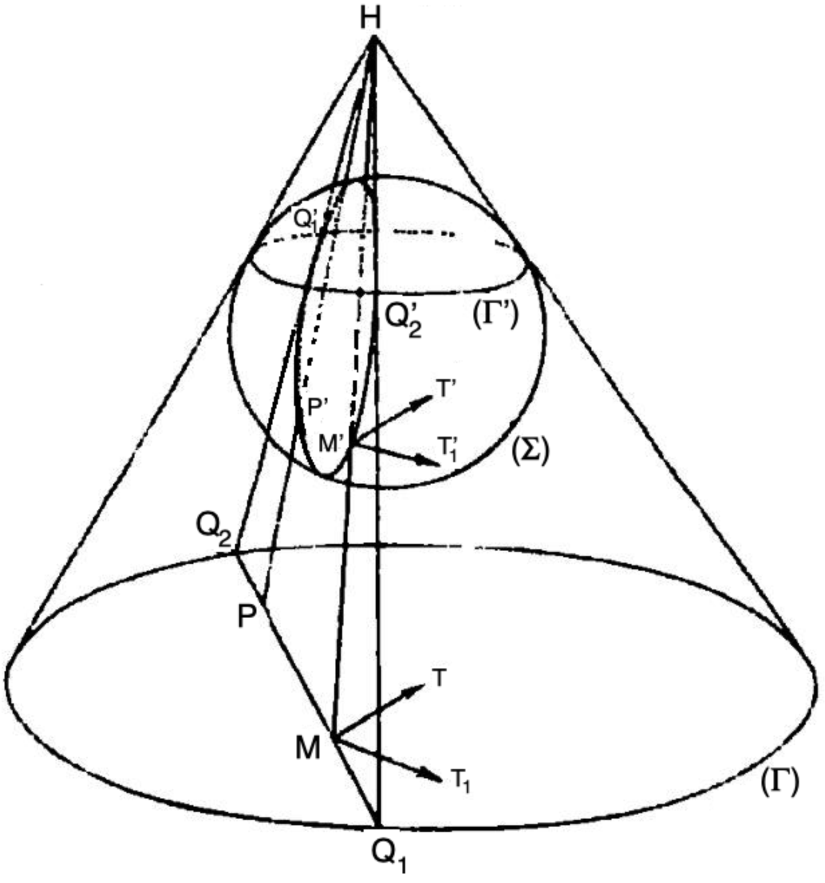
[3] "une conique imaginaire situee dans le plan de l'infini, qui est dite l'ombilicale, ou cercle imaginaire de l'infini, et par laquelle passent toutes les spheres de l'espace." (Cours de gomtrie, pure et applique de l'cole polytechnique, par Maurice d'Ocagne. Ocagne, Maurice d', 1862-1938.)

**Figure 9**



hyperbolic plane on the cap considered of the sphere; the points of $(\Gamma)$ project onto $(\Gamma')$, the circle $(\Gamma')$ will be called moreover the *absolute.*

Let us prove that this representation conserves angles, that is to say, that the (ordinary) angle at which two curves issuing from $M'$ on the sphere intersect is equal to the (non Euclidean) angle at which the corresponding curves intersect on the plane $(\Pi)$. Let $MT, MT_1$ be on the plane $(\Pi)$; $M'T', M'T_1'$, on the plane tangent at $M'$ to the sphere, the tangents to the curves considered. The non Euclidean angle $\widehat{TMT_1}$ depends on the cross ratio of the lines $MT, MT_1$ and the two tangents from $M$ to $(\Gamma)$; this cross ratio is the same as that of the lines $M'T', M'T_1'$ and the two tangents from $M'$ to the curve of section of the cone with vertex $H$ by the plane tangent at $M'$ to the sphere. Now, according to the theorem of Dandelin, this section is a conic where $M'$ is one of the foci; consequently the two tangent issuing from $M'$ are the ordinary isotropic lines issuing from $M'$ in the tangent plane. Consequently finally the ordinary angle $\widehat{T'M'T_1'}$

is equal to the (non Euclidean) angle $\widehat{TMT_1}$. This is what needed to be proved.

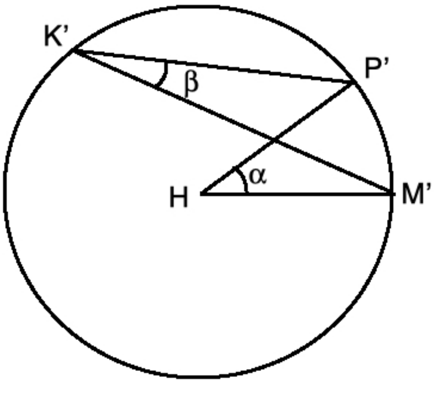
**132.** In the conformal representation considered, a line of the hyperbolic plane has as image the section of the sphere by a plane that passes through $H$; this section Is a circle *orthogonal to the absolute* $(\Gamma')$ at points $Q'_1$ and $Q'_2$, the projections of the points $Q_1$ and $Q_2$, where the line meets the absolute. If we take on this circle two points $M'$ and $P'$, arising from two points $M$ and $P$ of the line considered In the hyperbolic plane, we can propose to evaluate the (non-Euclidean) distance between these two points by means of the cross ratio $(M'P'Q'_1Q'_2)$ of the two given points $M', P'$ and the two points $Q'_1, Q'_2$ (Figure 9). Now we have (n° 125)

$$d = \frac{R}{2}\ \ln(MPQ_1Q_2) = \frac{R}{2}\ \ln(H.M'P'Q'_1Q'_2);$$

it is thus about comparing the cross ratio $(M'P'Q'_1Q'_2)$ to the cross ratio $(H.M'P'Q'_1Q'_2)$.

To make this comparison, perform an homographic transformation which sends the points $Q'_1$ and $Q'_2$ out to the two cyclic points at infinity; the point $H$ then becomes the centre of the circle (Figure 10) and we have

**Figure 10**



$$(H.M'P'Q'_1Q'_2) = e^{2i\alpha},$$

where $\alpha$ denotes the angle $\widehat{M'HP'}$. On the other hand. If $K'$ is any point on the circumference, we have

$$(M'P'Q'_1Q'_2) = (H.M'P'Q'_1Q'_2) = e^{2i\beta},$$

where $\beta$ denotes the angle $\widehat{H'K'P'}$. It follows that

$$(H.M'P'Q'_1Q'_2) = (M'P'Q'_1Q'_2)^2\,,$$

and consequently

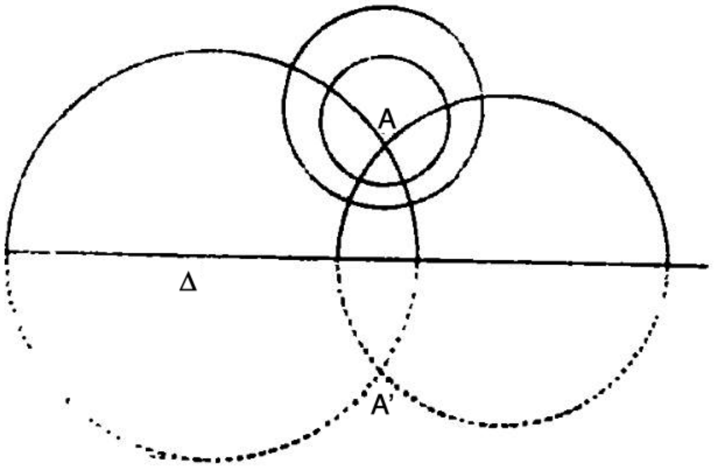
$$d = \frac{R}{2}\ \ln(H.M'P'Q'_1Q'_2) = R\ \ln(M'P'Q'_1Q'_2);$$

*this formula is the counterpart of formula (6.6) found in spherical geometry.*

**133.** It is now easy to see that the (non Euclidean) circumferences, the horocycles and the hypercycles of the plane (Π) are represented on the sphere by the circumferences (or arcs of circumferences). In fact, the cone with vertex $H$ that has as its base one of these curves, is bi-tangent to the cone of revolution with vertex $H$ circumscribed on the sphere; it is thus bi-tangent to the sphere; consequently *its intersection with the sphere decomposes into two plane curves*, that is to say, into two circles. If we begin from a circumference of the plane (Π), we obtain on the sphere two circles, but where only one is completely interior to the cap in use. If on the other hand we begin from an equidistant, we obtain two circles which intersect on $(\Gamma')$ and where we need only keep those arcs situated in the interior of the cap in use; these two arcs of a circle correspond to two parts of the equidistant separated by Its points of contact with the absolute.

**134.** We can move on now a conformal representation of the hyperbolic plane on the ordinary plane by performing an inversion that has its pole at a point of the sphere. If this pole is taken in the interior of the unused cap, the points of the hyperbolic plane are represented by the points interior to a certain real circle (the *absolute*), with the lines represented by the circles orthogonal to the absolute.

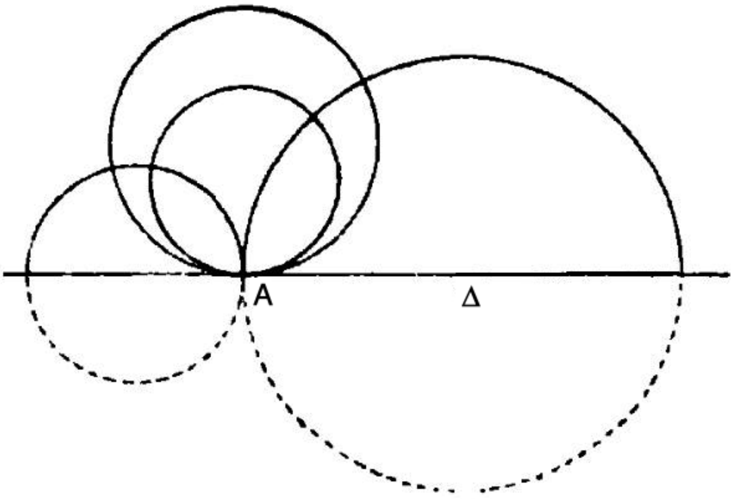
**Figure 11**



A more Interesting representation, and one that has been used by H. Poincare in his theory of Fuch functions, consists in performing an inversion whose pole is on the absolute $(\gamma')$. This absolute thus becomes a line and the points of the hyperbolic plane are represented by the points of one of the two half-planes bounded by the line (*the Poincare half-plane*). If we denote this line (absolute) by $\Delta$, the lines are represented by the semi-circumferences having their (ordinary) centre on $\Delta$ and situated on the Poincare half-plane.

We recover easily the property of the non-Euclidean circumferences of being represented by circumferences. Consider in fact the pencil of circumferences orthogonal to $\Delta$ and passing through a point $A$ of the Poincare half-plane (and the point $A'$ symmetric to $A$ with respect to $\Delta$).

**Figure 12**

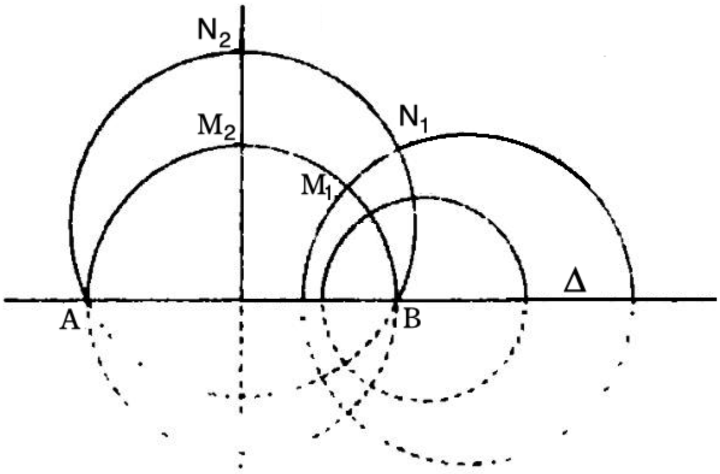


The orthogonal trajectories of this pencil will represent the non Euclidean circumferences whose (non Euclidean) centre is at $A$; we know that they form a pencil of circumferences where $A$ and $A'$ are the limit points (the points of Poncelet) (Figure 11).

Take now the bundle of circles (*parallel* non Euclidean lines) that pass through a fixed point $A$ of $\Delta$ and have their centres on $\Delta$; their orthogonal trajectories from another pencil of circles tangent at $A$ at $\Delta$ (Figure 12); they represent the horocycles normal to the parallel lines considered.

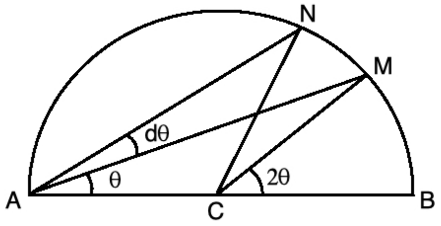
Finally take a pencil of circles that have their centres on $\Delta$ and admit two limit points $A$ and $B$ on $\Delta$ (Figure 13); their orthogonal trajectories are the circles that pass through $A$ and $B$, which thus represent the hypercycles; among them we find a (non Euclidean) line represented by the semi-circle of diameter $AB$; the segments of (non Euclidean) line $MN, M_1N_1, M_2N_2$ perpendicular to the (non Euclidean) line $AB$ are of constant length.

Figure 13



**135.** If we refer the Poincare half-plane to a system of rectangular coordinates, where the line $\Delta$ is taken as the axis $OX$, we can easily find the analytic expression of the linear element of the hyperbolic plane. Consider in fact two nearby points $M$ and $N$ (Figure 14); let $A$ and ? be the points where it cuts $\Delta$. We can express rationally the coordinates of a point $M$ of the circumference by means of a parameter $\theta$ that denotes the angle $BAM$. The values of $t$ corresponding to the points

Figure 14



$$M, \quad N, \quad A, \quad B$$

are respectively

$$\tan\theta, \quad \tan(\theta + d\theta), \quad \infty, \quad 0,$$

and we have

$$(MNAB) = \frac{\tan(\theta + d\theta)}{\tan\theta} = 1 + \frac{d\tan\theta}{\tan\theta} = 1 + \frac{2\,d\theta}{\sin 2\theta}\ ;$$

consequently

$$ds = R \ln \left( 1 + \frac{2\, d\theta}{\sin 2\theta} \right) .$$

Let then $r$ be the ordinary radius of the circle $AMNB$, and $C$ its centre; we have

$$\frac{d(2\theta)}{\sin 2\theta} = \frac{r\, d(2\theta)}{r\, \sin 2\theta} = \frac{\sqrt{dX^2 + dY^2}}{Y} ;$$

consequently

$$ds^2 = R^2 \, \frac{dX^2 + dY^2}{Y^2} = -\frac{1}{K} \, \frac{dX^2 + dY^2}{Y^2} . \tag{6.7}$$

**136.** In summary, we have found, for elliptic geometry, the noteworthy linear element

$$ds^2 = \frac{dX^2 + dY^2 + K(X\, dY - Y\, dX)^2}{[1 + K(X^2 + Y^2)]^2} ; \tag{6.4}$$

The coordinates $X, Y$ are chosen In such a way that every line is represented by an equation of first degree; all the points of the space are represented analytically by means of these coordinates, except for the points of one line, namely the polar line of the point $(X = Y = 0)$.

Spherical geometry admits the linear element

$$ds^2 = \frac{dX^2 + dY^2}{\left[ 1 + \frac{K}{4}(X^2 + Y^2) \right]^2} ; \tag{6.5}$$

the coordinates $X, Y$ are chosen so as to realise on the Euclidean plane a conformal representation; all the points of the sphere are represented analytically by means of these coordinates, except for one only, the antipode of the point $X = Y = 0$.

Hyperbolic geometry admits three linear elements of note, namely the two which we obtained by giving, in the above formulae, a negative value to $K$; in both cases, all the points of the hyperbolic plane are represented analytically by the coordinates $X, Y$, provided that they satisfy, in the first case, the inequality

$$1 + K(X^2 + Y^2) > 0;$$

and in the second case, the inequality

$$1 + \frac{K}{4}(X^2 + Y^2) > 0.$$

There exists furthermore one other noteworthy form of the linear element (conformal representation on the Poincare half-plane), namely

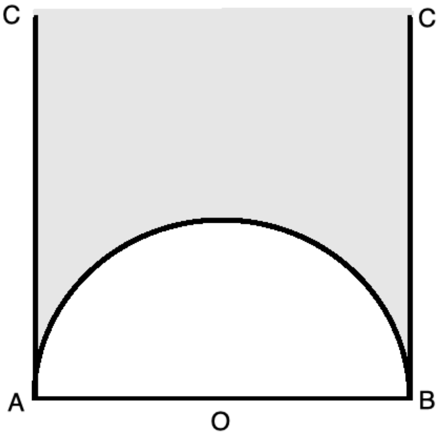
$$ds^2 = -\frac{1}{K} \, \frac{dX^2 + dY^2}{Y^2} ; \tag{6.7}$$

all the points are represented analytically by the coordinates $X, Y$ (with the condition $Y > 0$). The (non Euclidean) element of area is

$$d\sigma = R^2\,\frac{dX\,dY}{Y^2}\ , \tag{6.8}$$

and it is easy to show that a triangle whose three vertices are at infinity on the absolute, as indicated in Figure 15, and which is bounded by a semi-circumference with centre $O$ and of (ordinary) radius $a$ and by two half lines $AC$ and $BC$, has area $\pi R^2$. If in fact, we have $OA = OB = a$, we need to calculate the integral

**Figure 15**



$$I = R^2 \iint \frac{dX\,dY}{Y^2}$$

extended to the domain defined by the inequalities

$$-a \le X \le a,$$
$$X^2 + Y^2 \ge a^2;$$

the integration gives

$$I = R^2 \int_{-a}^{+a} \frac{dX}{\sqrt{a^2 - X^2}} = \pi R^2.$$

we see clearly in the Figure that the three angles of this triangle are zero.

## V.– The group of displacements of non Euclidean geometries

**137.** Each of the three non Euclidean spaces of two dimensions (spherical, elliptic, and hyperbolic) satisfy the axiom of free mobility and admit a continuous family of direct isometric transformations (non Euclidean displacements) that depend on three parameters; it admits also a three parameter family of inversive isometries.

Leave aside the case of the sphere, which is well known, as well as that of the elliptic plane, and let us consider only hyperbolic displacements.

If we represent the non Euclidean plane (of Lobachevsky) geodesically on the projective plane, we see immediately that *every homographic transformation which conserves the absolute represents an isometric non Euclidean transformation.* It Is moreover not necessary to specify that the homography conserves the set of points interior to the absolute, because these points are characterised by the invariant projective property that the tangents from one of these points to the absolute are imaginary.

We can, as we know, express the homogeneous coordinates of a point of the absolute conic as real rational functions of a real parameter $t$; every homographic transformation that conserves the absolute will establish a homographic transformation of the parameter $t$. Conversely, every homography on the absolute entails a homography of the entire plane; this is due to the fact that every point of the plane is completely defined by the points of contact of the tangents issuing from this point to the absolute and every line of the plane by its points of intersection with the absolute. This is also clear by calculation if we choose, which is always possible, the system of projective coordinates of the plane by the condition that we have on the absolute,

$$\frac{x}{t^2} = \frac{y}{t} = \frac{z}{1} \ .$$

If we perform on the absolute the homography

$$t' = \frac{at + b}{a't + b'} \ ,$$

we will deduce, in the plane itself,

$$\begin{aligned} \rho x' &= a^2 x + 2aby + b^2 z, \\ \rho y' &= aa'x + (ab' + ba')y + bb'z, \\ \rho z' &= a'^2 x + 2a'b'y + b'^2 z. \end{aligned}$$

We have then, to classify non Euclidean displacements, only to classify the real homographic transformations of one variable. We have first of all a major division, according as we suppose that $ab' - ba'$ is positive or negative. The first case corresponds to displacement properly so called, the second case to displacements followed by a symmetry.

**138.** *Displacements properly so called* will be classified according to the nature of the double points of the homography established on the absolute.

1° *The double points are imaginary.* – The displacement then leaves invariant the real line that joins the two double points (a line exterior to the absolute) as well as its pole $A$ (interior to the absolute). The displacement thus obtained is clearly a *rotation around the fixed point $A$.* In a continuous rotation around $A$, the different points of the plain describe (non Euclidean) circumferences with centre $A$.

2$^o$ *The double points are real and distinct.* – If $A$ and $B$ are these two double points, the displacement leaves invariant the line $AB$, each point of this line displaces by a segment of constant (non Euclidean) length. We have here what could be called a (non Euclidean) *translation.* In a continuous translation with axis $AB$, the points that are not on the axis do not describe lines, but hypercycles (of constant curvature less than $1/R$).

1$^o$ *The double points coincide.* – If $A$ is the unique double point, the corresponding displacement leaves fixed the point $A$ and consequently transforms among themselves the lines of the pencil of parallels of Lobachevsky with summit $A$. In a continuous displacement around $A$, each point of the plane describes a horocycle, represented by a conic that hyperosculates the absolute at $A$.

*Displacements followed by symmetry* can occur only if the homography of the conic has real and distinct double points. The equation that gives the double points is

$$a't^2 + (b' - a)t - b = 0;$$

its discriminant is

$$(b' - a)^2 + 4ba' = (b' + a)^2 - 4(ab' - ba') > 0.$$

If $A$ and $B$ are the double points of the homography, the corresponding transformation results from a translation of the axis $AB$ followed by a symmetry with respect to $AB$.

**139.** Let us move on now to the conformal representation and take, to simplify, the representation on the Poincare half plane. According to the definition obtained for the non Euclidean distance between two points, every transformation that will exchange circles while conserving the absolute $\Delta$ will be an isometric transformation. No, in the *conformal plane*, there exist two classes of point transformations that changes circles into circles (the *Kreisverwandschaften* of Möbius). Represent a point of the plain by its affixed $z = X + iY$ with respect to two rectangular coordinate axes (we will choose $\Delta$ as real axis). The first class of transformations (direct transformations) is define by the formula

$$z = \frac{\alpha z + \beta}{\alpha' z + \beta'} \ ,$$

where $\alpha, \beta, \alpha', \beta'$ are arbitrary *complex* constants. The second class (inverse transformations) is defined by

$$z = \frac{\alpha z_0 + \beta}{\alpha' z_0 + \beta'} \ ,$$

where $z_0$ denotes the quantity $X - iY$ the conjugate imaginary of $z$.

The transformations of the first class that leave the real axis invariant *and* *which conserve each of the two half planes* bounded by $\Delta$ are obtained by giving $\alpha, \beta, \alpha', \beta'$ real values $a, b, a', b'$:

$$z = \frac{az + b}{a'z + b'} \ . \tag{6.9}$$

The corresponding non Euclidean displacements are still classified according to the nature of the double points of the homography (6.9).

If the double points are (conjugate) imaginaries, those are the affixes of a point $A$ of the Poincare half plane and of the point $A'$ symmetric with respect to $\Delta$. The displacement is a (non Euclidean) *rotation* around $A$.

If the double points are real, they are the affixes of two points $A$ and $B$ situated on the absolute. The corresponding displacement is a translation having as axis the semi circle (non Euclidean line) described on $AB$ as diameter.

If finally the double points merge, we obtain on the absolute a point $A$ and we have a displacement that transforms between themselves the (non Euclidean) lines, that is to say, the circles orthogonal to $\Delta$ and passing through $A$. If in particular the point $A$ is taken to infinity, the corresponding pencil of Lobachevsky parallels is represented by the ordinary pencil of parallels to $OY$: the horocycles which are the orthogonal trajectories are represented by the parallels to $OX$, and the corresponding non Euclidean displacement is represented by an ordinary translation parallel to $OX$.

Let us move on to displacements followed by a symmetry; these are represented analytically by formulae of the form

$$z = \frac{az_0 + b}{a'z_0 + b'} \ , \tag{6.10}$$

where the coefficients $a, b, a', b'$ are real and such that $ab' - ba' < 0$ (so that the two half planes are conserved by the transformation). Such a transformation leaves invariant two real points of the absolute and we recover the interpretation indicated earlier: translation followed by a symmetry with respect to the exist of the translation.

## VI.– Three dimensional non Euclidean spaces: projective representation

**140.** We could begin, as we did for $n = 2$, from the *spherical space*, where each point is defined by four coordinates $x, y, z, t$ that satisfy the relation

$$x^2 + y^2 + z^2 + t^2 = R^2,$$

and the linear element is

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2.$$

Such a space is said to be of curvature $1/R^2$.

Let us approach things from a general projective point of view. Consider, in a projective space of three dimensions referred to projective coordinates $x, y, z, t$, a quadric (the absolute) with equation

$$f(x, y, z, t) = 0.$$

Multiply the projective coordinates of a point by a factor such that we have

$$f(x, y, z, t) = \frac{1}{K}\,, \tag{6.11}$$

where $K$ is a given constant, and define the linear element

$$ds^2 = f(dx, dy, dz, dt). \tag{6.12}$$

For the Riemannian space thus defined to have real points and for its linear element to be a positive definite form, *it is necessary at the outset that the polar plane of the point* $(x, y, z, t)$ *with respect to the absolute does not intersect the absolute*; in fact the four quantities $dx, dy, dz, dt$ are subject to the single relation

$$f_x dx + f_y dy + f_z dz + f_t dt = 0,$$

which states that the point $(dx, dy, dz, dt)$ is in the polar plane of the point $(x, y, z, t)$. If the polar plane intersects the absolute, the differential form $ds^2$ will not have a constant sign.

*The above condition excludes at the outset ruled quadrics*, because every plane intersects such a quadric in a real curve.

There remain therefore only two possible hypotheses.

I. *The absolute is an imaginary quadric (with real equation).* It is necessary then that the form $f$ be *positive definite*; consequently the constant $K$ is necessarily positive. We obtain the *elliptic space of positive curvature* $K$.

II. *The absolute is a real quadric that is not ruled*, for example and ellipsoid. It is the planes exterior to the quadric that do not intersect it; so $f$ must be positive in the exterior to the quadric. Since the points $(x, y, z, t)$ of the Riemannian space are in the interior of the quadric (so that their polar planes are exterior), the constant $K$ is necessarily negative. We obtain the *hyperbolic space of negative curvature* $K$. The form $f$ is decomposable into three positive squares and one negative.

**141.** We can obtain directly the projective interpretation of the elementary distance between two points $M$ and $M'$. Every point of the line $MM'$ has coordinates of the form

$$x + \lambda\, dx, \qquad y + \lambda\, dy, \qquad z + \lambda\, dz, \qquad t + \lambda\, dt\,;$$

the values of $\lambda$ that correspond to its points of intersection $P_1, P_2$ with the absolute are given by the equation

$$f(x + \lambda\, dx, ..., t + \lambda\, dt) \equiv \frac{1}{K} + \lambda^2 ds^2 = 0,$$

from which

$$\lambda_1 = \frac{1}{\sqrt{-K}\ ds}\ , \qquad \lambda_1 = -\frac{1}{\sqrt{-K}\ ds}\ .$$

The cross ratio $(MM'P_1P_2)$ is equal to the cross ratio of the four values of $\lambda$ corresponding to the four points considered

$$(MM'P_1P_2) = (0, 1, \lambda_1, \lambda_2) = \frac{\lambda_1(1-\lambda_2)}{\lambda_2(1-\lambda_1)} = \frac{1-\dfrac{1}{\lambda_2}}{1-\dfrac{1}{\lambda_1}} = \frac{1+ds\sqrt{-K}}{1-ds\sqrt{-K}}\ ;$$

by passing to logarithms and confining them to their principal parts, we have

$$ds = \frac{1}{2\sqrt{-K}}\ \ln(MM'P_1P_2).$$

We can deduce, by integration along a segment of the finite line, the expression for the non Euclidean distance $d$ between two points $M$ and $M'$, *a distance counted along the line*,

$$d = \frac{1}{2\sqrt{-K}}\ \ln(MM'P_1P_2), \tag{6.13}$$

from which, by a calculation analogous to that of n° 122, the formula

$$\frac{1}{2}(x'\,f'_x + y'\,f'_y + z'\,f'_z + t'\,f'_t) = \frac{1}{K}\ \cos(d\sqrt{K}). \tag{6.14}$$

We will come later (n° 145) to the direct proof of the property of the line of being a geodesic.

**142.** In the above, we forced the projective coordinates of a point to satisfy the relation $f = 1/K$. We could consider more generally any set of four coordinates $x, y, z, t)$ not all zero and call this set an *analytic point*; two *analytic points* whose four coordinates are proportional, but not all equal, will be regarded as distinct, while occupying the same position in the space. We shall call the *scalar square* of such a point the quantity $f(x, y, z, t)$, and we shall cal more generally the *scalar product* of two analytic points $(x, y, z, t)$, $(x', y', z', t')$ the quantity

$$\frac{1}{2}\left(x'\ \frac{\partial f}{\partial x} + y'\ \frac{\partial f}{\partial y} + z'\ \frac{\partial f}{\partial z} + t'\ \frac{\partial f}{\partial t}\right).$$

If $\boldsymbol{M}$ and $\boldsymbol{N}$ denote any two analytic points, the scalar square of the point $\lambda\boldsymbol{M} + \mu\boldsymbol{N}$, whose coordinates are deduces from those of $\boldsymbol{M}$ and $\boldsymbol{N}$ by multiplication respectively by $\lambda$ and $\mu$ and addition, is equal to

$$\lambda^2\boldsymbol{M}^2 + 2\lambda\mu\boldsymbol{M}\cdot\boldsymbol{N} + \mu^2\boldsymbol{N}^2.$$

Two points whose scalar product is zero are conjugate to one another with respect to the absolute; each of the two is in the polar plane of the other.

The infinitesimal vector defined by two infinitely close analytic points $\boldsymbol{M}, \boldsymbol{M}'$ is defined analytically by the four quantities $dx, dy, dz, dt$. Let us restrict the coordinates of the points considered by the condition $f(x, y, z, t) = 1/K$ (in other words let us suppose that $\boldsymbol{M}^2 = 1/K$); the for numbers $dx, dy, dz, dt$ represent an analytic point of the projective space which is situated in the polar plane of $\boldsymbol{M}$ with respect to the absolute, and whose scalar square $f(dx, dy, dz, dt) = ds^2$ is equal to the square of the length of the vector. We shall call it the *the representative point of the vector.*

More generally, every vector issuing from a point $\boldsymbol{M}$ of the elliptic or hyperbolic space will be represented by an analytic point situated on the polar plane of $\boldsymbol{M}$ with respect to the absolute, and having as scalar square the square of the length of the vector; this representative point is moreover on the line issuing from $\boldsymbol{M}$ in the direction of the vector. We show easily that *the scalar product of two vectors issuing from $\boldsymbol{M}$ is equal to the scalar product of their representative points.*

**143.** The coordinates considered up until now are arbitrary projective coordinates. There are some that generalise the rectangular coordinates of the Euclidean space.

Start from any point $\boldsymbol{A}$ (with scalar square $1/K$) of the elliptic or hyperbolic space, and consider the representative points $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of three rectangular unit vectors issuing from $\boldsymbol{A}$. Every analytic point $\boldsymbol{M}$ of the projective space can be expressed in the form

$$\boldsymbol{M} = t\,\boldsymbol{A} + x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_+ z\,\boldsymbol{e}_3,$$

and we will have, by virtue of the obvious relations

$$\boldsymbol{A}\cdot\boldsymbol{e}_i = 0, \quad \boldsymbol{e}_i^2 = 1, \quad \boldsymbol{e}_i\cdot\boldsymbol{e}_j = 0 \quad (i \neq j; i, j = 1, 2, 3),$$

the following expression for the scalar square of $\boldsymbol{M}$:

$$\boldsymbol{M}^2 = x^2 + y^2 + z^2 + \frac{1}{K}\,t^2.$$

If the point $\boldsymbol{M}$ is a point of the elliptic or hyperbolic space with scalar square $1/K$:

$$x^2 + y^2 + z^2 + \frac{1}{K}\,t^2 = \frac{1}{K}\ ,$$

we will have for the linear element of the space,

$$ds^2 = d\boldsymbol{M}^2 = dx^2 + dy^2 + dz^2 + \frac{1}{K}\,dt^2.$$

The coordinates obtained in this way, subject to the relation

$$x^2 + y^2 + z^2 + \frac{1}{K}\,t^2 = \frac{1}{K}\ ,$$

carry the name, according to Killing, of *Weierstrass coordinates.*

If we make $K$ tend towards zero, the coordinate $t$ takes the value 1, and the $ds^2$, as we see easily by passing to the limit, reduces to $dx^2 + dy^2 + dz^2$.

The Weierstrass coordinates are defined only up to a sign. If in the elliptic space we agree to regard as *distinct* two points whose Weierstrass coordinates are equal and opposite, we will have the spherical space. In the case of the hyperbolic space, we could equally regard as distinct the two points $(x, y, z, t)$ and $(-x, -y, -z, -t)$; but *it is impossible to pass continuously from a point with coordinate $t$ positive to a point with coordinate $t$ negative*, because of the relation

$$t^2 = 1 - K(x^2 + y^2 + z^2) \geq 1.$$

the convention that allows us to pass from the elliptic space to the spherical space will here lead to a space that is *not connected*, that is to say, to two manifolds that are *completely separated* the one from the other. We exclude such a possibility. We see however, according to the above, that we can always suppose, in the hyperbolic space, that the coordinate $t$ is *positive.*

**144.** The passage from a system of Weierstrass coordinates to another is done by means of a linear substitution. If the first system is fixed and the second variable, we obtain the most general linear substitution that leaves invariant the form $x^2 + y^2 + z^2 + t^2/K$. All these substitutions also leave invariant the linear element

$$dx^2 + dy^2 + dz^2 + \frac{1}{K}\, dt^2;$$

they thus define the group of isometric transformations of the space. This group is of six parameters, the number of arbitrary quantities which enter into the most general system of Weierstrass coordinates, that is to say basically in the most general tetrahedron which is conjugate with respect to the absolute.

In the elliptic space, we obtain the same isometric transformation by changing all the signs of the coefficients; in the spherical space we obtain in on the contrary two distinct displacements. In the hyperbolic space, cannot be changed in sign if we restrict the coordinates $t$ and $t'$ to be positive.

We show easily that the determinant of the coefficients is equal to $+1$ or to $-1$. The displacements properly so called correspond to the case where the determinant of the coefficients of the substitution is equal to $+1$; the displacements followed by a symmetry to the case where the determinant is equal to $-1$.

**145.** It is easy to determine an Euclidean metric that osculates at a given point $A$ the metric of the elliptic or hyperbolic space. Take in fact a system of Weierstrass coordinates $x, y, z, t$ with origin $A$. In the neighbourhood of the point $A$, we have

$$\begin{aligned} t^2 &= 1 - K(x^2 + y^2 + z^2), \\ t &= 1 - \frac{1}{2}\, K(x^2 + y^2 + z^2) + \cdots . \end{aligned}$$

By expressing the linear element of the space by means only of the variables $x, y, z$, and by neglecting in the coefficients the terms of order greater than the first, there remains simply $dx^2+dy^2+dz^2$, which is an Euclidean linear element. We thus obtain a representation of the given space on one of its osculating Euclidean spaces at $A$ by representing the point $(x,y,z,t)$ by the point with rectangular coordinates $x,y,z$. We get another more convenient representation by introducing the non homogeneous coordinates

$$X=\frac{x}{t}, \qquad Y=\frac{y}{t}, \qquad Z=\frac{z}{t}\,;$$

we have, to the same degree of approximation as just now,

$$ds^2=dX^2+dY^2+dZ^2.$$

In the representation on this osculating Euclidean space at $A$, the absolute is represented by the *sphere*

$$X^2+Y^2+Z^2+\frac{1}{K}=0.$$

Since a line of the projective space (in which is localised the elliptic or hyperbolic space) is represented, in the osculating Euclidean space at one of its points $A$, by a line and that this line has a zero Euclidean curvature at $A$, it follows that all lines of the projective space have a zero (non Euclidean) curvature at each of its points and is consequently a geodesic of the non Euclidean space. Consequently the planes of the projective space are totally geodesic surfaces; the axiom of the plane is thus satisfied in the elliptic or hyperbolic space.

We see also another important property. Given a line issuing from $A$, the *parallel* line (in the sense of Levi-Civita) issuing from an infinitely close point $A'$ must be represented, in the osculating Euclidean space at $A$, by a line parallel (in the ordinary sense) to the first; consequently, in the projective space, *it intersects the first at a point of the plane* $t=0$ *polar to the point* $A$ *with respect to the absolute.* We thus obtain a simple geometric construction of this line.

We can easily calculate the $ds^2$ of the space in non-homogeneous coordinates $X,Y,Z$. The Weierstrass coordinates are supplied by the equations

$$x=tX, \qquad y=tY, \qquad z=tZ, \qquad \text{with } t^2=\frac{1}{1+K(X^2+Y^2+Z^2)}\ .$$

We have

$$ds^2=t^2\left[dX^2+dY^2+dZ^2+\right.$$
$$\left.+2\,\frac{dt}{t}(X\,dX+Y\,dY+Z\,dZ)+\frac{dt^2}{t^2}\left(X^2+Y^2+Z^2+\frac{1}{K}\right)\right].$$

By using the relation

$$\frac{dt}{t}=-\frac{K(X\,dX+Y\,dY+Z\,dZ)}{1+K(X^2+Y^2+Z^2)}\ ,$$

we find, after simplification,

$$ds^2=\frac{dX^2+dY^2+dZ^2+K\left[(YdZ-ZdY)^2+(ZdX-XdZ)^2+(XdY-YdX)^2\right]}{\left[1+K(X^2+Y^2+Z^2)\right]^2}. \qquad (6.15)$$

This formula generalises formula (6.4) established for spaces of two dimensions.

**146.** The generalisation of *Cartesian* coordinates is made naturally by starting from any point $A$ (with scalar square $1/K$) of the elliptic or hyperbolic space and from any three vectors issuing from $A$ and represented by any three analytic points $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of the polar plane of $A$ with respect to the absolute. If we put

$$\boldsymbol{M} = t\boldsymbol{A} + x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3,$$

we have

$$\boldsymbol{M}^2 = \frac{1}{K}t^2 + g_{11}x^2 + g_{22}y^2 + g_{33}z^2 + 2G_{23}yz + 2g_{31}zx + 2g_{12}xy,$$

where the coefficients $g_{ij}$ denote the scalar products $\boldsymbol{e}_i \cdot \boldsymbol{e}_j$.

The above Cartesian coordinates are, aw well as in the Euclidean space, suited to the theory of curvilinear coordinates. If we have chosen in a non Euclidean space of $n$ dimensions any system of coordinates $u^1, ..., u^n$, we will attach to each point $M$ of the space a system of Cartesian coordinates defined by their origin $M$ (with scalar square $1/K$) and the representative points of the vectors $\boldsymbol{e}_i = \partial\boldsymbol{M}/\partial u^i$.

The geometric functions $\boldsymbol{M}, \boldsymbol{e}_i$ of $u^1, ..., u^n$ satisfy the relations

$$\boldsymbol{M}^2 = \frac{1}{K}, \quad \boldsymbol{M} \cdot \boldsymbol{e}_i = 0, \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij}\,,$$

with

$$ds^2 = g_{ij}du^i du^j.$$

Conversely, knowing $ds^2$ of the elliptic or hyperbolic space allows, as in the case of the Euclidean space, the local reconstruction of this space. We have in fact relations of the form

$$\left.\begin{array}{rcl} d\boldsymbol{M} &=& du^i\boldsymbol{e}_i, \\ d\boldsymbol{e}_i &=& \omega^0{}_i\boldsymbol{M} + \omega^k{}_i\boldsymbol{e}_k. \end{array}\right\} \tag{6.16}$$

The Pfaffian expressions

$$\omega^0{}_i = \Gamma^0{}_{ir}du^r, \quad \omega^k{}_i = \Gamma^k{}_{jr}du^r$$

are determined by taking into account the relations

$$\boldsymbol{M}^2 = \frac{1}{K}, \quad \boldsymbol{M} \cdot \boldsymbol{e}_i = 0, \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = g_{ij}\,,$$

which differentiated give

$$\omega^0{}_i = -Kg_{ik}du^k \quad \text{or} \quad \Gamma^0{}_{ik} = -Kg_{ik}, \tag{6.17}$$

$$dg_{ij} = g_{jk}\omega^k{}_i + g_{ik}\omega^k{}_j = \omega_{ij} + \omega_{ji}. \tag{6.18}$$

The conditions of integrability for the first of equations (6.16) gives finally

$$\Gamma^0{}_{ij}\boldsymbol{M} + \Gamma^k{}_{ij}\boldsymbol{e}_k = \Gamma^0{}_{ji}\boldsymbol{M} + \Gamma^k{}_{ji}\boldsymbol{e}_k,$$

from which

$$\Gamma^k{}_{ij} = \Gamma^k{}_{ji}. \tag{6.19}$$

We see that the quantities $\Gamma^k{}_{ij}$ and the forms $\omega^k{}_i$ are precisely those that have been determined in the general theory of Riemannian spaces. We deduce in particular, for the *absolute*, or *covariant*, elementary displacement of the vectors $\boldsymbol{e}_i$, the formulae

$$D\boldsymbol{e}_i = \omega^k{}_i\boldsymbol{e}_k = d\boldsymbol{e}_i + Kg_{ik}du^k\boldsymbol{M}. \tag{6.20}$$

**147.** Let us look now for the conditions that must be satisfied by the coefficients $g_{ij}$ in order for a given $ds^2$ to be locally elliptical or hyperbolic with constant curvature $K$. Equations (6.16) must be totally integrable. By expressing, as in n° 43, these conditions of integrability, we find the following formulae, which generalise formulae (2.28) of n° 43,

$$\frac{\partial\Gamma^k{}_{ir}}{\partial u^s} - \frac{\partial\Gamma^k{}_{is}}{\partial u^r} + (\Gamma^h{}_{ir}\Gamma^k hs - \Gamma^h{}_{is}\Gamma^k hr) = K(\varepsilon^k{}_s g_{ir} - \varepsilon^k{}_r g_{is}, \tag{6.21}$$

where $\varepsilon^\beta{}_\alpha$ is equal to 1 if $\alpha = \beta$ and to 0 if $\alpha \neq \beta$.

We can put it into a condensed form that generalises equations (2.30) of n° 45, namely

$$d\omega^k{}_i - [\omega^h{}_i\omega^k{}_h] = -Kg_{ih}[du^h\, du^k]. \tag{6.22}$$

## VII.– Three dimensional non Euclidean spaces: conformal representation

**148.** We can obtain a conformal representation of non Euclidean spaces of three dimensions on the ordinary Euclidean space, or rather on the anallagmatic space. This space is none other than the Euclidean space, but completed by a single *point at infinity*, instead of being, like the projective space, by an infinity of points forming a plane at infinity. On denoting by $X, Y, Z$ the ordinary rectangular coordinates, every point of the anallagmatic space is represented analytically by five homogeneous coordinates not all zero $x_0, x_1, x_2, x_3, x_4$ defined by the relations

$$\frac{x_0}{1} = \frac{x_1}{X} = \frac{x_2}{Y} = \frac{x_3}{Z} = \frac{x_4}{X^2+Y^2+Z^2}\ ; \tag{6.23}$$

these coordinates are in addition related by the quadratic relation

$$\Omega(x) \equiv x_1^2 + x_2^2 + x_3^2 - x_0 x_4 = 0. \tag{6.24}$$

The points whose coordinate $x_0$ is not zero are the points at finite distance of the Euclidean space; if the coordinate $x_0$ is zero, the relation $\Omega = 0$ shows that $x_1, x_2$ and $x_3$ are all zero and we obtain a point whose only non zero coordinate is $x_4$: this is *the point at infinity.*

Every sphere is represented by a linear equation

$$a_0 x_4 - 2a_1 x_1 - 2a_2 x_2 - 2a_3 x_3 + a_4 x_0 = 0; \tag{6.25}$$

the coefficients $a_0, a_1, a_2, a_3, a_4$ are called the homogeneous coordinates of the sphere; if $a_0 \neq 0$, we have a sphere properly so called, if $a_0 = 0$, we have a plane. The planes are thus to be regarded as particular spheres, characterised by the property of containing the point at infinity: for equation (6.25) to be satisfied by $x_0 = x_1 = x_2 = x_3 = 0, x_4 \neq 0$, it is necessary and sufficient that $a_0$ be zero.

The linear substitutions performed on the running coordinates $x_i$ and leaving invariant the form $\Omega(x)$ are the *anallagmatic transformations*. The change spheres ( and planes) into spheres (and planes). These transformations leave invariant the differential quadratic form $\Omega(dx)$. For $x_0 = 1$, this form reduces to $dX^2 + dY^2 + dZ^2$; in the general case, a simple calculation gives

$$\Omega(dx) = x_0^2(dX^2 + dY^2 + dZ^2), \tag{6.26}$$

consequently the anallagmatic transformations are conformal transformations: *they conserve angles.*

We prove (the theorem of Liouville) that every conformal transformation of the space of three dimensions is either a displacement accompanied or not by a symmetry, or a transformation by similitude, direct or inverse, or such a transformation followed by an inversion. These transformations decompose into two distinct connected families (*direct or inverse* anallagmatic transformations); they form a group of 10 parameters.

**149.** Consider a fixed sphere, to which we assign the role played by the absolute in the projective representation of the non Euclidean geometries.

Take for example the sphere with equation

$$x_0 + \frac{K}{4}\, x_4 = 0, \quad \text{or} \quad X^2 + Y^2 + Z^2 + \frac{4}{K} = 0.$$

This sphere is real if $K < 0$, imaginary but with real equation if $K > 0$.

This sphere being fixed, we will normalise the homogeneous coordinates $x_i$ by the condition

$$x_0 + \frac{K}{4}\, x_4 = 1, \quad \text{or} \quad x_0 \left[1 + \frac{K}{4}(X^2 + Y^2 + Z^2)\right] = 1, \tag{6.27}$$

and we will put

$$x_0 - \frac{K}{4}\, x_4 = u;$$

we deduce

$$x_0 x_4 = \frac{1-u^2}{K}\,.$$

The form $\Omega(x)$ can thus be written

$$\Omega(x) = x_1^2 + x_2^2 + x_3^2 + \frac{u^2}{K} - \frac{1}{K}\,,$$

so that we have, for the coordinates of a point,

$$x_1^2 + x_2^2 + x_3^2 + \frac{1}{K}\, u^2 = \frac{1}{K}\,,$$

and the quadratic differential form becomes

$$\Omega(dx) = dx_1^2 + dx_2^2 + dx_3^2 + \frac{1}{K}\, du^2\,.$$

According to what was said in n$^{\text{o}}$ 143, $x, y, z, u$ can be regarded as Weierstrass coordinates of a non Euclidean space of curvature $K$, and the form $\Omega(dx)$ as its linear element. Now according to (6.26) and the value of $x_0$ given by (6.27), we have for this differential form the expression

$$ds^2 = \frac{dX^2 + dY^2 + dZ^2}{\left[1 + \frac{K}{4}(X^2 + Y^2 + Z^2)\right]^2}\,. \tag{6.28}$$

This is one of the expressions for $ds^2$ of the non Euclidean space of curvature $K$ in its conformal representation. This expression, already pointed out by Riemann for any number of dimensions of the space, is the generalisation of formula (6.5) of n$^{\text{o}}$ 129.

In formula (6.28), $K > 0, X, Y, Z$ are the rectangular coordinates of any point of the anallagmatic space, although the formula fails by default for the point at infinity. If $K$ is negative, the point $(X, Y, Z)$ is assumed to be interior to the absolute; but we can also regard the non Euclidean space as realised by the *exterior* of the absolute including the point at infinity; in the first case the coordinate $x_0$ is essentially positive, in the second case it is essentially negative.

**150.** There is another form of $ds^2$ of the non Euclidean space with negative curvature $K$ by taking as the absolute, not a sphere properly so called, but a plane, for example the plane $Z = 0$. Fo this we will normalise the homogeneous coordinates $x_i$ by the condition

$$x_3 = \frac{1}{-\sqrt{K}}, \quad \text{from which} \quad x_0 = \frac{1}{-\sqrt{K}\ Z}.$$

We have then

$$\Omega(x) = x_1^2 + x_2^2 - x_0 x_4 - \frac{1}{K} \ , \tag{6.29}$$

and the $ds^2$ of the non Euclidean space of curvature $1/K$ is the form $\Omega(dx)$ which gives here, according to the value of $x_0$,

$$ds^2 = -\frac{1}{K} \ \frac{dX^2 + dY^2 + dZ^2}{Z^2} \ . \tag{6.30}$$

We can moreover pass from formula (6.28) to formula (6.30) by an anallagmatic transformation, for example an inversion, that transforms the sphere $1 + \frac{K}{4}(X^2 + Y^2 + Z^4) = 0$ in the plane $Z = 0$, or rather transforming the linear form $x_0 + \frac{K}{4} \ x_1$ into the linear form $\sqrt{-K} \ x_3$.

With one or the other choice of the absolute, the non Euclidean planes are represented analytically by a linear homogeneous equation in $x_1, x_2, x_3, u = x_0 - \frac{K}{4} \ x_4$ (first case) or in $x_1, x_2, x_3, x_4$ (second case). In all cases we obtain the *spheres orthogonal to the absolute*. It follows that the non Euclidean lines are represented by the circumferences (or lines) orthogonal to the absolute. The non Euclidean distance between two points $M, M'$, calculated on the circumference (or line) orthogonal to the absolute which joins them is given by the formula

$$d = \frac{1}{\sqrt{-K}} \ \ln(MM'P_1P_2) \ , \tag{6.31}$$

where $P_1$ and $P_2$ are the two points where this circumference cuts the absolute.

This formula is the generalisation of formulae (6.6) and (6.6') of n° 130 and 131, valid in the conformal representation of the non Euclidean spaces of dimension 2.

**151.** Formula (6.14) of n° 141, which involves the polar form of the left hand side of the equation of the absolute in the projective representation, also has its analog in the conformal representation, which involves the polar form $\Omega(x)$. We have, with $x_i$ and $x'_i$ are the normalised coordinates of two points $M$ and $M'$ of the space, and $d$ is the non Euclidean distance between them, the relation

$$\frac{1}{2} \ x'_i \ \frac{\partial \Omega}{\partial x_i} = \frac{1}{K} \left[ \cos(d\sqrt{K}\,) - 1 \right]$$

or

$$x_1x'_1 + x_2x'_2 + x_3x'_3 - \frac{1}{2} \ x_0x'_4 - \frac{1}{2} \ x_4x'_0 = \frac{1}{K} \left[ \cos(d\sqrt{K}\,) - 1 \right] .$$

Consider for example, for $K < 0$, the conformal representation on the half-space $Z > 0$. The equation of the non Euclidean sphere of radius $R$ and centre the non Euclidean point $(a_i)$, can be written, by noting that $a_3x_3 = -\frac{1}{K}$ in normalised coordinates,

$$a_1x_1 + a_2x_2 + a_3 \cosh(r\sqrt{-K}\,)x_3 - \frac{1}{2} \ a_0x'_4 - \frac{1}{2} \ a_4x_0 = 0,$$

or, in non-homogeneous coordinates,

$$X^2 + Y^2 + Z^2 - 2aX - 2bY - 2c\cosh(r\sqrt{-K}\,)Z + a^2 + b^2 + c^2 = 0. \quad (6.32)$$

Interpreted in the language of elementary geometry, we see that the non-Euclidean sphere with non Euclidean centre ?$(a, b, c)$, non Euclidean radius $r$, is the locus of points whose ratio of the distances to the point $A$ and its symmetric point $A'$ with respect to the absolute is constant and equal to $\tanh\left(r\frac{\sqrt{-K}}{2}\right)$. The points $A$ and $A'$ are thus the Poncelet points of the pencil of spheres formed by the sphere considered and the absolute. This result is immediately explained in part if we note that the non Euclidean lines that pass through the non Euclidean centre of the sphere cut the spheres of this pencil orthogonally, in other words, that the circumferences passing through the points $A$ and $A'$ are all orthogonal to the sphere considered (*cf.* n° 134).

We have finally an anallagmatic form of the $ds^2$ of the Euclidean space by assigning to a sphere of zero radius, for example the one that has the origin as its centre, the role of the absolute. Putting

$$x_4 = 1, \qquad \text{or} \quad x_0 = \frac{1}{X^2 + Y^2 + Z^2}\,,$$

we get

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_0\,dx_4 = \frac{dX^2 + dY^2 + dZ^2}{(X^2 + Y^2 + Z^2)^2}\,.$$

The planes now have a linear homogeneous equation $x_1, x_2, x_3, x_4$; they are represented by the spheres that pass through the origin; the lines are represented by the circumferences that pass through the origin. We get a geometric interpretation of this form of the Euclidean geometry by an inversion that has the origin as its pole. The classical geometry could moreover be interpreted by noting that the point at infinity of the anallagmatic space plays the role of the *absolute*.

## VIII.– Locally spherical or hyperbolic normal Riemannian spaces

**152.** If the linear element of a Riemannian space satisfies the conditions found in n° 147 for it to be locally elliptic or hyperbolic of given curvature $K$, we can *develop* onto the non Euclidean space, elliptic or hyperbolic, any sufficiently small portion of this Riemannian space. If the space has a metric that is everywhere regular and is normal, we can repeat the arguments that have been made for the Euclidean space (Chap. III). These arguments relied purely on the two following properties of the Euclidean space: the first is the admitting a group of everywhere regular isometric transformations; the second is of being *simply connected*, in the sense that every closed contour drawn in this space can, by continuous deformation, be reduced to a point.

The first property belongs to spherical space, elliptic space and to hyperbolic space. The second belongs clearly to hyperbolic space. We will see that it belongs also to *spherical* space.

In fact, consider the spherical space whose every point is defined analytically by four numbers $x, y, z, t$ required to satisfy the relation

$$x^2 + y^2 + z^2 + t^2 = R^2.$$

Perform a stereographic projection of this space by putting

$$\frac{x}{R-t} = X, \qquad \frac{y}{R-t} = Y, \qquad \frac{z}{R-t} = Z;$$

we can deduce by squaring, adding and taking into consideration the relation that exists between $x, y, z, t$,

$$\frac{R+t}{R-t} = X^2 + Y^2 + Z^2,$$

from which we deduce

$$\left.\begin{aligned} t &= R\,\frac{X^2+Y^2+Z^2-1}{X^2+Y^2+Z^2+1} \\ x &= R\,\frac{2X}{X^2+Y^2+Z^2+1} \\ y &= R\,\frac{2Y}{X^2+Y^2+Z^2+1} \\ z &= R\,\frac{2Z}{X^2+Y^2+Z^2+1} \end{aligned}\right\} \tag{6.33}$$

Regards $X, Y, Z$ as the rectangular coordinates of a point in ordinary space; to any point of the spherical space, with the exception of a single point $(x = y = z = 0, t = R)$, there corresponds a point and one only of the Euclidean space[4]; conversely to each point of the Euclidean space there corresponds a point and only one of the spherical space.

This said, consider any closed contour drawn in spherical space; we can suppose that it does not pass through the point

$$(x = y = z = 0,\ t = R).$$

It has as image in the Euclidean space a closed contour which we can reduce to a point by continuous deformation; consequently, the contour originally given in the spherical space can also, by continuous deformation, be reduced to a point.

Spherical space is thus simply connected; but it is not the same for the elliptic space, as we will see in a moment.

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[4] We can agree that the point of the spherical space which is the exception corresponds to a point at infinity of the Euclidean space; spherical space thus does not differ from from the anallagmatic space of n° 148.

**153.** The theory of locally Euclidean normal spaces will thus generalise without change to locally spherical spaces and to locally hyperbolic spaces. By development of the given Riemannian space either onto the spherical space or onto the hyperbolic space, the (spherical or hyperbolic) space will be covered entirely once and only once. If the given space is simply connected, it will be identical to the spherical space or to the hyperbolic space. If not it will be represented in one of the two spaces by a fundamental polyhedron defined by an holonomy group generated by isometric transformations of the space and subject to satisfying the two following conditions:

**1°** The group is discontinuous.
**2°** None of its operations (except the identity) leaves invariant a point of the space.

Let us leave aside the case of locally hyperbolic spaces. For $n = 2$, the determination of which of them are orientable reduces to that of Fuchsian groups no operation of which leaves invariant an *interior* point of the Poincare half plane, that is to say with are formed exclusively of *parabolic*[5] or *hyperbolic* substitutions; it could happen moreover that the fundamental polygon has an infinite number of sides.

**154.** The case of locally spherical spaces gives rise to interesting theorems. Any point of a spherical space of $n-1$ dimensions of curvature 1 (we can always reduce it to this case) is defined by $n$ numbers $x_1, ..., x_n$ that satisfy

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

The group of displacements (accompanied or not by a symmetry) is that of *orthogonal linear substitutions* in $n$ variables. Let

$$x'_i = a_{ik} x_k \qquad (i = 1, ..., n) \tag{6.34}$$

be such a substitution. The equality

$$x'^2_1 + \cdots + x'^2_n = x_1^2 + \cdots + x_n^2$$

leads to the following relations between the coefficients

$$\left.\begin{array}{lcll} \sum_k a_{ki}^2 & = & 1 & (i = 1, ..., n) \\ \sum_k a_{ki} a_{kj} & = & 0 & (i \neq j;\ i, j = 1, ..., n) \end{array}\right\} \tag{6.35}$$

These relations allow us to solve the equations of the substitution with respect ro the $x_i$, which gives

$$x_i = a_{ki} x'_k \qquad (i = 1, ..., n). \tag{6.36}$$

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[5] In his book entitled : *Einführung in die Grundlagen der Geometrie,* (Paderborn, 1983), Killing stated incorrectly the theorem that the substitutions of the group must be exclusively hyperbolic.

We will obtain a remarkable canonical form for the equations of the orthogonal substitution (6.34). First see if there exists a system of values not all zero $(x_1, ..., x_n)$ such that the transformed values $(x'_1, ..., x'_n$ can be deduced from the original values by multiplication by the same factor $\lambda$. The relations

$$\lambda x_i = a_{ik} x_k$$

leads to an equation of degree $n$ (the characteristic equation) for $\lambda$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

A system of values of the $x_i$ not all zero defines, in the $n$-dimensional Euclidean space, a vector at the origin. The orthogonal substitution (6.34) changes a vector into another vector, but of the same length and, more generally, it conserves the scalar product of any two vectors.

Suppose that the characteristic equation has an imaginary root $\lambda$. There will exists a vector $\boldsymbol{e}$ (necessarily imaginary) which will be transformed into the vector $\lambda\boldsymbol{e}$,

$$\boldsymbol{e}' = \lambda\boldsymbol{e};$$

by replacing in this equation each quantity by its complex conjugate quantity (represented by means of the subscript index 0), we will have

$$\boldsymbol{e}'_0 = \lambda_0\boldsymbol{e}_0;$$

from which we deduce

$$\boldsymbol{e}\cdot\boldsymbol{e}'_0 = \lambda\lambda_0\boldsymbol{e}\cdot\boldsymbol{e}_0,$$

from which, since the scalar product $\boldsymbol{e}\cdot\boldsymbol{e}_0$ is essentially positive,

$$\lambda\lambda_0 = 1.$$

*The root $\lambda$ is thus of the form $e^{i\alpha}$*. We have furthermore

$$\boldsymbol{e}^2 = \lambda^2\boldsymbol{e}^2,$$

from which, since $\lambda^2$ cannot be equal to 1,

$$\boldsymbol{e}^2 = 0.$$

Let then

$$\boldsymbol{e} = \boldsymbol{e}_1 + i\boldsymbol{e}_2,$$

where the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are real; we have

$$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2, \qquad \boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 0.$$

The two vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ thus have the same length and are orthogonal to each other. Since the vector $\boldsymbol{e}$ is defined only up to a factor, we can suppose that the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are unit vectors *and choose them in the Euclidean space as the first two basis vectors.*

Replace, in formulae (6.34), the $x_i$ and $x'_i$ by the projections $(1, i, 0, ..., 0)$ and $(e^{i\alpha}, ie^{i\alpha}, 0, ..., 0)$ of the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$; we get

$$\begin{array}{lllll} a_{11} = \cos\alpha, & a_{21} = -\sin\alpha, & a_{31} = 0, & ..., & a_{n1} = 0, \\ a_{12} = \sin\alpha, & a_{22} = \cos\alpha, & a_{32} = 0, & ..., & a_{n2} = 0\ . \end{array}$$

The first relations (6.35), applied to the coefficients of the substitution (6.36) inverse of (6.34), gives us, for $i = 1.2$,

$$\begin{array}{lll} a_{13} = 0, & ..., & a_{1n} = 0, \\ a_{23} = 0, & ..., & a_{2n} = 0. \end{array}$$

Finally, the orthogonal substitution takes the form

$$\left.\begin{array}{l} x'_1 = \quad x_1 \cos\alpha + x_2 \sin\alpha, \\ x'_2 = -x_1 \sin\alpha + x_2 \cos\alpha, \\ x'_3 = a_{33}x_3 + a_{34}x_4 + \cdots + a_{3n}x_n, \\ \quad\vdots \\ x'_n = a_{n3}x_3 + a_{n4}x_4 + \cdots + a_{nn}x_n; \end{array}\right\} \tag{6.37}$$

it is the result of two orthogonal substitutions, one of determinant $+1$ acting only on the variables $x_1$ and $x_2$, the other acting on the variables $x_3, ..., x_n$.

If the characteristic equation in $\lambda$ admits $p$ pairs of imaginary roots, we will be able to continue the decomposition into $p$ orthogonal substitutions of determinant 1 acting on $p$ different pairs of coordinates $(x_1, x_2; x_3, x_4; ...; x_{2p-1}, x_{2p})$, and one orthogonal substitution acting on the $n - 2p$ last coordinates.

The characteristic equation with respect to this last substitution admits only real roots, which we show easily to be equal to $+1$ of $-1$.

We will show then, as before, and step by step, that we can decompose the orthogonal substitution acting on $x_{2p+1}, ..., x_n$ into $n - 2p$ substitutions that involve only one coordinate:

$$x'_\alpha = \lambda_\alpha x_\alpha \qquad (\alpha = 2p + 1, ..., n;\ \lambda_\alpha = \pm 1).$$

The canonical form to which we can reduce any orthogonal substitution is thus, in rectangular coordinates,

$$\left.\begin{array}{l} x'_{2i-1} = \quad x_{2i-1} \cos\alpha_i + x_{2i} \sin\alpha_i, \\ x'_{2i} = -x_{2i-1} \sin\alpha_i + x_{2i} \cos\alpha_i, \\ x'_{2p+j} = -x_{2p+j}, \\ \quad\vdots \\ x'_{2p+q+k} = \quad x_{2p+q+k}, \end{array}\right\} \tag{6.38}$$

where the index $i$ varies from 1 to $p$, the index $j$ from 1 to $q$, and the index $k$

from 1 to $n - 2p - q$. One of the integers $p, q, n - 2p - q$ (or even two of them) can be zero.

The displacements themselves of the spherical space of dimension $n - 1$ correspond to orthogonal substitutions of determinant +1., that is to say which contain an *even* number of roots $\lambda = -1$; displacements accompanied by a symmetry correspond to an *odd* number of such roots.

**155.** That said, any holonomy group of a locally spherical normal space should only contain operations that do not leave any point fixed. This amounts to saying that *the characteristic equation must not admit the root* $\lambda = 1$. In fact all points obtained by setting to zero the coordinates pertaining to roots different from 1 will be fixed by the operation considered. The converse moreover is exact. We will thus always have $n = 2p + q$.

*Suppose first the space has an even number of dimensions*, that is to say $n$ is *odd.* Since the root $\lambda = 1$ will never show up, the root $\lambda = -1$ will show up an odd number of times, consequently *all the non-identical operations of the holonomy group are displacements accompanied by a symmetry.* If $S$ is one of these operations, the operation $S^2$ which represents a displacement, must reduce to the identity operation; this requires that the squares $e^{2i\alpha}$ of the imaginary roots all be equal to 1, which is impossible. Thus all the roots of the characteristic equation are equal to $-1$, and the only non-identical operation of the holonomy group is

$$x'_i = -x_i \qquad (i = 1, ..., n).$$

If therefore the space considered is not the spherical space, we obtain it by regarding as identical two points of the spherical space at antipodes the one from the other. *We thus obtain an elliptic space.* The proof itself shows that this space is non orientable. We have the following theorem, whose first part is due to Killing:

*Any locally spherical normal space with an even number of dimensions is identical either to the spherical space, or to the elliptic space. The first is orientable, the second is not.*

In particular the elliptic plane (or the projective plane which is topologically identical to it) is not orientable.

*Suppose now that the space has an odd number of dimensions*, that is to say $n$ even. The canonical form of the orthogonal substitutions shows that if the root $\lambda = 1$ does not show up, the root $\lambda = -1$ shows up an *even* number of times; consequently the determinant of the coefficients is equal to 1. The holonomy group therefore only contains displacements. Whence the following theorem, due to Killing:

*Any locally spherical normal space with an odd number of dimensions is orientable.*

In particular the elliptic space with an odd number of dimensions is orientable; it corresponds to the holonomy group formed by the operation $x'_i = -x_i$, and the identity operation.

## IX.– Riemannian spaces of three dimensions that satisfy the axiom of the plane.

**156.** We will now show that the only three-dimensional Riemannian spaces that satisfy the axiom of the plane are locally Euclidean, locally elliptical or locally hyperbolic spaces.

Any sufficiently small region of such a space admits, as we have seen (n$^o$ 114), a geodesic representation on ordinary space. Perform this representation. The isotropic directions from a point $M$ generate a cone (imaginary) of second order with vertex $M$, which we will denote by $(\Gamma_M)$. Consider any two points $A$, $B$ (belonging to the region in ordinary space on which the representation is made), as well as the straight line that joins them. We have seen (n$^o$ 103) that any two planes containing $AB$ intersect at $A$ and at $B$ at the same angle (defined by the metric of the Riemannian space, that is to say by the isotropic cones with vertices $A$ and $B$). The involution defined by any pair of orthogonal planes passing through $AB$ is thus the same at $A$ and at $B$; thus the double planes of this involution are also the same. Consequently the planes tangent to the cone $(\Gamma_A)$ through $AB$ are also tangent to the cone $(\Gamma_B)$; said differently *any two isotropic cones admit two common tangent planes.*

We can, from now on, follow a geometric path or an analytic path. Let us follow the geometric path.

Let $A, B, C$ be any three points not on a straight line. Consider the two common tangent planes $P_A, P'_A$ to the two cones $(\Gamma_B)$ and $(\Gamma_C)$; the two common tangent planes $P_B, P'_B$ to the two cones $(\Gamma_C)$ and $(\Gamma_A)$; the two common tangent planes $P_C, P'_C$ to the two cones $(\Gamma_A)$ and $(\Gamma_B)$; finally consider a plane $\Pi_A$ tangent to the cone $(\Gamma_A)$ and different from $P_B, P'_B, P_C, P'_C$; and similarly two analogous planes $\Pi_B$ and $\Pi_C$.

The nine planes that we have defined are tangent to the same quadric $(Q)$. We will show that *the isotropic cone with vertex A id circumscribed by this conic.* In fact the cone with summit $A$ circumscribed by $(Q)$ and the cone $(\Gamma_A)$ have *five* common tangent planes, namely $P_B, P'_B, P_C, P'_C, \Pi_A$; thus they merge. We prove similarly that the isotropic cones with vertices $B$ and $C$ are circumscribed by $(Q)$.

Now let $M$ be any point. The cone with vertex $M$ circumscribed by $(Q)$ and the cone $(\Gamma_M)$ have in common six tangent planes, namely the two common tangent planes to $(\Gamma_M)$ and $(\Gamma_A)$, the two common tangent planes to $(\Gamma_M)$ and $(\Gamma_B)$, and the two common tangent planes to $(\Gamma_M)$ and $(\Gamma_C)$. Consequently the cone $(\Gamma_M)$ is circumscribed by the quadric $(Q)$.

It follows from this that the different isotropic cones are the cones circumscribed by the quadric $(Q)$. This quadric can degenerate into a conic (necessarily imaginary); we prove easily that it cannot degenerate into a system of two points. If it is not degenerate, it is either imaginary, or real not ruled, because were it real and ruled, through any point $M$ there would pass real isotropic directions. Moreover if it is real and not ruled, the points $M$ of ordinary space which represent the points of the Riemannian space are necessarily interior to the quadric.

In the three possible cases, we can define an Euclidean, elliptic or hyperbolic metric (and even an infinity), for which the quadric $(Q)$ plays the role of the absolute. The Euclidean metric corresponds to the case where the quadric $(Q)$ reduces to a conic, that an homography can always transform into the umbilical. In each case, the metric (Euclidean or no Euclidean) depends on an unit of length that we can choose arbitrarily. In the metrics defined by the quadric $(Q)$ *the angle of two directions from the smae point $M$ is the same as in the metric of the given Riemannian space*, because the cone of isotropic directions is the same.

In conclusion, *the Riemannian space we sought have the property of admitting a geodesic and conformal representation on an Euclidean space or on a non Euclidean space, elliptic or hyperbolic.*

**157.** It is now easy to see that *if two Riemannian spaces admit the one on the other a representation at once geodesic and conformal, the linear elements of these spaces differ only by a constant factor.* Parallel transport of a vector $\boldsymbol{x}$ with origin $M$ to an infinitely close point $M'$ gives in the two spaces two vectors $\boldsymbol{x}'$ and $\overline{\boldsymbol{x}}'$ *with the same direction*; in fact, according to the theorem of Severi, to perform this transport, we trace the geodesic $MM'$, we construct the geodesic surface at $M$ tangent at $M$ to $MM'$ and to $\boldsymbol{x}$, and we take at $M'$ a vector tangent to this geodesic surface and that makes with $MM'$ extended the angle that $\boldsymbol{x}$ at $M$ makes with $MM'$. The construction gives in the two spaces the same final direction.

We will now show that the two vectors $\boldsymbol{x}'$ and $\overline{\boldsymbol{x}}'$ not only have the same direction at $M'$, but furthermore are identical.

In fact denote by $(u^i)$ and $(u^i + du^i)$ the coordinates of the points $M$ and $M'$, and by $X^i$ the components of the vector $\boldsymbol{x}$. Those of $\boldsymbol{x}'$ and $\overline{\boldsymbol{x}}'$ are respectively

$$X^i - X^k \Gamma_{k\ h}^{\ i} du^h$$

and

$$X^i - X^k \overline{\Gamma}_{k\ h}^{\ i} du^h$$

where we denote by $\Gamma_{k\ h}^{\ i}$ and $\overline{\Gamma}_{k\ h}^{\ i}$ the Christoffel symbols of the two spaces.

We have to find for $\boldsymbol{x}'$ and $\overline{\boldsymbol{x}}'$ the same components, up to a factor $1 + \overline{\varpi}$ infinitely close to 1. We have consequently

$$X^k (\overline{\Gamma}_{k\ h}^{\ i} - \Gamma_{k\ h}^{\ i}) du^h = \overline{\varpi} X^i.$$

The factor $\overline{\varpi}$ is independent of the vector $\boldsymbol{x}$, otherwise it would be equal to $n$

homographic functions of the variables $X^i$ with distinct denominators, which is absurd. It results immediately in the equality of the coefficients ${\Gamma_k}^i{}_h$ and ${\overline{\Gamma}_k}^i{}_h$ every time that the upper index $i$ is different from one of the lower indices $k$ and $h$. We have further, for any index $i$,

$$\overline{\omega} = ({\overline{\Gamma}_i}^i{}_i - {\Gamma_i}^i{}_i)du^i;$$

this is possible only if the quantity $\overline{\omega}$ is zero, with ${\overline{\Gamma}_i}^i{}_i = {\Gamma_i}^i{}_i$.

Since the Christoffel symbols are the same for the two spaces, *the law of parallel transport is also the same in the two spaces.*

That said, since the representation of the two spaces on each other is conformal, we will have $ds_1^2 = k^2 ds_2^2$, where $k$ is a finite factor (function of $u^1, ..., u^n$). Let us begin from a given point $A$; a vector $\boldsymbol{x}$ with origin $A$ is measured in the two spaces by two numbers whose ratio is equal to the numerical value $k_0$ of $k$ at point $A$. Parallel transport this vector $\boldsymbol{x}$ from $A$ to $M$ following any path, but fixed; it will give a vector $\boldsymbol{x}'$ which, in each of the two spaces, has the same length as the vector $\boldsymbol{x}$; consequently the ratio of the numbers that measure this vector in the two spaces is still equal to $k_0$. We thus have $k = k_0 =$ const. This is what we wanted to prove.

The linear element of any Riemannian space that satisfies the axiom of the plane thus differs only by a constant factor from a Euclidean or non-Euclidean (elliptic or hyperbolic) linear element; it is consequently itself Euclidean or non-Euclidean, and we arrive at the following theorem:

*Any Riemannian space that satisfies the axiom of the plane is locally Euclidean, or locally elliptic with positive constant curvature, or locally hyperbolic with negative constant curvature.*

We recover, as a consequence, the theorem proved directly (n° 109) according to which the axiom of the plane entrains the axiom of free mobility.

The preceding proof, for the case $n = 3$, does not apply to the case of two dimensional Riemannian spaces which admit a geodesic representation on the plane. It is however correct that these spaces are locally Euclidean, elliptic or hyperbolic, but the proof cannot be done except by calculation.[6]

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[6] See É. Cartan, *Sur les variété à connexion projective.* (*Bull. Soc. Math. France*, t. 52, 1924, p 226-228); *Leçons sur la théorie des espaces à connexion projective* (Paris, Gauthier-Villars, 1937, p. 247-254).

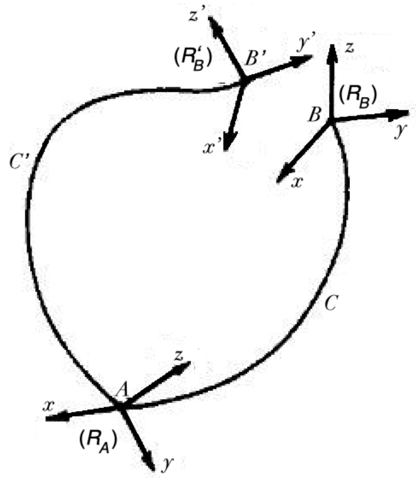
# 7 Riemannian Curvature

## I. – The displacement associated with a cycle.

**158.** We return now to the most general Riemannian spaces, so as to bring out that which differentiates them from Euclidian space.

We saw (n° 92) that given in a Riemannian space an arc of a curve *acb* at each point of which we can suppose attached the corresponding natural Cartesian frame of reference $(R)$, it was possible to *develop* onto the Euclidian space (for example onto the Euclidian space tangent at $a$) this arc of curve as well as all the corresponding frames of reference. If the space considered is not locally Euclidean and if the two points $a$ and $b$ are not too distant one from the other, the development of another arc of curve $ac'b$ starting from the same point $a$ and ending at the same point $b$ will give, if we set out, in the Euclidean space, from the same initial position $A$ and $(R_A)$, two different results, the point $b$ and its frame of reference will come to occupy respectively, on one hand the positions $B$ and $(R_B)$, on the other hand the positions $B'$ and $(R'_B)$.

Figure 1



We can also express the above result in the following way. Suppose that on setting out, in the Euclidean space, from the initial position $B'$ and $(R'_B)$, we develop the closed contour or *cycle* $bc'acb$; after describing the cycle, we will obtain the final position $B$ and $(R_B)$ for the starting point $b$ of the cycle and the corresponding frame of reference. To return to the initial position, you will need to perform, *in the Euclidian space*, to perform a certain displacement, the one that carries $B$ to $B'$ and $(R_B)$ to $(R'_B)$; this displacement is said to be the *associated* with the given cycle. It depends of course on the initial position that we gave to $B'$ and $(R'_B)$; but we can say that, *with respect to the frame of reference* $(R'_B)$, it is well defined. More vividly, we can say that, in the Euclidian space tangent at $B'$, it is a well defined displacement.

Let us add the following two almost obvious remarks.

**Remark I.** — *If in the Riemannian space, we transport by parallelism along the cycle* $bc'acb$ *the body of vectors with origin* $b$*, it undergoes on arrival a rotation identical to that which would result from the displacement associated with the cycle, performed in the Euclidian space tangent at* $b$*.*

**Remark II.** — *If, in the Euclidean space, we extend the vector integral* $\int D\boldsymbol{X}$ *to the contour* $B'C'ACB$*, where* $\boldsymbol{X}$ *denotes a given field of vectors in the Riemannian space (for example the vector field* $\boldsymbol{e}_i$*), we obtain the geometric variation that the vector of the field at the point* $b$ *undergoes when we apply to it the displacement inverse of the one associated with the cycle* $bc'acb$*.*

**159.** It would be very difficult to study directly the displacement *associated* with an arbitrary cycle that begins from a given point and returns to it. We will confine ourselves to an *infinitesimal* cycle, and furthermore we will assume a cycle of a particular form.

Consider for this two distinct systems of differentiation denoted respectively by the symbols $d$ and $\delta$. The quantities $du^i$ may be regarded as the products of a constant (infinitesimal) parameter $\alpha$ with given functions $\xi^i(u^1, ..., u^n)$ (or left indeterminate):

$$du^i = \alpha\ \xi^i(u^1, ..., u^n).$$

We have similarly

$$\delta u^i = \beta\ \eta^i(u^1, ..., u^n).$$

Then let $m$ be any point, with coordinates $(u^i)$, in the Riemannian space; let $m_1$ be the point with coordinates $(u^i + du^i)$, and $m_2$ the point with coordinates $(u^i + \delta u^i)$. The vector $\overrightarrow{mm_1}$ defines an elementary displacement $d$; the vector $\overrightarrow{mm_2}$ an elementary displacement $\delta$. Perform now on the point $m_1$ the elementary displacement $\delta$: we will obtain a point $m_3$ with coordinates

$$u^i + du^i + \delta(u^i + du^i) = u^i + du^i + \delta u^i + \delta du^i.$$

Perform similarly on $m_2$ the elementary displacement $d$; we will obtain the point $m'_3$ with coordinates

$$u^i + \delta u^i + d(u^i + \delta u^i) = u^i + \delta u^i + du^i + d\delta u^i$$

We see that the point $m'_3$ will coincide with the point $m_3$ if we have

$$d\delta u^i = \delta du^i,$$

in other words *if the two differentiations commute*. We know then that if $f$ is any function of the coordinates, we have

$$d\delta f = \delta df.$$

We will suppose the two differentiations considered commute. Then every point $m$ of the space is one of the vertices of an *elementary cycle* $mm_1m_3m_2$, which we will suppose is travelled in the order indicated by the preceding notation (the direction of rotation thus being that which brings the direction of the displacement $d$ into that of $\delta$).

These are the elementary cycles that we will consider. In the particular case where we have

$$\begin{array}{ll} \xi^r = 1 & \text{all the other } \xi \text{ being zero,} \\ \eta^s = 1 & \text{all the other } \eta \text{ being zero,} \end{array}$$

the points $m_1$, $m_3$, $m_2$ have the same coordinates as $m$, except that for the point $m_1$ the coordinate $u^r$ is augmented by $\alpha$, for the point $m_2$ the coordinate $u^s$ is augmented by $\beta$, and for the point $m_3$ the two coordinates $u^r$ and $u^s$ are augmented respectively by $\alpha$ and $\beta$.

**160.** When we develop on the Euclidian space the side $mm_1$ of the cycle, by starting from a given position $(M, e_i)$, the point $M$ and the vectors $e_i$ of the frame of reference that is attached to it undergo elementary geometric variations given by the formulae

$$\begin{aligned} dM &= du^k e_k, \\ de_i &= \omega^k{}_i(d)\ e_k. \end{aligned} \tag{7.1}$$

Similarly, the development of $mm_2$ gives the variations

$$\begin{aligned} \delta M &= \delta u^k e_k, \\ \delta e_i &= \omega^k{}_i(\delta)\ e_k. \end{aligned} \tag{7.2}$$

Develop now the contour $mm_1m_3$. It is necessary, when setting out from $M_1$ and from the frame of reference $(R_1)$, to apply the operation $\delta$ defined by formulae (7.2); but formulae (7.2) give the operation $\delta$ performed at point $M$, whereas it should be performed by replacing everywhere the coordinate $u^i$ by $u^i + du^i$. In other words the point $M_3$ and the vectors $(e_i)_3$ are deduced from the point $M_2$ and the vectors $(e_i)_2$ by applying to them the operation $d$. We thus have, in the Euclidian space,

$$M_3 = M + \delta M + d(M + \delta M);$$
$$(e_i)_3 = e_i + \delta e_i + d(e_i + \delta e_i).$$

The development of the contour $mm_2m_3$ will give on the contrary

$$M_3' = M + dM + \delta(M + dM),$$
$$(e_i)_3' = e_i + de_i + \delta(e_i + de_i).$$

Consequently, in the displacement associated with the cycle $mm_1m_3m_2$, the point $M_3$ and the vectors $(e_i)_3$ undergo displacements

$$\nabla M_3 = M_3' - M_3 = \delta dM - d\delta M;$$
$$\nabla(e_i)_3 = (e_i)_3' - (e_i)_3 = \delta de_i - d\delta e_i.$$

In reality the above quantities express only the *principal part* of the displacements considered, a principal part which is of order $\alpha\beta$, that is to say of the order of the area bounded by the cycle. These displacements are moreover referred to the frame of reference $(R)$ attached to the point $M$; but, to the degree of approximation considered, we will be able, in the final formulae, to regard the vectors $e_i$ as the vectors of the frame of reference $(R_3')$ attached to $M_3'$.
Calculation gives

$$\begin{aligned}
\delta dM - d\delta M &= (\delta du^k - d\delta u^k)e^k + (du^k\ \delta e_k - \delta u^k\ de_k)\\
&= [du^k\omega^h{}_k(\delta) - \delta u^k\omega^h{}_k(d)]e_k = (\Gamma^h{}_{rs} - \Gamma^h{}_{sr})du^r du^s e_h;\\
\delta de_i - d\delta e_i &= [\delta\omega^k{}_i(d) - d\omega^k{}_i(\delta)]e^k + [\omega^k{}_i(d)\omega^h{}_k(\delta) - \delta\omega^k{}_i(\delta)\omega^h{}_k(d)]e_h\\
&= \Omega^k{}_i(\delta, d)e_k,
\end{aligned}$$

by putting

$$\Omega^k{}_i(\delta, d) = \delta\omega^k{}_i(d) - d\omega^k{}_i(\delta) - [\delta\omega^k{}_i(\delta)\omega^h{}_k(d) - \omega^k{}_i(d)\omega^h{}_k(\delta)].$$

By using the condensed notation already introduced (n° 45), we see that the alternating bilinear form $\Omega_i{}^k$ is equal to

$$\Omega_i{}^k = d\omega_i{}^k - [\omega_i{}^h\omega_h{}^k]. \tag{7.3}$$

Finally, we have therefore, for the displacement associated with the elementary cycle considered,

$$\left.\begin{array}{rcl}\nabla M & = & 0,\\ \nabla e_i & = & \Omega_i{}^k(\delta, d)e_k.\end{array}\right\} \tag{7.4}$$

We see that *this displacement leaves fixed the origin of the cycle and it reduces consequently to a rotation around this point.*

**161.** We can conduct the calculation differently. Attach (ideally) to each point $m$ of the Riemannian space a point $p$ defined by its Cartesian coordinates $(x^i)$ referred to the natural frame of reference $(R)$. In the development of Euclidian space, this point $p$ will take naturally the position $P$ defined by its Cartesian coordinates. In the development of the side $mm_1$ of the cycle previously considered, the point $P$ undergoes an absolute elementary displacement whose contravariant components are (n° 44)

$$Dx^i = dx^i + du^i + x^k {\omega_k}^i(d).$$

Similarly, in the development of $mm_2$, it undergoes a geometric displacement with components

$$\Delta x^i = \delta x^i + \delta u^i + x^k {\omega_k}^i(\delta).$$

The development of the contour $mm_1m_3$ gives then a point $P_3$ which, referred to the frame of reference $(R)$ of the point $M$, has as coordinates

$$x^i + Dx^i + \Delta x^i + D\Delta x^i;$$

the development of the contour $mm_2m_3$ gives on the contrary a point $P_3'$ with coordinates

$$x^i + \Delta x^i + Dx^i + \Delta Dx^i.$$

The displacement undergone by the point $P_3$ in the displacement associated with the cycle thus has contravariant components

$$\nabla x^i = \Delta Dx^i - D\Delta x^i.$$

Calculation gives

$$\nabla x^i = \delta Dx^i + Dx^k\ {\omega_k}^i(\delta) - d\Delta x^i - \Delta x^k\ {\omega_k}^i(d).$$

Now,

$$\begin{aligned}\delta Dx^i + Dx^k\ {\omega_k}^i(\delta) = {} & \delta dx^i + \delta du^i + \delta x^k\ {\omega_k}^i(d) + x^k\ \delta{\omega_k}^i(d) \\ & + dx^k\ {\omega_k}^i(\delta) + du^k\ {\omega_k}^i(\delta) + x^h\ {\omega_h}^k(d){\omega_k}^i(\delta).\end{aligned}$$

It follows immediately that

$$\begin{aligned}\nabla x^i = {} & x^k \left\{\delta{\omega_k}^i(d) - d{\omega_k}^i(\delta) - \left[{\omega_k}^h(\delta){\omega_h}^i(d) - {\omega_k}^h(d){\omega_h}^i(\delta)\right]\right\} \\ & + \left[du^k{\omega_k}^i(\delta) - \delta u^k{\omega_k}^i(d)\right].\end{aligned}$$

The last sum is zero, and we obtain

$$\nabla x^i = {\Omega_k}^i(\delta, d)x^k. \tag{7.5}$$

The fact that the term independent of $x^i$ is zero on the right hand side shows clearly that the elementary displacement associated with the cycle is a rotation around the origin of the cycle.

*The quantities ${\Omega_k}^i(\delta, d)$ are the mixed components of the bivector which represents this rotation* (n° 20). The covariant components of this bivector are

$$\Omega_{ki}(\delta, d) = g_{ih}{\Omega_k}^h(\delta, d).$$

*We are certain in advance of the relations*

$$\Omega_{ij} + \Omega_{ji} = 0;$$

besides this is what we will verify later on by calculation.

## II. – The Riemann-Christoffel tensor.

**162.** Let us see how the cycle enters into the components of the rotation associated with it. The expressions ${\Omega_i}^j$ are alternating bilinear forms of the $du^k$; in other words, we have

$${\Omega_i}^j(d,\delta) = \frac{1}{2}\, {R_i}^j{}_{rs}(du^s\delta u^s - du^s\delta u^r) = \frac{1}{2}{R_i}^j{}_{rs}p^{rs},$$

with coefficients ${R_i}^j{}_{rs}$ that are antisymmetric with respect to the indices $r$ and $s$; they are thus linear with respect to the (contravariant) components $p^{rs}$ of the bivector represented by the parallelogram which forms the cycle. If then ${a_i}^j$ are the mixed components of the bivector which represents the rotation associated with the cycle, we have the formula

$${a_i}^j = \frac{1}{2}\, {R_i}^j{}_{sr}p^{rs} = -\frac{1}{2}{R_i}^j{}_{rs}p^{rs}. \tag{7.6}$$

*This formula is general*, in the sense that it is independent of the particular form of the contour of the cycle, provided that it encloses an element of surface equivalent to a bivector with components $p^{rs}$. We will accept this result without proof. (See however N$^{\text{os}}$ 190 and 192).

The quantities ${R_i}^j{}_{rs}$ define that which is called the *Riemann-Christoffel tensor.*

Calculation of these quantities presents no difficulty. We get

$${R_i}^j{}_{rs} = \frac{\partial {\Gamma_i}^j{}_s}{\partial u^r} - \frac{\partial {\Gamma_i}^j{}_r}{\partial u^s} - \left[{\Gamma_i}^k{}_r{\Gamma_k}^j{}_s - {\Gamma_i}^k{}_s{\Gamma_k}^j{}_r\right]. \tag{7.7}$$

**163.** Let us come now to the calculation of the covariant components of the bivector that represents the rotation associated with a given cycle. We have by definition, taking into account (7.4),

$$\begin{aligned}\Omega_{ij} = g_{jk}{\Omega_i}^k = g_{jk}\{d{\omega_i}^k - [{\omega_i}^h{\omega_h}^k]\} &= d(g_{jk}{\omega_i}^k) - [dg_{jk}{\omega_i}^k] - [{\omega_i}^h\omega_{hj}]\\ &= d\omega_{ij} - [\omega_{jk}{\omega_i}^k] - [\omega_{kj}{\omega_i}^k] - [{\omega_i}^k\omega_{kj}];\end{aligned}$$

$$\Omega_{ij} = d\omega_{ij} + [{\omega_i}^k\omega_{jk}] = d\omega_{ij} + [\omega_{ik}{\omega_j}^k] \tag{7.8}$$

The two different expressions obtained are equivalent, because we have

$$[{\omega_i}^k\omega_{jk}] = g_{kh}[{\omega_i}^k{\omega_j}^h] = [\omega_{ih}{\omega_j}^h].$$

Formula (7.8) shows clearly that changing $i$ into $j$ and $j$ into $i$ changes the sign of $\Omega_{ij}$; the equation

$$dg_{ij} = \omega_{ij} + \omega_{ji}$$

gives in fact, by noting that $d\delta g_{ij} = \delta d g_{ij}$,

$$0 = d\omega_{ij} + d\omega_{ji}.$$

The actual calculation of the coefficients $R_{ijrs}$ of $\Omega_{ij}$ is easily performed by starting from equations (7.8). We have

$$R_{ijrs} = \frac{\Gamma_{ijs}}{\partial u^r} - \frac{\Gamma_{ijr}}{\partial u^s} + g^{kh}(\Gamma_{ikr}\Gamma_{jhs} - \Gamma_{iks}\Gamma_{jhr}).$$

Now

$$\frac{\Gamma_{ijs}}{\partial u^r} = \frac{\partial \begin{bmatrix} i\,j \\ s \end{bmatrix}}{\partial u^r} = \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial u^r \partial u^s} + \frac{\partial^2 g_{js}}{\partial u^i \partial u^r} - \frac{\partial^2 g_{is}}{\partial u^j \partial u^r}\right)$$

$$\frac{\Gamma_{ijr}}{\partial u^s} = \frac{\partial \begin{bmatrix} i\,j \\ r \end{bmatrix}}{\partial u^r} = \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial u^r \partial u^s} + \frac{\partial^2 g_{jr}}{\partial u^i \partial u^s} - \frac{\partial^2 g_{ir}}{\partial u^j \partial u^s}\right)$$

Thus

$$\begin{aligned} R_{ijrs} = \frac{1}{2}&\left(\frac{\partial^2 g_{ir}}{\partial u^j \partial u^s} + \frac{\partial^2 g_{js}}{\partial u^i \partial u^r} - \frac{\partial^2 g_{is}}{\partial u^j \partial u^r} - \frac{\partial^2 g_{jr}}{\partial u^j \partial u^s}\right) \\ &+ g^{kh}\left\{\begin{bmatrix} i\,r \\ k \end{bmatrix}\begin{bmatrix} j\,s \\ h \end{bmatrix} - \begin{bmatrix} i\,s \\ k \end{bmatrix}\begin{bmatrix} j\,r \\ h \end{bmatrix}\right\} \end{aligned} \tag{7.9}$$

The formula thus obtained shows the remarkable symmetry properties of the Riemannian symbols $R_{ijrs}$. We have the relations

$$R_{ijrs} = -R_{jirs} = -R_{ijsr}, \tag{7.10}$$

$$R_{ijrs} = R_{rsij}, \tag{7.11}$$

$$R_{ijkh} + R_{ikhj} + R_{ihjk} = 0. \tag{7.12}$$

We can show moreover that relations (7.10) and (7.12) imply (7.11).

## III. – Riemannian curvature of two-dimensional spaces.[1]

**164.** In the case of two dimensions, the displacement associated with an elementary cycle reduces to a rotation around the origin $M$ of the cycle. If we call the area bounded by the cycle $d\sigma$, and the angle of this rotation $K\ d\sigma$, *counted*

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[1] One might fruitfully consult the *Théorie des surfaces* of G. Darboux (Volume III, Book VI).

*positively in the direction of travel of the cycle*, the coefficient $K$ is called *the Riemannian curvature* of the space at the point $M$.

If $a^{12} = -R^{12}_{12} p^{12}$ is the contravariant component of the bivector that represents the rotation, we have

$$K\ d\sigma = \sqrt{g}\, a^{12} = -R^{12}_{12} \sqrt{g}\, p^{12} = -R^{12}_{12}\ d\sigma,$$

consequently,

$$K = -R^{12}_{12} = -\ \frac{1}{g}\ R_{1212}$$

Take for example a surface in ordinary space, considered as a Riemannian space of two dimensions. To have its Riemannian curvature at a point $A$, we can refer the surface to three rectangular coordinate axes $A_x$, $A_y$, $A_z$ that have this point as origin, with the $z$-axis normal to the surface. By taking $x$ and $y$ as coordinates, and by denoting as usual by $p, q, r, s, t$ the partial derivatives of the first two orders of the function $z$ of $x$ and $y$, which defines the surface, we will have to calculate the form

$$\Omega_{12} = d\omega_{12} + {\omega_1}^i \wedge \omega_{2i},$$

where the index of summation $i$ takes the values 1 and 2; now, the sum on the right-hand side has all of its coefficients zero at point $A$, since $dx^2 + dy^2$ is an Euclidian linear element osculating that of the surface at $A$. We have on the other hand

$$\begin{aligned} ds^2 &= (1+p^2)dx^2 + 2pq\ dxdy + (1+q^2)dy^2 \\ \omega_{12} &= \Gamma_{121}\ dx + \Gamma_{122}\ dy = \begin{bmatrix} 1\,1 \\ 2 \end{bmatrix} dx + \begin{bmatrix} 1\,2 \\ 2 \end{bmatrix} dy \\ &= \left( \frac{\partial g_{12}}{\partial x} - \frac{1}{2}\ \frac{\partial g_{11}}{\partial y} \right) dx + \frac{1}{2}\ \frac{\partial g_{22}}{\partial x}\ dy \\ &= q(r\ dx + s\ dy) = q\ dp; \end{aligned}$$

consequently,

$$d\omega_{12} = [dq\, dp] = [(s\, dx + t\, dy)\ (r\, dx + s\, dy)] = (s^2 - rt)[dx\ dy].$$

We thus have at the point $A$

$$K = -R_{1212} = rt - s^2.$$

But if we choose as axes $x$ and $y$ the principal tangents at $A$, we have, at this point,

$$r = \frac{1}{R_1}, \qquad t = \frac{1}{R_2}, \qquad s = 0,$$

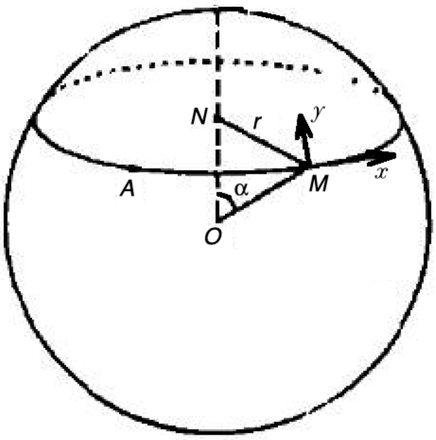
where we denote by $1/R_1$ and $1/R_2$ the two principal curvatures. We thus have

$$K = \frac{1}{R_1 R_2};$$

*the Riemann curvature of a surface at a point is equal to its total curvature*, the product of its two principal curvatures. It is Gauss, as we know, who first proved that the quantity $1/R_1R_2$ depends only on the linear element of the surface.

**165.** There is interest in understanding in a particular case the displacement associated with a cycle. Take a sphere of radius $R$ and a small circle with pole $P$ (Figure 26). Denote by $\alpha$ the half-angle at the vertex of a cone that has as vertex the centre of the sphere and the small circle as base. The development of the circle on the plane is obtained by developing the developable circumscribing the sphere along the circle; this developable is a cone. Attach to each point $M$ of the circle two rectangular axes, $M_x$ tangent to the circle in the direction of travel chosen, $M_y$ tangent to the meridian passing through $M$. The development will give an arc of a circle with centre $S$ and radius $R\tan\alpha$ (n° 101), where the length of this arc of circle is that of the small circle of the sphere, namely $2\pi R\sin\alpha$; if then $\varphi$ denotes the angle $A'SA$, we have

**Figure 26**



$$R\tan\alpha(2\pi-\phi)=2\pi R\sin\alpha$$

where

$$\varphi=2\pi(1-\cos\alpha).$$

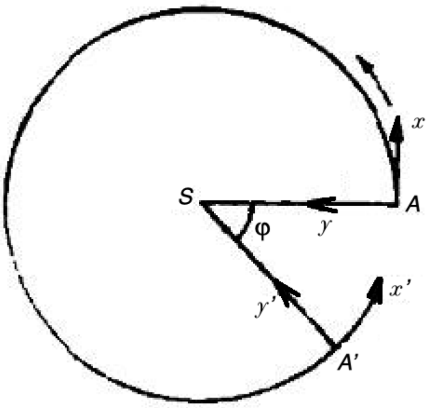
The area bounded on the sphere by the circle is, on the other hand,

$$S=\iint R^2\sin\psi\;d\theta\,d\varphi=2\pi R^2\int_0^\alpha\sin\theta\,d\theta=2\pi R^2(1-\cos\alpha);$$

we then have

$$\varphi=\frac{S}{R^2}\,.$$

Figure 27



We see that the rotation which brings the frame of reference $x'Ay'$ to be parallel to the frame of reference $xAy$ is precisely the rotation by angle $\phi$, *performed in the direction of travel of the cycle.* This rotation is thus equal to the product of the area $S$ bounded by the cycle with the curvature $1/R^2$ of the sphere.

To this rotation we is added a translation, that which brings $A'$ to $A$, and which has as magnitude $2R \tan\alpha \sin(\varphi/2)$. *It is not zero*; but if the cycle is infinitesimal, *this translation is infinitely small with respect to the rotation*; its principle part $\pi R\alpha^3$, by taking as principal infinitesimal the area bounded by the cycle, is zero.

**166.** We can give, in the case of any surface, a simple geometric interpretation of the Riemannian curvature, an interpretation which is related to a famous theorem of Gauss on the sum of the angles of a geodesic triangle.

Consider on the surface the cycle formed by a very small geodesic triangle. If we develop it on the plane tangent to one of its vertices $A$, we obtain a rectilinear triangle $ABCA'$, where the point $A'$ is, to the approximation considered, coincident with $A$. The side $AB$, from which we began the development, is tangent to the side $ab$ of the geodesic triangle; as for the direction of the side $ac$, to find it rotate the direction $AC$ by the angle $Kd\sigma$. Now, in the development, the angles $\widehat{B}$ and $\widehat{C}$ have been conserved; the angle $\widehat{BAC}$ of the rectilinear triangle is thus equal to $\pi - \widehat{B} - \widehat{C}$; we must find the angle $\widehat{A}$ of the geodesic triangle by turning $AC$ by angle $Kd\sigma$; consequently, we have

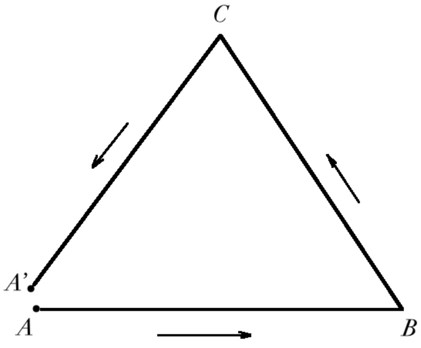
$$\pi - \widehat{B} - \widehat{C} + Kd\sigma = \widehat{A},$$

from which

$$\widehat{A} + \widehat{B} + \widehat{C} - \pi = Kd\sigma. \tag{7.13}$$

This is the theorem of Gauss.

**Figure 28**



In the particular case of the sphere of radius $R$, the left-hand side of equation (7.13) (the spherical excess) is effectively, according to the theorem of Albert Girard, equal to the area of the spherical triangle divided by $r^2$.

The Riemannian curvature at a point of a surface (or of any Riemannian space of two dimensions) can thus be defined by the limit of the ratio $\frac{A+B+C-\pi}{d\sigma}$, in which $A$, $B$, $C$ are the angles of an infinitesimal geodesic triangle of area $d\sigma$ that has the point $A$ as vertex.

**167.** More generally, we can consider any infinitesimal cycle; in the development of this cycle, the tangents at two neighbouring points $M$, $M'$ derived from two points $m$, $m'$ of the cycle make an angle between them equal to the *angle of the geodesic contingency* of the cycle, namely $ds/\rho_g$ where $\rho_g$ denotes the radius of geodesic curvature (n° 100). The direction of the tangent at the point of departure $A$ of the developed cycle has turned in total by angle $\int ds/\rho_g$; the angle that it must then be rotated to obtain the angle of rotation $2\pi$ is thus equal to $Kd\sigma$, and we havee the formula, also due to Gauss,

$$2\pi - \int \frac{ds}{\rho_g} = Kd\sigma;$$

in this form, it can be extended to any cycle and gives

$$2\pi - \int \frac{ds}{\rho_g} = \iint Kd\sigma.$$

The theorem on the geodesic triangle is only a special case. In the case of a geodesic polygon, the formula becomes, denoting by $s$ the sum of the angles of the polygon and by $n$ the number of its sides,

$$s - (n-2)\pi = \iint Kd\sigma. \tag{7.14}$$

**168.** The preceding formula leads to interesting conclusions if we apply it to a closed Riemannian space with an everywhere regular metric. If we decompose this space into geodesic polygons, and if we denote by $F$ is the number of polygons, $S$ that of their vertices, and $A$ that of their sides, then the sum of the left-hand sides of equations (7.14), written for all the polygons, gives

$$2\pi(F + S - A);$$

we then have

$$F + S - A = \frac{1}{2\pi} \iint K \, d\sigma,$$

where the integral is over all space.

We know that the integer $F + S - A$ depends only on those properties of the space that belong to the *Analysis situs*. If the space is homeomorphic to the surface of a sphere, this integer is equal to 2, and we have the formula

$$\iint K \, d\sigma = 4\pi,$$

which shows that *the least value of the Riemannian curvature of the space is equal to* $4\pi/S$*, where we denote by* $S$ *the total surface area of this space*; it is thus impossible to define on a manifold homeomorphic to the sphere a metric with a Riemannian curvature that is everywhere negative or zero.

If the space is homeomorphic to the elliptic (or projective) plane the average value of the Riemannian curvature is equal to $2\pi/S$.

If the space is homeomorphic to a torus, this average value is zero. In fact, we have seen that we can endow the torus with a metric that is everywhere Euclidian.

All these results are moreover obvious on the closed surfaces of ordinary space, where the quantity $\iint K \, d\sigma$ denotes, according to a theorem of Gauss, the oriented area of the spherical representation of the surface (*curvatura integra*).

## IV. – Riemannian curvature of three-dimensional spaces.

**169.** To study Riemannian curvature at a point $A$ of a space of three dimensions, we will suppose, with no loss of generality, that the frame of reference attached to the point $A$ is rectangular. The Riemann-Christoffel tensor involves six quantities which, to abbreviate, we will denote in the following way. Put

$$\left.\begin{array}{ccccc} & & R_{2323} & = & -K_{11} \\ & & R_{3131} & = & -K_{22} \\ & & R_{1212} & = & -K_{33} \\ R_{3112} & = & R_{1231} & = & -K_{23} \\ R_{1223} & = & R_{2312} & = & -K_{31} \\ R_{2331} & = & R_{3123} & = & -K_{13}. \end{array}\right\} \tag{7.15}$$

Any elementary cycle with origin $A$ can be defined by the vector supplementary to the bivector bounded by the cycle. If $\alpha$, $\beta$, $\gamma$ are the direction cosines of the normal to the plane element which contains the cycle, and if $d\sigma$ denotes the area bounded by the cycle, the quantities

$$\alpha\, d\sigma, \qquad \beta\, d\sigma, \qquad \gamma\, d\sigma$$

define the bivector bounded by the cycle.

The rotation associated with the cycle can also be represented by a vector placed on the axis of the rotation; denote its components by

$$p\, d\sigma, \qquad q\, d\sigma, \qquad r\, d\sigma.$$

We have

$$\begin{aligned} p\, d\sigma &= -(R_{2323}\alpha\, d\sigma + R_{2331}\beta\, d\sigma + R_{2312}\gamma\, d\sigma \\ &= (K_{11}\alpha + K_{12}\beta + K_{13}\gamma)d\sigma. \end{aligned}$$

Consequently we can write

$$\left.\begin{array}{lcl} p &=& K_{11}\alpha + K_{12}\beta + K_{13}\gamma, \\ q &=& K_{21}\alpha + K_{22}\beta + K_{23}\gamma, \\ r &=& K_{31}\alpha + K_{32}\beta + K_{33}\gamma. \end{array}\right\} \tag{7.16}$$

*We will call the normal projection of the vector $(p, q, r)$, which represents the rotation per unit surface area associated with this plane element, the Riemannian curvature $K$ of the space at $A$, in the direction of the plane element considered.* We then have

$$\begin{aligned} K &= p\alpha + q\beta \ + r\gamma \\ &= K_{11}\alpha^2 + K_{22}\beta^2 + K_{33}\gamma^2 + 2K_{23}\beta\gamma + K_{31}\gamma\alpha + K_{12}\alpha\beta. \end{aligned} \tag{7.17}$$

The right hand side of this formula is a quadratic form, which we will denote by $\Phi(\alpha, \beta, \gamma)$.

**170.** It is important to note that the knowledge at $A$ of the Riemannian curvature of the space in all plane directions leads to complete knowledge of the Riemann-Christoffel tensor, which is defined precisely by the six coefficients of the form $\Phi$.

The law of variation of the Riemannian curvature in a variable plane direction can be formulated geometrically in a simple way. Consider, in the Euclidian space tangent at $A$, the quadric with centre $A$ and equation equation

$$\phi(x, y, z) = 1$$

Call it the *indicatrix quadric* of Riemann. The Riemannian curvature in a given plane direction is obtained by bringing the normal at $A$ to this direction and taking one of the points $P$ where it meets the indicatrix quadric: we then have

$$K = \frac{1}{\overline{AP}^{\,2}}\,.$$

In fact, if the coordinates of the point $P$ are $\alpha\rho$, $\beta\rho$, $\gamma\rho$, we have

$$1 = \Phi(\alpha\rho, \beta\rho, \gamma\rho) = \rho^2\Phi(\alpha, \beta, \gamma) = K\rho^2$$

from which

$$K = \frac{1}{\rho^2} \,.$$

The indicatrix quadric can moreover be an imaginary ellipsoid (in which case the curvature of the space is positive in all directions), a real ellipsoid (in which case the curvature is negative in all directions); an hyperboloid, an elliptic or hyperbolic cylinder, or finally a system of two parallel planes. The quadric vanishes if the space is Euclidian.

The space is said to be *isotropic* at point $A$ if the indicatrix quadric is a sphere: we then say also that the space at $A$ has (locally) constant Riemannian curvature.

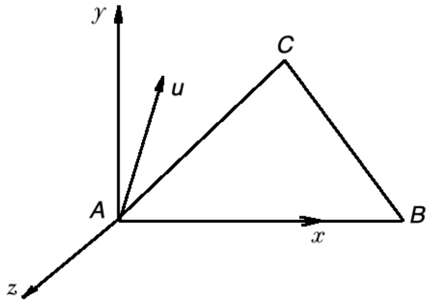
If we return now to formulae (7.16) which give the rotation per unit surface area associated with a cycle, we see easily *the axis of this rotation is perpendicular to the diametral plane conjugate of the the normal of the cycle with respect to the indicatrix quadric.* The rotation is performed around the normal to the cycle in the case, and only in the case, where this normal is one of the axes of the indicatrix quadric. *The directions of the axes of the indicatrix quadric are called the (*Ricci*) principal directions of the space at point $A$.* They are indeterminate if the space is isotropic at $A$.

**171.** We are now in a position to realise that *totally geodesic surfaces* only occur exceptionally in any Riemannian space. In fact, if a surface is totally geodesic, the normal to the surface, transported parallel to itself along a cycle drawn on the surface, before staying normal to the surface, must return to its initial position. Consequently the rotation associated with any elementary cycle drawn on the surface must be performed around the normal to the cycle. *The totally geodesic surface must thus be, at each of its points, normal to one of the principal directions at this point.*

This theorem, due to Ricci, clearly shows that in general there is no totally geodesic surface. If the principal directions are distinct, there exist at most three one-parameter families of totally geodesic surfaces: it is necessary for this that each of the equations in total differentials that express that the normal to the surface is a principal direction be *completely integrable*; these conditions are moreover not sufficient. A closer look shows that *the Euclidian space is the only one which admits a triply orthogonal system of totally geodesic surfaces.*

**172.** We can extend to spaces of three dimensions the theorem of Gauss about the sum of the angles of a geodesic triangle. Consider at a point $A$ in the Riemannian space and a plane element at this point; take through $A$ two very small arcs of geodesics $Ab$, $Ac$ tangent to this plane element and join the geodesic *bc*. Develop the cycle *Abc* onto the Euclidian space tangent at $A$ and let $ABC$ be the rectilinear triangle obtained. Take three rectangular axes with origin $A$,

**Figure 29**



where the axis $Ax$ is along $AB$, the axis $Ay$ is in the plane element considered, and the axis $Az$ is perpendicular to this plane element. Let us imagine finally the tangent $Au$ to the geodesic $Ac$ at $A$. We will pass from $Ac$ to $Au$ by the rotation $(p\,d\sigma, q\,d\sigma, r\,d\sigma)$ associated with the cycle *Abc*. Conversely, we will pass from $Au$ to $AC$ by the rotation $(-p\,d\sigma, -q\,d\sigma, -r\,d\sigma)$. Consider the angle that is made with $Ax$ by the direction $Au$ to which we apply this rotation. This angle has as its initial value $\widehat{A}$ and as its final value $\pi - \widehat{B} - \widehat{C}$. Now the rotation $(-p\,d\sigma, -q\,d\sigma, -r\,d\sigma)$ can be decomposed into three others, one taken on $Au$, another on $AB$ and the last $(-r\,d\sigma$ on $Az$; each of the first two does not change the angle of $Au$ with $Ax$; the last decreases it by $r\,d\sigma$; we thus have, by accepting that $AC$ is very nearly in the plane $xAy$,

$$\widehat{A} - (\pi - \widehat{B} - \widehat{C}) = r\,d\sigma$$

or

$$\widehat{A} + \widehat{B} + \widehat{C} - \pi = r\,d\sigma.$$

Now $r$ is precisely the curvature $K$ of the space in the direction of the plane element considered. We thus recover the generalised formula of Gauss,

$$\widehat{A} + \widehat{B} + \widehat{C} - \pi = K\,d\sigma.$$

**173.** There exists finally a generalisation of the theorem according to which the Riemannian curvature of a surface embedded in Euclidean space is equal to Its total curvature.

Consider in any Riemannian space a surface $(S)$. Choose on this surface any system of co-ordinates $u, v$. Any point $P$ of the space, sufficiently close to $(S)$, can then be defined by the co-ordinates $u, v$ of the point $M$ of the surface obtained by dropping the normal geodesic $PM$ from $P$, and by the length $w$ of the arc $MP$. With this system of coordinates, the linear element of the space takes the form

$$ds^2 = g_{11}du^2 + 2g_{12}du\,dv + g_{22}dv^2 + dw^2.$$

The linear element of the surface is deduced by making $w = 0$ everywhere:

$$ds^2 = g_{11}du^2 + 2g_{12}du\,dv + g_{22}dv^2.$$

We see easily that at any point of the surface, the quantities $g^{ij}$ $(i, j = i, 2)$ have the same numerical values, whether we calculate them from the linear element of the ambient space, or we calculate them from the linear element of the surface. We have, in fact, in the two cases,

$$\frac{g^{11}}{g_{22}} = \frac{g^{11}}{-g_{12}} = \frac{g^{22}}{g_{11}} = \frac{1}{g_{11}g_{22} - g_{12}^2}\,.$$

It is the same for the quantities $\Gamma_{ikj}$ and $\Gamma_i{}^k{}_j$ for which the indices take the values 1 or 2: the quantity

$$\Gamma_{ikj} = \begin{bmatrix} i\,j \\ k \end{bmatrix}$$

in fact involves only the derivatives, with respect to $u$ or $v$, of the coefficients $g^{11}, g^{12}$ and $g^{22}$; as for $\Gamma_i{}^k{}_j$, they are deduced from the $\Gamma_{ikj}$ by means of the coefficients $g^{11}, g^{12}$ and $g^{22}$.

That said, the intrinsic Riemannian curvature of the surface is given by the form $\Omega_{12}$ calculated in the two-dimensional Riemannian space formed by the surface; denoting it by $(\Omega_{12})_i$ we have

$$(\Omega_{12})_i = d\omega_{12} + [\omega^k{}_1\ \omega_{2k}],$$

where the summation is over indices $k = 1, 2$.

The Riemannian curvature of the ambient space, in the direction of the plane element tangent to the surface, is given by the form

$$(\Omega_{12})_e = d\omega_{12} + [\omega^k{}_1\ \omega_{2k}],$$

where the summation is over indices $k = 1, 2, 3$; therefore we now have

$$(\Omega_{12})_e - (\Omega_{12})_i = [\omega^3{}_1\ \omega_{23}].$$

It follows from this formula that the difference between the two Riemannian curvatures at a point $M$ of the surface depends only on the numerical values at this point of the coefficients of $\omega^3{}_1$ and $\omega_{23}$, and consequently *does not change if we replace the given metric by a metric that osculates at* $M$. Take for example an osculating *Euclidean* metric; the difference $K_i - K_e$ reduces then to the Riemannian curvature of the surface, which is equal to the product $\frac{1}{R_1R_2}$ of its principal curvatures. Now the principal curvatures are the same in Riemannian space as in the osculating Euclidean space; consequently, we have the general formula

$$K_i - K_e = \frac{1}{R_1 R_2}\,. \tag{7.18}$$

It states that *the intrinsic Riemannian curvature of a surface at one of its points $M$ is equal to the Riemannian curvature of the ambient space at $M$ in the direction of the plane element tangent to the surface, augmented by the total curvature (product of the principal curvatures) of the surface at $M$.*

In particular *the Riemannian curvature of the space at a point $M$ in the direction of a given plane element is equal to the intrinsic Riemannian curvature at $M$ of the surface geodesic at $M$ tangent to this plane element.*

The definition given by Riemann of the Riemannian curvature can be related to the above considerations. Riemann considers precisely the geodesic surface at $M$ tangent to a given plane element and the total curvature, in the sense of Gauss, attached to the linear element of this surface; it is this that he calls the curvature of the space at $M$ in the direction of the given plane element.

## V. – Riemannian curvature of spaces with dimension greater than three. Spaces with constant Riemannian curvature.

**174.** In the case of a space of any number $n$ of dimensions, the rotation associated with an infinitesimal cycle is represented by a bivector. The Riemannian curvature of the space in the direction of the plane element of the cycle is obtained by taking the scalar product of the preceding bivector with the bivector bounded by the cycle, and by dividing the number obtained by the square of the measure of the area bounded by the cycle. It is easy to see that this definition coincides, for $n = 3$, with that which was given above. The scalar product of the bivector $(\alpha\, d\sigma, \beta\, d\sigma, \gamma\, d\sigma)$ bounded by the cycle with the bivector $(p\, d\sigma, q\, d\sigma, r\, d\sigma)$ which represents the rotation associated with the cycle, is in fact

$$(p\alpha + q\beta + r\gamma) d\sigma^2 = K\, d\sigma^2.$$

By representing in the most general case by $p^{ij}$ the contravariant components of the bivector bounded by the cycle, and by $a_{ij}$ as the covariant components of the associated rotation, we have

$$a_{ij} = -\frac{1}{2}\, R_{ijkh} p^{kh},$$

and consequently

$$K\, d\sigma^2 = -\frac{1}{4}\, R_{ijkh} p^{ij} p^{kh},$$

or finally

$$K = -\frac{\frac{1}{4}\,R_{ijkh}p^{ij}q^{kh}}{\frac{1}{2}\,p^{ij}p_{ij}}\ . \tag{7.19}$$

More generally, we can define, with E. Bompiani,[2] the mixed curvature at a point of two directions of oriented plane elements defined by the simple bivectors $p^{ij}$ and $q^{ij}$ by the expression

$$-\frac{\frac{1}{4}\,R_{ijkh}p^{ij}p^{kh}}{\sqrt{\frac{1}{2}\,p^{ij}p_{ij}}\sqrt{\frac{1}{2}\,q^{ij}q_{ij}}}\ .$$

**175.** It is Important to note that *knowledge at a point $M$ of the Riemannian curvature of the space in the different plane directions at $M$ leads to knowledge at $M$ of the entire Riemann-Christoffel tensor*. In other words, the identity

$$R_{ijkh}p^{ij}p^{kh} = \overline{R}_{ijkh}p^{ij}p^{kh} \tag{7.20}$$

if satisfied by all simple bivectors $p^{ij}$, leads to the equality one by one of all the coefficients $R_{ijkh}$ and $\overline{R}_{ijkh}$. We can say moreover that if the Riemannian curvature of the space at a point $M$ is zero in all the plane directions at $M$, all of the components of the Riemann-Christoffel tensor are zero at this point.

This theorem is far from obvious, because in the identity (7.20), assumed satisfied, *the variables $p^{ij}$ are not independent*, since the components of a simple bivector are related by non-identical quadratic relations. Introduce independent variables by defining a simple bivector by means of two arbitrary vectors $X^i$ and $Y^i$, and identity (7.20) becomes

$$R_{ijkh}X^iY^jX^kY^h = \overline{R}_{ijkh}X^iY^jX^kY^h,$$

now valid whatever the variables $X^i$ and $Y^i$. By equating successively to each other the coefficients of $(X^i)^2(Y^i)^2$, of $(X^i)^2Y^jY^k$, of $X^iX^kY^jY^h$, where the indices shown are distinct, we get

$$\left.\begin{array}{rcl} R_{ijij} & = & \overline{R}_{ijij}, \\ R_{ijik}+R_{ikij} & = & \overline{R}_{ijik}+\overline{R}_{ikij}, \\ R_{ijkh}+R_{kjih}+R_{ihkj}+R_{khij} & = & \overline{R}_{ijkh}+\overline{R}_{kjih}+\overline{R}_{ihkj}+\overline{R}_{khij}\,. \end{array}\right\} \tag{7.21}$$

The relations proved in n° 163

$$R_{ijkh} = R_{khij}$$

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[2] E. Bompiani, *Spazi Riemanniani luoghi di varieta totalmente geodetiche* (Rend. Circ. matem. Palermo, t. 46, 1924).

show first that the components $R_{ijij}$ and $R_{ijik}$ are the same in the two spaces. We have furthermore, according to (7.21), and (6.11),

$$R_{ijkh} + R_{ihkj} = \overline{R}_{ijkh} + \overline{R}_{ihkj}$$

or

$$R_{ijkh} - \overline{R}_{ijkh} = R_{ihkj} - \overline{R}_{ihkj}\,,$$

from which, by cyclic permutation of the indices $j, k, h$

$$R_{ikhj} - \overline{R}_{ikhj} = R_{ijkh} - \overline{R}_{ijkh} = R_{ihjk} - \overline{R}_{ihjk}\,.$$

The common value of these three differences is zero, since their sum is zero by virtue of the relations

$$R_{ijkh} + R_{ikhj} + R_{ihjk} = 0,$$

which are satisfied (n° 163) by the components $\overline{R}$ and $R$. The theorem is therefore proved.

We can note that the proof involves only the properties of the $R_{ijkh}$ expressed in the equalities (7.11), (7.12) and (7.10)

$$R_{ijkh} = -R_{jikh} = -R_{ijhk}.$$

**176.** Consider in particular the case where the space is *isotropic* at the point considered, that is to say it has the same Riemannian curvature $K$ in all the plane directions at this point. Equality (7.19) then becomes

$$R_{ijkh}p^{ij}p^{kh} = -K(g_{ik}g_{jh} - g_{ih}g_{jk})p^{ij}p^{kh}.$$

Since the coefficients of the quadratic form in $p^{ij}$ which is on the right hand side satisfy precisely equations (7.10), (7.11) and (7.12), we have

$$R_{ijkh} = -K(g_{ik}g_{jh} - g_{ih}g_{jk}), \tag{7.22}$$

or, more intuitively,

$$\frac{1}{2}\,R_{ijkh}p^{kh} = -Kp_{ik}\,. \tag{7.23}$$

This equality states that the covariant components $a_{ij}$ of the rotation associated with an elementary cycle $p^{ij}$ are of the form

$$a_{ij} = Kp_{ij}\,. \tag{7.24}$$

Geometrically, *the rotation is represented by a simple bivector situated in the plane element of the cycle, and whose measure is obtained by multiplying by $K$ the area bounded by the cycle*; the rotation is performed in the direction of the cycle if $K$ is positive, and in the opposite direction if $K$ is negative. The number $K$ measures the Riemannian curvature of the space at the point $M$, without needing to specify in which direction it is taken.

**177.** The preceding property admits a converse. *If the rotation associated with any elementary cycle with origin $M$ is represented by a simple bivector situated in the plane element of the cycle, the space is isotropic at $M$.* The relations

$$\frac{a_{12}}{p_{12}} = \frac{a_{13}}{p_{13}} = \cdots = \frac{a_{ij}}{p_{ij}} = \cdots$$

which occur, by hypothesis, for any bivector, provide in fact a sequence of rational fractions with respect to $2n$ independent variables $X_1, ..., X_n; Y_1, ..., Y_n$ which allow the definition of any covariant simple bivector. These rational fractions, each reduced to their simplest form, must give the same irreducible fraction. Now the denominators $X_iY_j - X_jY_i$ are obviously relatively prime; consequently the common value of the functions considered is an integer polynomial, necessarily of degree zero, that is to say *a constant.* We thus have the general relation (7.24), where $K$ denotes a conveniently chosen constant.

**178.** There is a case where one is sure in advance of the isotropy of the space at a given point $M$; it is the one where the space enjoys free mobility around this point. There then exists in fact an isometric transformation that leaves fixed the point $M$ and transforms any plane element at $M$ into any another plane element; this transformation obviously conserves the Riemannian curvature (which depends only on the linear element), and it follows that the space must have the same Riemannian curvature in all plane directions around $M$.

If the space satisfies the axiom of free mobility, then it is therefore isotropic at each of its points, and its Riemannian curvature is moreover the same at all the points; it is said to be with *constant Riemannian curvature.*

Locally elliptic or hyperbolic spaces must have this property, because they satisfy the axiom of free mobility. Verification by a calculation can be carried out in the following way:

Consider the absolute in a projective space. Denote by $M$ a point (with scalar square equal to $\frac{1}{K}$, where $K$ is the *curvature* of the space, in the sense that was given to this word in Chapter VI). Moreover, let $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ be the points, situated on the polar hyperplane of $M$ with respect to the absolute, which are defined analytically in the same way as the basis vectors of the natural frame of reference attached to the point $M$. We have proved (n° 146) the formulae

$$\left.\begin{array}{rcl} d\boldsymbol{M} & = & du^k \boldsymbol{e}_k, \\ d\boldsymbol{e}_i & = & -Kg_{ik}du^k\ \boldsymbol{M} + \omega_i{}^k \boldsymbol{e}_k, \end{array}\right\} \tag{7.25}$$

in which the symbols $d\boldsymbol{M}$ and $d\boldsymbol{e}_i$ indicate *true differentials.*

To calculate the covariant components $\Omega_{ij}$ of the rotation associated with an elementary cycle, first calculate $d\omega_{ij}$. We have

$$d\omega_{ij} = d\boldsymbol{e}_i \cdot \boldsymbol{e}_j\ ;$$

consequently

$$d\omega_{ij} = [d\boldsymbol{e}_j \; d\boldsymbol{e}_i] = [\omega_j{}^k \; \omega_{ik}] + K[g_{jk}du^k \; g_{ik}du^k]. \tag{7.26}$$

We deduce immediately

$$\Omega_{ij} = -K[g_{jk}du^k \; g_{ik}du^k],$$

from which

$$a_{ij} = Kp_{ij},$$

by calling $p_{ij}$ the covariant components of the bivector bounded by the cycle.

The space thus has constant *Riemannian* curvature $K$.

The preceding calculation leads to another important conclusion. The integrability conditions of equations (7.25), where $M$ and $e_i$ are unknown geometric functions in a projective plane where we have defined an elliptic or hyperbolic metric of given curvature $K$, are precisely equations (7.26), which state that the Riemannian curvature has the constant value $K$. *Consequently, any space whose Riemannian curvature is constant is locally elliptic (if $K$ is positive), hyperbolic (if $K$ is negative), or Euclidean (if $K$ is zero).* The development of this space onto the elliptic, hyperbolic or Euclidean space of curvature $K$ can in fact be performed, since equations (7.25) which give this development are completely integrable.

**179.** We can now prove the theorem according to which the axiom of the plane is satisfied only for spaces that are isotropic at each of their points. Suppose, which is legitimate (n° 115), that all manifolds of dimension $n-1$ which are geodesic at a given point $A$ are totally geodesic. Consider one of these manifolds and an elementary cycle from $A$ drawn on this manifold; the bivector that represents the rotation associated with the cycle can be decomposed into simple bivectors situated in the plane element of dimension $n-1$ tangent to the manifold and into a simple bivector situated in a plane element that contains the normal to the manifold. The rotations that represent the first bivectors leave invariant the unit vector normal to the manifold; it follows that it must be the same with the rotation represented by the last bivector; this last bivector is therefore zero. That said, if we take any elementary cycle with origin $A$, the rotation associated with it will necessarily be represented by a simple bivector situated in the plane element of the cycle, otherwise we could find a vector that is normal to this plane element and *not invariant under the rotation*: the manifold $V_{n-1}$ geodesic at $A$ normal to this vector would not then be totally geodesic.

Since the rotation associated to any cycle is represented by a simple bivector situated in the plane element of the cycle, the space is isotropic at $A$ (n° 177). *If then the axiom of the plane is satisfied in this space, it is isotropic at each of its points.* We will see in the following chapter (n° 199) that this property leads to that of being of constant Riemannian curvature.

## VI. – The contracted curvature tensor. Principal directions.

**180.** The Riemann-Christoffel tensor admits a contracted tensor

$$R_{ij} = R_i{}^k{}_{jk}\,.$$

We show easily, with the help of formulas (7.11), the symmetry property of this tensor

$$R_{ij} = R_{ji}\,.$$

The second contracted tensor $R = R^i_i = R^{ij}_{ij}$ is called the *scalar Riemannian curvature* of the space.

The set of directions from a point that satisfy the relation

$$R_{ij}\ du^i du^j = 0$$

defines a cone of second order that is intrinsically attached to this point and that we will call *the Ricci cone*. The principal directions of this cone are those which Ricci calls the principal directions of the space at the point considered.[3]

The principal directions are indeterminate if the space is isotropic at the point considered, but the converse is not always true. There exists in any case, when this circumstance presents itself, an isotropy of a second kind, larger than the isotropy considered up to now. The two isotropies are identical for $n = 3$. We return later to this concept (n° 200).

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[3] C. Ricci, *Direzioni e invarianti principali di una varieta qualunque* (*Atti R. istit. Veneto*, t. 63, 1904, pp 1233-1239).

# 8 The Bianchi Identities

## I. – Exterior Differential Forms[1]

**181.** Forms that occur under the summation sign in multiple integrals are called differential forms with exterior multiplication or, more briefly, exterior differential forms. They obey certain rules of calculation which we now review rapidly.

Take for example, in the ordinary space of three dimensions, a double integral over a piece of surface

$$I = \iint P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy\,.$$

In the differential form

$$\varpi = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy\,,$$

the terms $dy\,dz,\ dz\,dx,\ dx\,dy$ are not exactly identical to ordinary products. If we express the coordinates of a point on the surface of integration as a function of two parameters $\alpha, \beta$, we can regard $\alpha, \beta$ as the coordinates of a point on an auxiliary plane; the integral $I$ can then be reduced to an ordinary double integral over a certain region of this plane. To carry out this reduction, replace respectively the symbols

$$dy\,dz, \qquad dz\,dx, \qquad dx\,dy$$

by the quantities

$$\frac{\partial(y,z)}{\partial(\alpha,\beta)}\,d\alpha\,d\beta\,, \qquad \frac{\partial(z,x)}{\partial(\alpha,\beta)}\,d\alpha\,d\beta\,, \qquad \frac{\partial(x,y)}{\partial(\alpha,\beta)}\,d\alpha\,d\beta\,.$$

So we see clearly that $dy\,dz$ must not be confused with $dz\,dy$, which must be regarded as equal and *opposite* to $dy\,dz$.

We can present things as follows. Introduce two symbols of differentiation $d_1$ and $d_2$, by putting

$$d_1 u = \frac{du}{d\alpha}\,d\alpha, \qquad d_2 u = \frac{du}{d\beta}\,d\beta;$$

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[1] See also É. Cartan *Leçons sur les invariants intégraux* (Paris, Hermann, 1922.)

these two symbols of differentiation commute with each other. With this notation, we have

$$dy\,dz = \begin{vmatrix} d_1y & d_1z \\ d_2y & d_2z \end{vmatrix}, \qquad dz\,dx = \begin{vmatrix} d_1z & d_1x \\ d_2z & d_2x \end{vmatrix}, \qquad dx\,dy = \begin{vmatrix} d_1x & d_1y \\ d_2x & d_2y \end{vmatrix}.$$

The quantities $dy\,dz,\ dz\,dx,\ dx\,dy$ are certainly products, if you like, but products with *exterior multiplication* (Grassmann products), where the sign of the product changes when we change the order of the factors.

More generally, we can introduce any two commuting symbols of differentiation $d_1$ and $d_2$. If the results of the two operations $d_1$ and $d_2$ are infinitely small, we can decompose the surface of integration into a net of small curvilinear parallelograms each of which has vertices respectively

$$\begin{gathered} x, \quad y, \quad z; \\ x + d_1x, \quad y + d_1y, \quad z + d_1z; \\ x + d_1x + d_2d_1x, \quad y + d_1y + d_2d_1y, \quad z + d_1z + d_2d_1z; \\ x + d_2x, \quad y + d_2y, \quad z + d_2z; \end{gathered}$$

and the integral $I$ will be the sum of the quantities

$$P(d_1y\,d_2z - d_1z\,d_2y) + Q(d_1z\,d_2x - d_1x\,d_2z) + R(d_1x\,d_2y - d_1y\,d_2x)$$

over all the elementary parallelograms. In fact, the quantities $dy\,dz,\ dz\,dx,\ dx\,dy$ behave like the components of a simple bivector.

To avoid any confusion, we will agree to enclose an outer product of differentials in square brackets, when such a product does not occur under an integral sign. (*cf.* n$^{\text{os}}$ 45, 147, 160, 163, 178).

**182.** The preceding considerations generalise to multiple integrals in any number of dimensions and lead to differential forms that reduce to sums of terms such as

$$A[dx_1\,dx_2\,...\,dx_\nu];$$

the exterior product in square brackets takes the place of a determinant of order $p$ that involves $p$ commuting symbols of differentiation. Such a product changes sign when we swap two of its factors (of course, the coefficient $A$ is not to be considered as a factor from this point of view).

Given two exterior differential forms $\varpi_1$ and $\varpi_2$, one of order $p$, the other of order $q$, we define the exterior product $[\varpi_1\,\varpi_2]$ of the two forms as the form of order $p+q$ obtained by performing in all possible way the exterior product of a term of the first form with a term of the second, and respecting the order in which the differentials occur. If, for example, we have

$$\varpi_1 = a_i dx^i, \qquad \varpi_2 = b_{ij}[dx^i\,dx^j],$$

we will have

$$[\varpi_1 \varpi_2] = a_i b_{jk}[dx^i\,dx^j\,dx^k].$$

**183.** There is a whole series of important formulae that allow us to transform a multiple integral of order $p$ over a closed domain into a multiple integral of order $p+1$ over a domain that admits the first as its boundary. The simplest of these formulae, known as the Cauchy-Green formula, is

$$\int P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy; \tag{8.1}$$

then we have the Stokes formula

$$\begin{aligned}&\int P\,dx + Q\,dy + R\,dz \\ &= \iint \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,\end{aligned} \tag{8.2}$$

then Ostrogradsky's formula

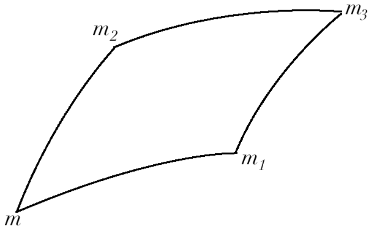
$$\iint P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iiint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz. \tag{8.3}$$

In spaces of more than three dimensions, there are similar formulae.

The procedure that enables us to form of all these formulae can be presented in a very simple form. Take first the case of a simple integral $\int \omega(d)$ over a closed contour $(C)$. Let $(S)$ be a portion of surface (in $n$-dimensional space) bounded by $(C)$. Introduce into $(S)$ two commuting symbols of differentiation $d_1$ and $d_2$ and partition $(S)$ into the corresponding network of infinitesimal parallelograms.

If $m$ is the vertex of one of these parallelograms (Figure 1), and if $m_1$ and $m_2$ are the vertices which we deduce from the operations $d_1$ and $d_2$, we have

**Figure 1**



$$\int_m^{m_1} \varpi = \varpi(d_1), \qquad \int_m^{m_2} \varpi = \varpi(d_2),$$

$$\int_{m_1}^{m_3} \varpi = \int_m^{m_2} \varpi + d_1 \int_m^{m_2} \varpi = \varpi(d_2) + d_1\varpi(d_2),$$

$$\int_{m_2}^{m_3} \varpi = \varpi(d_1) + d_2\varpi(d_1);$$

consequently the integral $\int \varpi$ over the contour of the parallelogram is equal to

$$\begin{aligned}&\varpi(d_1) + [\varpi(d_2) + d_1\varpi(d_2)] - [\varpi(d_1) + d_2\varpi(d_1)] - \varpi(d_2)\\ &\quad = d_1\varpi(d_2) - d_2\varpi(d_1)\end{aligned}$$

The expression obtained on the right hand side is that which we call the *bilinear covariant* of the expression $\varpi$ (n° 45). For example, if $P\,dx$ is a term of $\varpi$, we have

$$d_1(P\,d_2x) - d_2(P\,d_1x) = d_1P\,d_2x - d_2P\,d_1x = [dP\,dx].$$

We thus have the Stokes formula

$$\int P\,dx + Q\,dy + R\,dz = \iint dP\,dx + dQ\,dy + dR\,dz,$$

which generalises to any number of variables.

If we denote the differential form under the integral sign $\int$ by $\varpi$, we will denote the differential form under the sign $\iint$ by $d\varpi$, and we will call it the *exterior differential*[2] of the form $\varpi$.

We note that *the exterior differential of $\varpi$ is zero if $\varpi$ is an exact differential.*

To transform a double integral into a triple integral, we introduce three commuting differentiation symbols into the three-dimensional domain of integration, allowing it to be decomposed it into a network of elementary parallelepipeds. We will show that the double integral $\iint \varpi$ over the surface which bounds one of these parallelepipeds is equal to

$$d_1\varpi(d_2, d_3) - d_2\varpi(d_1, d_3) + d_3\varpi(d_1, d_2).$$

Let then $A\,dx\,dy$ be one of the terms in $\varpi$. We verify easily the identity

$$\begin{aligned}&d_1\left(A\begin{vmatrix} d_2x & d_3x\\ d_2y & d_3y\end{vmatrix}\right) + d_2\left(A\begin{vmatrix} d_3x & d_1x\\ d_3y & d_1y\end{vmatrix}\right) + d_3\left(A\begin{vmatrix} d_1x & d_2x\\ d_1y & d_2y\end{vmatrix}\right)\\ &\quad = \begin{vmatrix} d_1A & d_2A & d_3A\\ d_1x & d_2x & d_3x\\ d_1y & d_2y & d_3y\end{vmatrix},\end{aligned}$$

and consequently we have

$$\iint A\,dx\,dy = \iiint dA\,dx\,dy.$$

More generally if

$$\varpi = A_{ij}dx_i\,dx_j,$$

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[2] This notation is due to E. Kähler. *Einführung in die Theorie der Systeme von Differentialgleichungen* (*Hamburger Math. Einzelschriften*, 1934). It is preferable in many ways to the notation $\varpi'$ used by E. Cartan and which was used in the first edition of this work.

we have

$$\iint \varpi = \iiint dA_{ij}\, dx_i\, dx_j = \iiint d\varpi,$$

where $d\varpi$ denotes the *exterior differential* of the form $\varpi$.

The process of exterior differentiation given for forms of order 1 and 2 generalises to forms of any order: the exterior differential of the form

$$\varpi = A[dx_1\, dx_2 \ldots dx_p]$$

is

$$d\varpi = [dA\, dx_1\, dx_2 \ldots dx_p].$$

At least, it is a convention that we can make and which will be justified a little further on, at the end of n° 185.

**184.** The integrals which appear in the formulae of Green, Stokes, Ostrogradsky, and generally in the relation [3]

$$\int \varpi = \int d\varpi,$$

where $\varpi$ denotes an exterior differential form of degree $p$, with the right hand side over a domain $D$ of the space of dimension $p+1$, and the left hand side over the closed manifold $V$ of dimension $p$ which is the boundary of this domain, makes sense only if we have oriented the domain $D$ and the closed manifold $V$ which bounds it. We can orient the domain by agreeing that a certain $(p+1)$-hedron formed by $p+1$ independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{p+1}$ at a fixed point is direct; the manifold $V$ will be oriented in a consistent way by choosing a vector $\boldsymbol{e}'$ at each point $M$ of $V$ that is exterior to the domain $D$ and $p$ vectors $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \ldots, \boldsymbol{e}'_p$ tangent to $V$ in such a way that the $(p+1)$-hedron formed by the $p+1$ vectors $\boldsymbol{e}, \boldsymbol{e}'_1, \boldsymbol{e}'_2, \ldots, \boldsymbol{e}'_p$ is direct; the orientation of $V$ will then be obtained by agreeing that the $p$-hedron formed by the $p$ vectors $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \ldots, \boldsymbol{e}'_p$ is direct. It is easy to check that the Green's formula (and Stokes's, which is a generalisation of it) is valid with the preceding conventions; similarly for Ostrogradsky's formula. For example in the case of Green's formula, if the area on the $xy$-plane over which the double integral is taken is oriented anticlockwise, the direction in which we must take the integral $\int P\,dx + Q\,dy$ over the contour, simple or multiple, of the area, will be the direction of travel of an observer who has this area on his left.

**185.** The exterior derivative has certain properties, which are very simple.

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[3] The formula

$$[f(x)]_a^b = \int_a^b df(x)$$

is a particular case of this general relation, where $f(x)$ can be regarded as a form of degree zero.

First let $\varpi$ be any exterior differential form, $d\varpi$ its exterior differential and $m$ a factor, a given function of the variables. We have

$$d(m\varpi) = m\,d\varpi + [dm\,\varpi]. \tag{8.4}$$

In fact, if we take in $\varpi$ any term such as

$$A[dx_1 \dots dx_p],$$

the term in $m\varpi$ that corresponds to it is

$$mA[dx_1 \dots dx_p],$$

whose exterior differential is clearly

$$m[dA\,dx_1 \dots dx_p] + A[dm\,dx_1 \dots dx_p];$$

adding all similar terms to it proves the theorem.

A more general formula is the following. Let $\varpi_1$ and $\varpi_2$ be two exterior differential forms, of orders $p$ and $q$ respectively. Consider the form $[\varpi_1\,\varpi_2]$ of order $p+q$. Let

$$A[dx_1 \dots dx_p], \qquad B[dy_1 \dots dy_q]$$

be any two terms, one in $\varpi_1$, the other in $\varpi_2$. The term that corresponds to them in $[\varpi_1\,\varpi_2]$ is

$$AB[dx_1 \dots dx_p\,dy_1 \dots dy_q],$$

whose exterior differential is

$$B[dA\,dx_1 \dots dx_p\,dy_1 \dots dy_q] + A[dB\,dx_1 \dots dx_p\,dy_1 \dots dy_q].$$

The second term can be written as

$$(-1)^p A[dx_1 \dots dx_p\,dB\,dy_1 \dots dy_q].$$

This leads to the formula

$$d[\varpi_1\,\varpi_2] = [d\varpi_1\,\varpi_2] + (-1)^p[\varpi_1\,d\varpi_2] \tag{8.5}$$

which can be generalised to the product of any number of factors. It generalises the ordinary rule for differentiation of a product.

We can now prove that the operation of exterior differentiation, as defined in the general case at the end of n$^{\text{o}}$ 183, is covariant for any change of variables in the sense that, if we go from variables $x_i$ to variables $y_i$, and if, by this change of variables, the form $\varpi(x, dx)$ transforms into $\psi(y, dy)$, the exterior differential $d\varpi$, calculated by regarding the $x_i$ as independent variables, transforms into $d\psi$, calculated by regarding the variables $y_i$ as independent variables. In fact, the term $A[dx_1\,dx_2 \dots dx_p]$, assumed to be expressed by means of the variables $y_i$, has its exterior differential equal to $[dA\,dx_1\,dx_2 \dots dx_p]$, according to the rule for differentiation of a product, since the factors $dx_1, dx_2, ..., dx_p$, being exact differentials, have zero exterior differentials (n° 183).

**186.** There is an important theorem, due to H. Poincare, on successive exterior differentials of any differential form: *the second exterior differential is identically zero.*

In the case where we are in ordinary space and where we start with a linear form

$$\varpi = P\,dx + Q\,dy + R\,dz,$$

the theorem is obvious. In fact, consider any closed surface $(S)$ and the integral $\iint d\varpi$ over this closed surface. Divide the surface into two parts $(S_1)$ and $(S_2)$ by a closed curve $(C)$. The integral $\iint d\varpi$ over $(S_1)$ is equal to the integral $\int \varpi$ over the curve $(C)$ traversed in a certain direction; the integral $\iint d\varpi$ over $(S_2)$ is equal to the integral $\int \varpi$ over the curve $(C)$ *traversed in the opposite direction.* It follows that the integral $\iint d\varpi$ over the surface $(S)$ is zero, whatever the surface. The exterior differential of $d\varpi$ is thus identically zero.

The general analytic proof is very simple. Let

$$A[dx_1\,dx_2\,...\,dx_p]$$

be a term of the given form $\varpi$; the corresponding term of $d\varpi$ is

$$[dA\,dx_1\,dx_2\,...\,dx_p].$$

To find the exterior differential this form, we can consider it as the exterior product of $p+1$ factors, each of which is an exact differential; the exterior differential of the product is thus zero.

The theorem of Poincare has a converse, but we will not use it.

## II. – Tensorial Differential Forms

**187.** Besides the *scalar* differential forms considered so far, we need to consider *tensorial* differential forms.

Consider first an Euclidean space referred to a fixed system of Cartesian coordinates. Consider a $p$-dimensional domain of integration, to each element of which we assign an infinitely small *tensor*; each component of this tensor is assumed to be a differential form of degree $p$. If, for example, it is a mixed tensor with two indices, each of its components will be a differential form $\varpi_i{}^j$. The geometric sum of all the infinitely small tensors considered will be a tensor of the same nature with components $\int \varpi_i{}^j$.

We can take the exterior derivative a tensorial differential form, where the exterior differential of $\varpi_i{}^j$ is $d\varpi_i{}^j$.

**188.** If the Euclidean space is referred to any system of curvilinear coordinates, there will be a natural Cartesian coordinate system attached to each point of the space. To obtain the *absolute* exterior differential of the contravariant vectorial form $\varpi^i$, we can use an *uniform* field of vectors with components $X_i$ and consider the sum $X_i \varpi^i$; the exterior differential of this sum, *which is a scalar quantity*, will be

$$X_i d\varpi^i + dX_i\, \varpi^i = X_i(d\varpi^i + [\omega_k{}^i\, \varpi^k]).$$

The absolute exterior differential that we seek is thus

$$D\varpi^i = d\varpi^i + [\omega_k{}^i\, \varpi^k]. \tag{8.6}$$

We obtain similarly

$$D\varpi_i = d\varpi_i - [\omega_i{}^k\, \varpi_k], \tag{8.7}$$

and, more generally, for a tensorial form with two indices $\varpi_i{}^j$,

$$D\varpi_i{}^j = d\varpi_i{}^j - \omega_i{}^k \varpi_k{}^j + \omega_k{}^j \varpi_i{}^k. \tag{8.8}$$

Let us add that, as in the case of ordinary tensors, the absolute differential of a product is obtained by applying the rule for differentiation of a product, but by replacing the ordinary differentiation by the absolute differentiation.

We have, for example,

$$D[a_i{}^j{}_k du^k] = [Da_i{}^j{}_k du^k], \qquad D[\varpi^i \chi_j] = [D\varpi^i\, \chi_j] + (-1)^p [\varpi^i\, D\chi_j],$$

where $p$ is the order of the form $\varpi^i$.

**189.** Consider now a Riemannian space. If we take any domain of integration in this space, the geometric sum of an infinite number of infinitely small tensors (for example, vectors) attached to elements of the domain of integration *makes no sense.* But if the entire domain of integration is infinitely close to a given point $A$ of the Riemannian space, we can substitute an osculating Euclidean line element at $A$ for the line element of the Riemannian space; the tensorial integral will then make sense, and the principal part of this integral, which is a tensor attached at the point $A$, is *independent of the chosen osculating Euclidean metric.*

Specifically, assume a $(p+1)$-dimensional domain and its $p$ dimensional boundary. The tensorial integral of the element $\varpi_i{}^j$ over this boundary will be equal to the integral of the element $D\varpi_i$ over the given domain; now, at the point $A$, the coefficients of $D\varpi_i{}^j$ involve the osculating Euclidean metric only through the coefficients $\Gamma_i{}^j{}_k$, *which are the same as for the Riemannian metric.*

We can thus define a tensorial integral over an infinitely small domain of a Riemannian space, and the operation of absolute exterior differentiation is performed according to the same laws as in an Euclidean space. The theorems on the exterior differentiation of a product (n° 185) also generalise to Riemannian spaces.

**190.** For example, consider the vectorial integral $\int d\boldsymbol{M}$ over a very small cycle. We have here

$$\varpi^i = du^i,$$
$$D\varpi^i = [\omega_k{}^i\, du^k] = \frac{1}{2}(\Gamma_k{}^i{}_h - \Gamma_h{}^i{}_k)[du^h\, du^k] = 0.$$

Consequently, *the geometric sum of the vectors* $\overrightarrow{\boldsymbol{MM'}}$ *which join a point of the cycle to an infinitely close point is zero.*

This result can be related to the considerations developed in the preceding chapter. In fact, it proves that if we develop the cycle in an Euclidean space, the geometric sum of the corresponding vectors $\overrightarrow{\boldsymbol{MM'}}$ is zero, and consequently that the displacement associated with an infinitely small cycle *of any shape* reduces to a rotation.

## III. – The Bianchi Identities

**191.** We start from the formulae given in n$^{\text{os}}$ 160 and 163, which give the forms $\Omega_i{}^j$ or $\Omega_{ij}$ that define the Riemannian curvature

$$\Omega_i{}^j = d\omega_i{}^j - [\omega_i{}^k\, \omega_k{}^j], \tag{8.9}$$
$$\Omega_{ij} = d\omega_{ij} + [\omega_{ik}\, \omega_j{}^k]. \tag{8.10}$$

Find the exterior derivative the two sides of equations (8.9); taking into account these equations themselves, we get the new relations

$$d\Omega_i{}^j = -[\Omega_i{}^k\, \omega_k{}^j] + [\omega_i{}^k\, \Omega_k{}^j]. \tag{8.11}$$

If we refer to formula (8.8), we see that *relations (8.11) state that the absolute exterior differential of the tensorial differential form* $\Omega_i{}^j$ *is zero*, which we will write as

$$D\Omega_i{}^j = 0. \tag{8.11$'$}$$

Since the form $\Omega_i{}^j$ is of second degree, $D\Omega_i{}^j$ is of third degree and relations (8.11') can be expressed, with the notation of the absolute differential calculus, in the form [4]

$$R_i{}^j{}_{\alpha\beta|\gamma} + R_i{}^j{}_{\beta\gamma|\alpha} + R_i{}^j{}_{\gamma\alpha|\beta} = 0 \qquad (i, j, \alpha, \beta, \gamma = 1, 2, ..., n). \tag{8.12}$$

Relations (8.12), which only translate equations (8.11'), are what are known as the *Bianchi identities*.[5]

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[4] Recall [n° 41, note [1]] that a small vertical bar placed in front of one or more indices is the symbol of an absolute, or covariant, derivative, or of several successive covariant derivatives.

[5] L. Bianchi, *Sui simboli a quattro indici e sulla curvatura di Riemann* [*Rendic. Acad. Lincei* (5), Vol. 11, 1902, p. 3-7].

Since the form $\Omega_{ij}$ is only the form $\Omega_i{}^j$ written in its covariant form, its absolute exterior differential is also zero, which gives the identities

$$R_{ij\alpha\beta|\gamma} + R_{ij\beta\gamma|\alpha} + R_{ij\gamma\alpha|\beta} = 0, \tag{8.13}$$

which can moreover be deduced directly from (8.12).

The tensor $\Omega_{ij}$, or rather the opposite tensor $-\Omega_{ij}$, represents the bivector which defines the rotation associated with a surface element of the space. From this, and from n° 189, we deduce immediately the geometric meaning of the Bianchi identities:

*If we consider an elementary three-dimensional domain of the space, the bivectors that represent the rotations associated with the surface elements which bound this volume, have a geometric sum of zero.*

## IV. – Poincare's theorem in Riemannian spaces

**192.** We saw (n° 186) that the second exterior differential of a differential form is identically zero; this is Poincare's theorem. *In Euclidean space*, this theorem obviously applies to any tensorial differential form. In general, this is no longer the case in a Riemannian space.

To clarify our ideas, start with a vectorial differential form with components $\varpi^i$. Its absolute exterior differential is (n° 188)

$$D\varpi^i = d\varpi^i + [\omega_k{}^i \varpi^k].$$

Take the absolute exterior differential once more

$$D^2\varpi^i = d(D\varpi^i) + [\omega_k{}^i\, D\varpi^k];$$

calculation gives immediately

$$D^2\varpi^i = [\Omega_k{}^i\, \varpi^k]. \tag{8.14}$$

Here we see the Riemannian curvature of the space introducing itself, which in general prevents the second absolute exterior differential of $\varpi^i$ from being zero.

Absolute differentiations consequently give

$$D^3\varpi^i = [\Omega_k{}^i\, D\varpi^k],$$
$$D^4\varpi^i = [\Omega_k{}^h\, \Omega_h{}^i\, \varpi^k].$$

We would have similar expressions starting from any tensorial form.

If in particular $\varpi^i = du^i$, the absolute exterior differential $D\varpi^i$ is zero, the second differential must therefore be zero, and consequently we must have, according to (8.14),

$$[du^k\, \Omega_k{}^i] = 0; \tag{8.15}$$

this relation gives again equations (8.12) (n° 163) which relate the components

$R_k{}^i{}_{h\ell}, R_h{}^i{}_{\ell k}, R_\ell{}^i{}_{kh}$ of the curvature tensor; *this can be regarded as a proof of these equations.*

Another interesting application of formula (8.14) is obtained by taking for $\varpi^i$ an ordinary field of contravariant vectors $X^i$. Consider a cycle $(C)$ that bounds an infinitely small area, that is, all of whose points are infinitely close to a point $A$. The integral $\int DX^i$ over the cycle is equal to the double integral $\iint X^k \Omega_k{}^i$ over the area. If then this area is equivalent to an infinitely small bivector $p^{ij}$, we have the relation

$$\int_{(C)} DX^i = \frac{1}{2}\, R_k{}^i{}_{rs} p^{rs} X^k.$$

According to Note II of n° 158, we deduce that the geometric variation $\nabla X^i$ due to the rotation associated with the cycle is

$$\nabla X^i = -\frac{1}{2}\, X^k R_k{}^i{}_{rs} p^{rs}\,.$$

The mixed components $a_i{}^j$ of the bivector that represents this rotation are then

$$a_i{}^j = -\frac{1}{2}\, R_i{}^j{}_{rs} p^{rs}\,.$$

This result is identical to that given by formula (8.6) of n° 162, proved in the special case where the cycle is an infinitely small parallelogram. We see now that *this result is valid for any infinitely small cycle, whatever its shape* (*Cf.* n° 190).

**Note.** — From the formula that gives the covariant exterior differential of a tensorial form, we deduce that this differential is zero if the bivectorial form of fourth degree $[\Omega_i{}^k\, \Omega_k{}^j]$ is zero. This is obviously true if the space is two- or three-dimensional. It is easily proved that the same is true if the space, of any number of dimensions, has constant curvature.

## V. – Vectorial Curvatures and their First Representation

**193.** Let us return to the geometric interpretation of the Bianchi identities. They state (n° 191) that if we consider a three-dimensional element of the space, the geometric sum of the bivectors that represent the rotations associated with the surface elements of the boundary of the domain is zero.

In this statement, we are dealing with *free* bivectors. Let us see what happens if we consider *applied* bivectors (n° 19). To each surface element would then be associated the applied bivector

$$\frac{1}{2}[\boldsymbol{M}\, \boldsymbol{e}_i\, \boldsymbol{e}_j]\Omega^{ij}.$$

The geometric sum of all these applied bivectors gives an applied bivector and a free trivector; according to the Bianchi identities, the former is zero; so all that remains is the free trivector. Now, the integral

$$\iint \frac{1}{2}[\boldsymbol{M}\,\boldsymbol{e}_i\,\boldsymbol{e}_j]\Omega^{ij}$$

clearly gives, by absolute exterior differentiation, the free trivector

$$\iiint \frac{1}{6}(du^i\Omega^{jk} + du^j\Omega^{ki} + du^k\Omega^{ij})[\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k].$$

We will agree to say that the trivector with components

$$\Omega^{ijk} = [du^i\Omega^{jk}] + [du^j\Omega^{ki}] + [du^k\Omega ij], \tag{8.16}$$

or rather its negative, represents the *trivectorial curvature* of the three dimensional element considered. The tensor provided by the coefficients has six indices, with

$$\begin{aligned}
R^{ijk}_{ijk} &= R^{jk}_{jk} + R^{ki}_{ki} + R^{ij}_{ij},\\
R^{ijk}_{ijh} &= R^{jk}_{jh} + R^{ik}_{ih} \qquad (k \neq h),\\
R^{ijk}_{ih\ell} &= R^{jk}_{h\ell},\\
R^{ijk}_{h\ell m} &= 0 \qquad\qquad\qquad (i, j, k, h, \ell, m \text{ all different}).
\end{aligned}$$

In these formulae there is no need to sum over twice repeated indices, which have fixed values.

**194.** Consider now ($n \geq 4$) an elementary four-dimensional domain of the space and the (free) trivectorial curvatures of the three-dimensional elements of its boundary. Their geometric sum will be given by the absolute exterior differential of the form $\Omega^{ijk}$; now this is zero, since each of the forms $du^i$ and $\Omega^{jk}$ has zero differential. So *the geometric sum of the free trivectorial curvatures of the boundary elements of an infinitely small four-dimensional domain is zero.*

If we consider the *applied* trivectorial curvatures, this is no longer the case, and we obtain a quadrivector with components

$$\begin{aligned}
\Omega^{ijkh} &= [du^i\,\Omega^{jkh}] - [du^j\,\Omega^{ikh}] + [du^k\,\Omega^{ijh}] - [du^h\,\Omega^{ijk}]\\
&= 2\left\{[du^i\,du^j\,\Omega^{kh}] + [du^j\,du^k\,\Omega^{ih}] + [du^k\,du^i\,\Omega^{jh}]\right.\\
&\qquad \left.+[du^i\,du^h\,\Omega^{jk}] + [du^j\,du^h\,\Omega^{ki}] + [du^k\,du^h\,\Omega^{ij}]\right\}.
\end{aligned} \tag{8.17}$$

*This quadrivector, or rather half of its negative, can be regarded as defining the (free) quadrivectorial curvature of a four-dimensional element of the space.*

We see how we could continue these operations step by step to define the (free or applied) $p$-vectorial curvature of a $p$-dimensional element of the space. We thus get the following general theorem:

**Theorem.** — *Given an infinitesimally small p-dimensional domain of a Riemannian space, the geometric sum of the free* $(p-1)$*-vectorial curvatures of its boundary elements is zero; the geometric sum of the applied* $(p-1)$*-vectorial curvatures of these same elements is equal (up to a numerical factor) to the free p-vectorial curvature of the domain.*

**195.** Consider what happens in particular in the case of an infinitely small $(n-1)$-dimensional domain. The $(n-1)$-vectorial curvature has components

$$\Omega^{i_1 i_2 \ldots i_{n-1}} = [du^{i_1} \ldots du^{i_{n-3}} \Omega^{i_{n-2} i_{n-1}}] + \cdots .$$

We have here

$$\begin{aligned} R^{i_1 i_2 \ldots i_{n-1}}_{i_1 i_2 \ldots i_{n-1}} &= \frac{1}{2} R^{i_\alpha i_\beta}_{i_\alpha i_\beta} , \\ R^{i_1 i_2 \ldots i_{n-2} i_{n-1}}_{i_1 i_2 \ldots i_{n-2} i_n} &= R^{i_k i_{n-1}}_{k i_n} ; \end{aligned}$$

on the left hand sides of these relations, do not sum over indices repeated twice; on the right hand side of the first relation, the summation indices $i_\alpha, i_\beta$ take values $i_1, i_2, ..., i_{n-1}$; on the right hand side of the second relation, the summation index $k$ takes values $1, 2, ..., n$.

Orient the space and denote by $\ell_i \, d\sigma$ the covariant components of the vector supplementary to the $(n-1)$-dimensional element considered.

Similarly, denote by $q_i \, d\sigma$ the vector supplementary to the $(n-1)$-vectorial curvature of the given element.

We have, by putting $R = R^{kh}_{kh}$ (n° 180),

$$q_i = \frac{1}{2} \, \ell_i R - R_{ik} \ell^k . \tag{8.18}$$

This introduces the Riemannian scalar curvature $R$ and the contracted tensor $R_{ij}$ (n° 180).

These formulae can be interpreted as follows.

In the tangent Euclidean space at a point of the space, consider the quadric that has this point as centre, and as equation

$$S_{ij} X^i X^j \equiv \frac{1}{2} \, R \, g_{ij} X^i X^j - R_{ij} X^i X^j = 1.$$

We will call this the *Einstein quadric.*

The curvature of the $(n-1)$-dimensional element of size $d\sigma$ can be represented by a vector $q_i \, d\sigma$, with

$$q_i = S_{ik} \ell^k , \tag{8.19}$$

where $\ell^k$ denotes the contravariant components of the unit vector normal to the element. We see that *the vector is normal to the diametral hyperplane conjugate to the direction* $\ell^i$ *with respect to the Einstein quadric.*[6]

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[6] Fr.: *normal à l'hyperplan diamétral conjugué de la direction* $\ell^i$ *par rapport à la quadrique d'Einstein.*

The general theorem of n° 194 then tells us that *the geometric sum of the vectors representing the curvatures of the boundary elements of an infinitely small n-dimensional domain is zero.*

Analytically, this theorem can be expressed by setting the divergence of the tensor $S_{ij}$ to zero, or

$$S^k_{i|k} = 0;$$

*for $n = 4$, these are the equations which, in Einstein's theory, express the theorem of the conservation of momentum and energy.* The vector that represents the curvature of a three dimensional element of the space (spacetime) in fact represents nothing other than the momentum and energy content of this element.

Note that formulae (8.19) give, as a special case, the formulae (8.16) (n° 169) found to represent the curvature of a three-dimensional space.

**196.** Ricci's *principal directions* (n° 180) are at the same time the principal directions of Ricci's cone and of Einstein's quadric. It is now easy to prove a general theorem due to Ricci and which we have already considered in the special case $n = 3$ (n° 171).

Consider a totally geodesic manifold $V_{n-1}$. The normal to this manifold remains normal when it is parallel-transported along any path in the manifold; consequently, the rotation associated with any cycle of the manifold leaves this normal fixed. The *bivector associated with such a cycle is thus entirely tangent to the manifold.* It follows immediately that the trivector associated with a three-dimensional element of the manifold is also entirely tangent to $V_{n-1}$, since it is a sum of tangent simple trivectors. This argument generalises to an element of $V_{n-1}$ of any number of dimensions. In particular, the $(n-1)$-vector representing the curvature of an $(n-1)$-dimensional element of $V_{n-1}$ is tangent to $V_{n-1}$, and the supplementary vector $q_i\, d\sigma$ is normal to $V_{n-1}$, that is, normal to the element. The normal to $V_{n-1}$ is thus a principal direction of the Einstein quadric, that is, of the space.

## VI. – Vectorial Curvatures and their Second Representation

**197.** In the preceding Section, we defined the Riemannian curvature of a $p$-dimensional element of the space and represented this curvature by a $p$-vector. There is a second way to represent it, by a supplementary $(n-p)$-vector, which we can also take to be *free* or *applied.* This second representation assumes a prior orientation of the space. We have already used it for $p = 3$ (n° 169) and $p = n-1$ (n° 195).

If we take the *free* $(n-p)$-vectors, the theorem of n° 194 states that *the sum of the free $(n-p)$-vectors that represent the curvature of the boundary elements of an infinitely small $(p+1)$-dimensional domain is zero.*

It is very remarkable that, contrary to what we found in the preceding Section, *the geometric sum of the same applied* $(n-p)$*-vectors is still zero.*

It will will be sufficient for us to prove this for $p=4, n=7$.

Let $\Theta^{123} = \dfrac{1}{\sqrt{g}}\Omega_{4567}$ be one of the components of the trivector attached to a four-dimensional element of the space. The free 4-vector that represents the geometric sum of the trivectors applied the boundary of a small five-dimensional domain has as the component $\Theta^{1234}$ the expression

$$\Theta^{1234} = [du^1\,\Theta^{234}] - [du^2\,\Theta^{134}] + [du^3\,\Theta^{124}] - [du^4\,\Theta^{123}] = \frac{1}{\sqrt{g}}[du^k\,\Omega_{k567}].$$

Now each of the terms that make up $\Omega_{k567}$ has as a factor either the form $\omega_k = g_{kh}du^h$, or one of the forms $\Omega_{ki}$ $(i=5,6,7)$. Now the sum

$$[du^k\,\omega_k] = g_{kh}[du^k\,du^h]$$

is zero, as well as the sum

$$[du^k\,\Omega_{k\ell}],$$

which is zero according to relation (8.15). This proves that the different components $\Theta^{ijkh}$ are all zero.

Q.E.D.

**198.** In the special case $p=n-1$, the preceding theorem states that *the vectors representing the curvatures of the boundary elements of an infinitely small* $n$*-dimensional domain can be regarded as a system of forces in equilibrium.*

For $n=4$, this theorem completes the physical interpretation of Einstein's equations of gravitation: the vectors that represent in mechanics the "momentum-energy" are in fact *applied* vectors and not free vectors.

For $n=3$, the theorem can take on a remarkable mechanical form.

Let $A$ be a point in a three-dimensional Riemannian space. Attach a rectangular system of reference to this point and consider a small domain surrounding the point $A$. The components $p\,d\sigma, q\,d\sigma, r\,d\sigma$ of the vector attached to a surface element of the boundary of the domain are of the form (n° 169)

$$\begin{aligned} p &= K_{11}\alpha + K_{12}\beta + K_{13}\gamma,\\ q &= K_{21}\alpha + K_{22}\beta + K_{23}\gamma,\\ r &= K_{31}\alpha + K_{32}\beta + K_{33}\gamma, \end{aligned}$$

where $\alpha, \beta, \gamma$ are the direction cosines of the normal to the element. These formulae are identical to those that express the elastic forces in a continuous material medium. We thus get the following theorem:

*If we consider a three-dimensional Riemannian space as a continuous material medium such that the elastic pressure exerted on each element of the surface is equal to the vector representing the Riemannian curvature of this element, this* *medium is in equilibrium under the action of its elastic forces.*

## VII. – The Theorem of F. Schur

**199.** The preceding considerations lead us very naturally to see what happens to the theorems on the vectorial curvature of a space which is *isotropic* at each of its points.

In this case the rotation associated with a surface element reduces to a bivector tangent to that element and equal to the product of that element with a scalar $K$. The contravariant components of this bivector are thus

$$-\Omega^{ij} = K[\,du^i\,du^j\,].$$

By expressing that its absolute exterior differential is zero, and by noting that that of the tensor $[\,du^i\,du^j\,]$ is itself zero (n° 190), we get

$$[\,dK\,du^i\,du^j\,] = 0.$$

If $n \geq 3$, all the derivatives $\dfrac{\partial K}{\partial u^k}$ are zero, hence $K$ is a constant. We therefore have the following theorem, due to F. Schur[7]:

*If a Riemannian space of dimension $n \geq 3$ is isotropic at each of its points, it has constant curvature.*

**200.** There is a more general theorem for $n = 4$, due to G. Herglotz,[8] for spaces whose principal directions are completely indeterminate, that is, for which the Einstein quadric is everywhere a hypersphere. *These spaces are also characterised by constant curvature at each point in the different $(n-1)$-dimensional directions.*

For a space with this property, the $(n-1)$-vectorial curvature of an $n-1$-dimensional element is represented by an $(n-1)$-vector situated in the same $(n-1)$-plane as this element and proportional to this element; its contravariant components are thus of the form

$$\Omega^{i_1 i_2 \dots i_{n-1}} = H[\,du^{i_1}\,du^{i_2} \dots du^{i_{n-1}}\,].$$

Since the absolute exterior differential of the tensorial form thus obtained is zero, it follows, since $H$ is a scalar tensor, that

$$[\,dH\,du^{i_1}\,du^{i_2} \dots du^{i_{n-1}}\,] = 0,$$

from which

$$\frac{\partial H}{\partial u^{i_n}} = 0.$$

The curvature $H$ is thus everywhere the same.

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[7] *Math. Ann.,* Vol. 27, 1886, p. 563.

[8] *Leipz. Ber.,* Vol. 68, 1916, p. 199-203.

We also have

$$S_{ij} = H\, g_{ij} = \frac{1}{2}\, R\, g_{ij} - R_{ij},$$

hence

$$\begin{aligned} R^{i}{}_{j} &= 0 \qquad (i \neq j), \\ R^{i}{}_{i} &= \sum_{k} R^{ik}_{ik} = \frac{1}{2}\, R - H, \end{aligned}$$

where the index $i$ in the last formula is not a summation index.

Then summing with respect to $i$, we get

$$\begin{aligned} R &= n\left(\frac{1}{2}\, R - H\right), \\ H &= \frac{n-2}{2n}\, R. \end{aligned}$$

*The scalar Riemannian curvature $R$ is therefore constant.*

For $n = 3$, the preceding theorem reduces to Schur's theorem.

It is comparable with the hydrostatic theorem which states that a perfect fluid in equilibrium under the action only of its elastic forces has constant pressure.

We conclude with an interesting remark. If the curvature of a Riemannian space is zero in all $p$-dimensional directions at a point, the Riemann-Christoffel tensor at that point has all its components zero. An exception is made for $p = n - 1$, in which case the hypothesis leads easily to the $\frac{n(n+1)}{2}$ relations

$$R_{ij} = 0\,,$$

which shows that the contracted curvature tensor is zero. The spaces for which these relations are satisfied everywhere have zero curvature, *but only in the $(n-1)$-dimensional directions.* This happens in Einstein's theory for an *vacuous* spacetime, where there is neither momentum nor energy. Spaces for which $R_{ij} = 0$ are called *Einstein spaces.*

# 9 The method of the moving frame. Manifolds embedded in Riemannian space.

## I. – General.

**201.** *Up until now we have used almost exclusively the natural frames of reference* attached at each point of the chosen system of coordinates in the space. But it can be convenient, especially in theoretical research, to use local Cartesian frames of reference more appropriate to the nature of the problems treated, and with no necessary connection to the chosen coordinates. Each of these frames of reference is defined by its origin $M$ and $n$ linearly independent basis vectors $e_1, e_2, ..., e_n$.

The laws of algebra and of tensor analysis are not changed, *provided that any tensor with origin $M$ is represented analytically by its components with respect to the frame of reference with origin $M$.* In particular the contravariant components of the vector $\overrightarrow{MM'}$ joining the point $M$ to a point $M'$ infinitely close to $M$ will no longer be the differentials $du^i$ of the coordinates, but will be independent linear combinations which we will denote by $\omega^i$. We will continue to denote by $g_{ij}$ the covariant components of the fundamental tensor, but noting that the fundamental form will be

$$ds^2 = g_{ij}\omega^i\omega^j. \tag{9.1}$$

We can consider also the covariant components $\omega_i = g_{ij}\omega^j$ of the vector $\overrightarrow{MM'}$.

Finally, we continue to denote by ${\omega_i}^j$ and $\omega_{ij} = g_{jk}{\omega_i}^k$ the forms which allow us to define the absolute differentials of the basis vectors $e_i$:

$$De_i = {\omega_i}^k e_k. \tag{9.2}$$

These forms are defined by two types of conditions:

1°. Those which arise from the differentiation of the relations $e_i \cdot e_j = g_{ij}$, namely

$$dg_{ij} = \omega_{ij} + \omega_{ji}; \tag{9.3}$$

2°. Those which express that the absolute exterior differential (n° 188) of the contravariant tensorial differential form $\omega^i$, or covariant $\omega_i$, is zero:

$$d\omega^i = [\omega^k \, {\omega_k}^i], \tag{9.4}$$

$$d\omega_i = [{\omega_i}^k \, \omega_k]. \tag{9.5}$$

We can moreover pass directly from (9.4) to (9.5) by using relations (9.3).

**202.** The quantities that replace the Christoffel symbols of the first and second kind are the coefficients $\gamma_{ikj}$ and $\gamma_i{}^k{}_j$ of the forms $\omega_{ik}$ and $\omega_i{}^k{}_j$ expressed linearly in terms of $\omega^1, \omega^2, ..., \omega^n$:

$$\omega_{ij} = \gamma_{ijk}\omega^k, \quad \omega_i{}^j = \gamma_i{}^j{}_k\, \omega^k, \quad (\gamma_{ijk} = g_{jr}\gamma_i{}^r{}_j\ ). \tag{9.6}$$

Finally the forms $\Omega_i{}^j$ and $\Omega_{ij}$ which define the Riemannian curvature of the space are obtained by the same formulae as before (n° 160 and 163), namely

$$\Omega_i{}^j = d\omega_i{}^j - [\,\omega_i{}^k\,\omega_k{}^j\,], \qquad \Omega_{ij} = d\omega_{ij} + [\,\omega_i{}^k\,\omega_{jk}\,]. \tag{9.7}$$

As regards the components of the Riemann-Christoffel tensor, these are the coefficients of the forms $\Omega_i{}^j$ or $\Omega_{ij}$, expressed as exterior quadratic forms in $\omega^1, \omega^2, ..., \omega^n$:

$$\Omega_i{}^j = \frac{1}{2}\, R_i{}^j{}_{kh}\ [\,\omega^k\,\omega^h\,], \qquad \Omega_{ij} = \frac{1}{2}\ R_{ijkh}[\,\omega^k\,\omega^h\,]. \tag{9.8}$$

Equations (9.4) or (9.5) and (9.7) are those which we shall call the *equations of structure* of the space.

**203.** A particularly simple case is that where the frames attached to different points of the space are all equal to each other; in this case the components $g_{ij}$ of the fundamental tensor are constants, and conversely. We then have the relations

$$\omega_{ij} = -\omega_{ji}, \qquad \gamma_{ijk} = -\gamma_{jik}.$$

It is like this, *for example*, when the basis vectors $\boldsymbol{e}_i$ are unit and rectangular. This allows us to apply to Riemannian geometry the method of the moving frame (of the moving trihedral for $n = 3$), such as was used by G. Darboux in the theory of surfaces and furthermore by G. Ricci in the study of many problems of Riemannian geometry[1]. In this last case the covariant components and the contravariant components of a tensor are the same and we can denote them indifferently by $\omega^i$ of $\omega_i$, etc.

**204.** To determine, in the case of rectangular frames, the components $\gamma_{ijk}$ (Ricci's rotation coefficients), we put

$$d\omega_i = \frac{1}{2}\ c_{khi}[\,\omega^k\,\omega^h\,] \quad (c_{khi} = -c_{hki}). \tag{9.9}$$

We then have, according to (9.3) and (9.5),

$$\gamma_{ijk} + \gamma_{jik} = 0, \qquad \gamma_{ikj} - \gamma_{ijk} = c_{jki}, \tag{9.10}$$

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[1] *Rend. Accad. Lincei*, 5° serie, t. 2, 1895, p. 276-332; *Rend. Accad. Lincei*, 5° serie, t. 19$^{\mathrm{I}}$, 1910, p. 181-187; t. 19$^{\mathrm{II}}$, 1910, p. 85-90, etc.

from which we get easily

$$\gamma_{ijk} = \frac{1}{2}(c_{ijk} - c_{jki} - c_{kij}). \tag{9.11}$$

We shall now apply this last method to the study of some problems in Riemannian geometry.

## II. – Supplements to the theory of surfaces embedded in a Riemannian space.

**205.** Consider a surface $S$ embedded in a Riemannian space of three dimensions and let us attach at each point $M$ of $S$ a trirectangular trihedral whose unit vector $\boldsymbol{e}_3$ is normal to the surface. Of the six forms $\omega_1, \omega_2, \omega_3, \omega_{23} = -\omega_{32}, \omega_{31} = -\omega_{13}, \omega_{12} = -\omega_{21}$, the third $\omega_3$ is identically zero, because every elementary displacement of the point $M$ of the surface is normal to $\boldsymbol{e}_3$. Equations (9.5) and (9.7) are written here

$$\left.\begin{aligned} d\omega_1 &= [\omega_2\ \omega_{21}],\\ d\omega_2 &= [\omega_1\ \omega_{12}],\\ d\omega_{12} &= -[\omega_{13}\ \omega_{23}] - K_e[\omega_1\ \omega_2]; \end{aligned}\right\} \tag{9.12}$$

$$\left.\begin{aligned} 0 &= [\omega_1\ \omega_{13}] + [\omega_2\ \omega_{23}],\\ d\omega_{13} &= [\omega_{12}\ \omega_{23}] - K_{13}[\omega_1\ \omega_2],\\ d\omega_{23} &= -[\omega_{21}\ \omega_{13}] - K_{23}[\omega_1\ \omega_2]. \end{aligned}\right\} \tag{9.13}$$

We have denoted by $K_e = -R_{1212}$ the Riemannian curvature of the ambient space in the direction of the plane element tangent at $M$ to the surface; $K_{13} = -R_{1312}$ denotes the mixed curvature (n° 174) at $M$ of two oriented plane elements containing the vector $\boldsymbol{e}_1$, one tangent, the other normal to the surface $S$, where the orientation s defined by the sense that it is necessary to rotate the vector $\boldsymbol{e}_1$ by $\frac{\pi}{2}$ in order to coincide respectively with the vector $\boldsymbol{e}_2$ and the vector $\boldsymbol{e}_3$, and finally $K_{23} = -R_{2312}$ denotes an analogous mixed curvature.

Equations (9.12) define the structure of the surface $S$ considered as a Riemannian space of two dimensions that has fundamental form

$$ds^2 = (\omega_1)^2 + (\omega_2)^2.$$

The second fundamental form $\Phi$ of the surface is given by the relation

$$\Phi = -D\boldsymbol{e}_3 \cdot d\overrightarrow{M} = \omega_1\omega_{13} + \omega_2\omega_{23} = \gamma_{131}(\omega_1)^2 + (\gamma_{132} + \gamma_{231})\,\omega_1\omega_2 + \gamma_{232}(\omega_2)^2;$$

but the first relation (9.13) shows that $\gamma_{132} = \gamma_{231}$. We will put

$$\gamma_{131} = a_{11}, \qquad \gamma_{132} = \gamma_{231} = a_{12} = a_{21}, \qquad \gamma_{232} = a_{22}.$$

Since $\omega_{13}$ and $\omega_{23}$ are vectorial differential forms, representing the vector $-D\boldsymbol{e}_3$, for the two dimensional Riemannian space formed by the surface $S$, and that it is the same for $\omega_1$ and $\omega_2$, which represent the vector $\overrightarrow{dM}$, the coefficients $a_{11}, a_{12}, a_{22}$ of the form $\Phi$, independent of the choice of frame $(M\boldsymbol{e}_1\boldsymbol{e}_2)$, are the components of a symmetric tensor with two indices. The numerical value of each of the components is determined uniquely as soon as we are given the point $M$ and the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$.

Consider in fact any curve $C$ drawn on the surface passing through $M$ and tangent at this point to the vector $\boldsymbol{e}_1$ $(\omega_1 = ds, \omega_2 = 0)$; The equality

$$D\boldsymbol{e}_1 = \omega_{12}\boldsymbol{e}_2 + \omega_{13}\boldsymbol{e}_3 \qquad \text{or} \qquad \frac{D\boldsymbol{e}_1}{ds} = \frac{\omega_{12}}{ds}\,\boldsymbol{e}_2 + a_{11}\boldsymbol{e}_3,$$

shows that $a_{11}$ is the normal curvature of the curve

$$a_{11} = \frac{1}{\rho_n} = \frac{\cos V}{\rho}, \tag{9.14}$$

where $V$ denotes the angle of the principal normal with the normal to the surface: this is the theorem of Meusnier affirming the constancy of the normal curvature for all curves tangent to each other.

The geodesic curvature is given by the ratio $\frac{\omega_{12}}{ds}$

$$\frac{\omega_{12}}{ds} = \frac{1}{\rho_g} = \frac{\sin V}{\rho}; \tag{9.15}$$

*it varies with the curve*, since $\omega_{12}$ is not a tensorial form.

The component $a_{12} = a_{21}$ is the geodesic torsion (n° 89) of the curve $C$, as is shown by the relation[2]

$$\frac{D\boldsymbol{e}_2}{ds} = \frac{\omega_{21}}{ds}\,\boldsymbol{e}_1 + a_{21}\boldsymbol{e}_3, \qquad \text{from which} \qquad a_{12} = \boldsymbol{e}_3 \cdot \frac{D\boldsymbol{e}_2}{ds},$$

$$a_{12} = \frac{1}{\tau_g} = \frac{dV}{ds} + \frac{1}{\tau}. \tag{9.16}$$

Finally we know that the principal curvatures $\frac{1}{R_1}, \frac{1}{R_2}$ are given by the relations

$$\frac{a_{11}\omega_1 + a_{12}\omega_2}{\omega_1} = \frac{a_{21}\omega_1 + a_{22}\omega_2}{\omega_2} = \frac{1}{R},$$

from which

$$\left(a_{11} - \frac{1}{R}\right)\left(a_{22} - \frac{1}{R}\right) - a_{12}^2 = 0;$$

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[2] With the notation of n° 89, the vector $\boldsymbol{e}_2$ is written $-\sin\theta\,\epsilon_1 + \cos\theta\,\epsilon_2$, the vector $\boldsymbol{e}_3$ is written $\boldsymbol{v}$; we verify immediately, with the help of formula (9.16) of this paragraph, that the scalar product

$$-\boldsymbol{e}_2 \cdot \frac{D\boldsymbol{e}_3}{ds} = (\sin\theta\,\epsilon_1 - \cos\theta\,\epsilon_2)\frac{D\boldsymbol{v}}{ds}$$

is equal to the geodesic torsion.

we deduce

$$\left.\begin{aligned} a_{11}+a_{22} &= \frac{1}{R_1}+\frac{1}{R_2}=L; \\ a_{11}a_{22} &= \frac{1}{R_1R_2}\,. \end{aligned}\right\} \tag{9.17}$$

We thus have an interpretation of the component $a_{22}$ as a function of the elements of the curve and of the elements of the surface

$$a_{22}=\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{\rho_n}. \tag{9.18}$$

Note that the third formula (9.12) gives again a theorem that we have already obtained (n$^{\text{os}}$ 164, 173); since the value of $d\omega_{12}$ is $-K_i[\omega_1\ \omega_2]$, where $K_i$ denotes the Riemannian curvature of the surface $S$ considered as a Riemannian two-dimensional space, we have

$$K_i=\frac{1}{R_1R_2}+K_e\,.$$

**206.** The classical theorems with respect to the normal curvature and the geodesic torsion of two curves tangent to one another, due basically to the tensorial character of the coefficients $a_{ij}$ of the second fundamental form, admit generalisations by considering the derivatives of the tensor $a_{ij}$.

The first derived tensor $a_{ijk}$ is defined by the relations[3]

$$\left.\begin{aligned} &Da_{11}=da_{11}-2a_{12}\omega_{12}=a_{111}\omega_1+a_{112}\omega_2\,, \\ &Da_{12}=Da_{21}=da_{12}+(a_{11}-a_{22})\omega_{12} \\ &\qquad\qquad\qquad =a_{121}\omega_1+a_{122}\omega_2=a_{211}\omega_1+a_{212}\omega_2\,, \\ &Da_{22}=da_{22}+2a_{12}\omega_{12}=a_{221}\omega_1+a_{222}\omega_2\,. \end{aligned}\right\} \tag{9.19}$$

The first relation (9.19) gives, if we move along one curve $C$ tangent at $M$ to the vector $\boldsymbol{e}_1$ $(\omega_1=ds,\omega_2=0)$,

$$a_{111}=\frac{da_{12}}{ds}-2a_{12}\frac{\omega_{12}}{ds}=\frac{d\dfrac{1}{\rho_n}}{ds}-\frac{2}{\tau_g}\,\frac{1}{\rho_g}\,; \tag{9.20}$$

the last part of this relation has the same value at $M$ for all the curves $(C)$ considered. This is a theorem due to E. Laguerre[4] in regard to Euclidean space.

The second relation (9.19) gives, under the same conditions,

$$a_{121}=a_{211}=\frac{da_{12}}{ds}+(a_{11}-a_{22})\frac{\omega_{12}}{ds}=\frac{d\dfrac{1}{\tau_g}}{ds}+\left(\frac{2}{\rho_n}-L\right)\frac{1}{\rho_g}\,; \tag{9.21}$$

the last part still has the same value at $M$ for all the curves $(C)$ considered.

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[3] We refrain here from placing a vertical bar in front of the index of derivation $k$.

[4] *OEuvres*, II, p. 129-130. *Cf.* E. Goursat, *Cours d'Analyse*, 3$^{\text{o}}$ éd., Paris, 1917, p. 641-642.

Let us move on to the second derived tensor $a_{ijkh}$. We have

$$\left.\begin{aligned} Da_{111} &= da_{111} - (a_{211} + a_{121} + a_{112})\omega_{12} = a_{1111}\omega_1 + a_{1112}\omega_2\,, \\ Da_{121} &= da_{121} - (a_{122} + a_{221} - a_{111})\omega_{12} = a_{1211}\omega_1 + a_{1212}\omega_2\,. \end{aligned}\right\} \tag{9.22}$$

To take advantage of these relations, it is necessary to be able to express $a_{112}, a_{122}$ and $a_{221}$ by means of the components $a_{111}$ and $a_{121}$. We get there by exterior differentiation of the relations

$$\begin{aligned} \omega_{13} &= a_{11}\omega_1 + a_{12}\omega_2\,, \\ \omega_{23} &= a_{21}\omega_1 + a_{22}\omega_2\,. \end{aligned}$$

We can regard $\omega_{13}$ and $\omega_{23}$ as the tensorial components of the vector $-De_3$ and performing the absolute exterior differentiations, which gives (n° 188)

$$\begin{aligned} d\omega_{13} - [\omega_{12}\ \omega_{23}] &= [Da_{11}\ \omega_1] + [Da_{12}\ \omega_2]\,, \\ d\omega_{23} - [\omega_{21}\ \omega_{13}] &= [Da_{21}\ \omega_1] + [Da_{22}\ \omega_2]\,, \end{aligned}$$

that is to say, according to (9.13),

$$\left.\begin{aligned} a_{112} - a_{121} &= K_{13}\,, \\ a_{212} - a_{221} &= K_{23}\,. \end{aligned}\right\} \tag{9.23}$$

We deduce, according to (9.22), by moving along the line $(C)$,

$$\begin{aligned} a_{1111} &= \frac{a_{111}}{ds} - (3a_{121} + K_{13})\frac{1}{\rho_g}\,, \\ a_{1211} &= \frac{a_{121}}{ds} - (2a_{221} - a_{111} + K_{23})\frac{1}{\rho_g}\,. \end{aligned}$$

By replacing finally $a_{221}$ by $\frac{L}{ds} - a_{111}$ and $a_{111}$ and $a_{121}$ by their values (9.20) and (9.21), we obtain finally

$$a_{1111} = \frac{d^2\frac{1}{\rho_n}}{ds^2} - \frac{2}{\tau_g}\,\frac{d\frac{1}{\rho_g}}{ds} - \frac{1}{\rho_g}\left[5\,\frac{d\frac{1}{\tau_g}}{ds} + K_{13}\right] - \frac{3}{\rho_g^2}\left(\frac{2}{\rho_n} - L\right), \tag{9.24}$$

$$a_{1211} = \frac{d^2\frac{1}{\tau_g}}{ds^2} + \left(\frac{2}{\rho_n} - L\right)\frac{d\frac{1}{\rho_g}}{ds} - \frac{1}{\rho_g}\left[5\,\frac{d\frac{1}{\rho_n}}{ds} - \frac{6}{\tau_g}\,\frac{1}{\rho_g} - 3\,\frac{L}{ds} - K_{23}\right]. \tag{9.25}$$

We thus arrive at the following theorem:

THEOREM 9.1 — *If at a point $M$ of a surface embedded in a Riemannian space of three dimensions we consider the different curves on the surface that have a given tangent at this point, all these curves have in common at this point the* *normal curvature, the geodesic torsion, and the four quantities*

$$\frac{d\,\frac{1}{\rho_n}}{ds} - \frac{2}{\tau_g}\,\frac{1}{\rho_g}\,,$$

$$\frac{d\,\frac{1}{\tau_g}}{ds} + \left(\frac{2}{\rho_n} - L\right)\frac{1}{\rho_g}\,,$$

$$\frac{d^2\,\frac{1}{\rho_n}}{ds^2} - \frac{2}{\tau_g}\,\frac{d\frac{1}{\rho_g}}{ds} - \frac{1}{\rho_g}\left(5\,\frac{d\frac{1}{\tau_g}}{ds} + K_{13}\right) - \frac{3}{\rho_g^2}\left(\frac{2}{\rho_n} - L\right),$$

$$\frac{d^2\,\frac{1}{\tau_g}}{ds^2} + \left(\frac{2}{\rho_n} - L\right)\frac{d\frac{1}{\rho_g}}{ds} - \frac{1}{\rho_g}\left[5\,\frac{d\frac{1}{\rho_n}}{ds} - \frac{6}{\tau_g}\,\frac{1}{\rho_g} - 3\,\frac{L}{ds} - K_{23}\right].$$

We have denoted by $\frac{1}{\rho_n}, \frac{1}{\rho_g}, \frac{1}{\tau_g}$ the normal curvature, the geodesic curvature and the geodesic torsion, by $L$ the mean curvature of the surface (sum of the principal curvatures), by $K_{13}$ the mixed Riemannian curvature at $M$, in the ambient space, of two plane elements tangent to the lines $(C)$, to one tangent, the other normal to the surface; finally by $K23$ the mixed Riemannian curvature at $M$ of the plane element tangent to the surface and of the plane element normal to the lines $(C)$.

The quantities $K_{13}$ and $K_{23}$ are zero if the Riemannian space is of constant curvature or if the normal direction to the surface is a principal direction of the ambient space.

## III. – Lines of curvature and asymptotic lines of a manifold embedded in a Riemannian space.

**207.** Let $V$ be a $p$-dimensional manifold embedded in a Riemannian space of $n$ dimensions. Let us attach to each point $M$ of the manifold a rectangular frame of reference $(R)$ defined by $p$ unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_p$ tangent to the manifold and $n-p$ unit vectors mutually rectangular to each other and normal to the manifold $\boldsymbol{e}_{p+1}, \boldsymbol{e}_{p+2}, ..., \boldsymbol{e}_n$. We will denote by the Latin letters $i, j, k, ...$ the indices $1, 2, ..., p$ and by the Greek letters $\alpha, \beta, \gamma, ...$ the indices $p+1, p+2, ..., n$. By moving on the manifold, the components $\omega_\alpha$ of the elementary displacement of the point $M$ are zero. The equations of structure of the space become

$$\left.\begin{array}{rcl} d\omega_i &=& [\omega_k\ \omega_{ki}], \\ 0 &=& [\omega_k\ \omega_{k\alpha}], \\ d\omega_{ij} &=& [\omega_{ik}\ \omega_{kj}] + [\omega_{i\lambda}\ \omega_{\lambda j}] + \frac{1}{2}R_{ijkh}[\omega_k\ \omega_h], \\ d\omega_{i\alpha} &=& [\omega_{ik}\ \omega_{k\alpha}] + [\omega_{i\lambda}\ \omega_{\lambda\alpha}] + \frac{1}{2}R_{i\alpha kh}[\omega_k\ \omega_h], \\ d\omega_{\alpha\beta} &=& [\omega_{\alpha k}\ \omega_{k\beta}] + [\omega_{\alpha\lambda}\ \omega_{\lambda\beta}] + \frac{1}{2}R_{\alpha\beta kh}[\omega_k\ \omega_h]. \end{array}\right\} \tag{9.26}$$

Let us begin with the case of an hypersurface ($p = n-1$). We have a generalisation of the second fundamental form of a surface in a space of three dimensions by considering the scalar product

$$-D\boldsymbol{e}_n \cdot \overrightarrow{dM} = \omega_i\omega_{in} = \gamma_{inj}\omega^i\omega^j\ ;$$

the coefficients $\gamma_{inj}$ are symmetric with respect to the first and last indices by virtue of the relation expressed by the second line of equations (9.26), where $\alpha = n$. The quotient of this form by $ds^2$ gives the *normal curvature* of any curve tangent to the direction $(\omega_1, \omega_2, ..., \omega_{n-1})$.

The *principal tangents* correspond to the stationary values of the normal curvature at a point; referred to the frame of reference formed by the unit vectors carried by the principal tangents, the second fundamental form is written

$$\frac{1}{R_1}\,\omega_1^2 + \frac{1}{R_2}\,\omega_2^2 + \cdots + \frac{1}{R_n}\,\omega_n^2\,,$$

by highlighting the *principal curvatures* $\frac{1}{R_i}$.

The *lines of curvature* are the lines tangent at each of their points to a principal tangent; when we move along a line of curvature, the unit vector normal to the hypersurface undergoes an absolute displacement parallel to the tangent to the line and this property characterises the lines of curvature.

The *asymptotic lines* are those that annul the second fundamental form.

**208.** If the manifold $V$ is of $p < n-1$ dimensions, things are more complicated. There exist $n-p$ quadratic differential forms that generalise the second fundamental form of a surface; these are the forms

$$\Phi_\alpha = -\left|D\boldsymbol{e}_\alpha \cdot \overrightarrow{dM}\right| = |\omega_i\omega_{i\alpha}| = \gamma_{i\alpha j}[\omega_i\ \omega_j] \qquad (\alpha = p+1, ..., n),$$

where the coefficients $\gamma_{i\alpha j}$ are symmetric with respect to their first and last indices.

The *asymptotic lines* are those which annul all these forms; it is possible that none exist.

We will define the *principal tangents* by a generalisation of a characteristic property of the principal tangents of a surface, for example the property according to which the absolute differential of the unit vector normal to the surface, when we move in the direction of a tangent to the surface, is parallel to that tangent. In the general case the tangent will be said to be principal if the absolute differential of any unit vector normal to $V$ has its tangential component parallel to the tangent considered. Since the $i^{\text{th}}$ tangential component of a vector whose $p$ first components $x_i$ are zero is the sum $X_\alpha \omega_{\alpha i}$, it is necessary and sufficient, for a tangent to be principal, that, whatever $\alpha > p$, there is proportionality between the forms $\omega_{\alpha 1}, \omega_{\alpha 2}, ..., \omega_{\alpha p}$ and the forms $\omega_1, \omega_2, ..., \omega_p$ when we move in the direction of this tangent. In other words it is necessary and sufficient that each of the quadratic indicatrices $\Phi_\alpha = 1$, situated in the plane element tangent to the manifold, admit the tangent considered as one of its axes.

THEOREM 9.2 — *A tangent is principal when it is a common axis of the $n-p$ indicatrices of the manifold.*

**209.** A *line of curvature* will be by definition a line whose tangent at each of its points is a principal tangent.

A notable class of $p$-dimensional manifolds is characterised by the property that the $n-p$ indicatrix quadrics have, at each point of the manifold, $p$ common rectangular axes. For the manifold $V$ to belong to this class, it is necessary and sufficient that we can choose the $p$ rectangular unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_p$ such that the forms $\Phi_\alpha$ contain only squared terms; if so the coefficients of rotation $\gamma_{i\alpha j}$ will be zero for $i \neq j$ and the form $\omega_{i\alpha}$ will be a multiple of $\omega_i$. It follows that each of the exterior quadratic forms $\omega_{h\alpha}\ \omega_{k\beta}]$ is zero, which gives us the relations

$$\gamma_{k\alpha i}\gamma_{k\beta j} - \gamma_{k\alpha j}\gamma_{k\beta i} = 0 \qquad (i, j = 1, 2, ..., p; \alpha, \beta = p+1, ..., n). \tag{9.27}$$

Conversely, if these $\frac{p(p-1)}{2}\ \frac{(n-p)(n-p-1)}{2}$ relations are satisfied then the manifold belongs to the class considered.

In fact, choose new basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_p$ such that the form $\Phi_{p+1}$ contains only squared terms; we will thus have $\gamma_{i,p+1,j} = 0$ for $i \neq j$. Setting now in relation (9.27) $\alpha = p+1, \beta = p+2$, we will have

$$\gamma_{i,p+2,j}[\gamma_{i,p+1,i} - \gamma_{j,p+1,j}] = 0 \qquad (i, j = 1, 2, ..., p);$$

if therefore the coefficients of $(\omega_i)^2$ and of $(\omega_j)^2$ in $\Phi_{p+1}$ are different, we will have $\gamma_{i,p+2,j} = 0$. Suppose for example that the coefficients of $(\omega_1)^2, (\omega_2)^2, ..., (\omega_h)^2$ in $\Phi_{p+1}$ are equal to each other but different from the following coefficients, we can take in the $h$-plane defined by $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_h$ another system of rectangular unit vectors in such a way as not to change the form $\Phi_{p+1}$, but to nullify the coefficients $\gamma_{i,p+2,j}$ for $i \neq j\ (i, j = 1, 2, ..., h)$. We proceed similarly for the following coefficients. We will thus come to reduce simultaneously the two forms $\Phi_{p+1}$ and $\Phi_{p+2}$ to have only squared terms. We continue thus for the form $\Phi_{p+3}$, and thus subsequently. *Equations (9.27) give therefore the necessary and sufficient conditions for the manifold to admit $p$ families of lines of curvature intersecting* *themselves orthogonally.* [5]

**210.** A particularly noteworthy case is that where all the lines of the manifold are lines of curvature. For this it is necessary and sufficient that all the forms $\Phi_\alpha$ are proportional to the $ds^2$ of the manifold. We can then choose the vectors $\boldsymbol{e}_{p+1}, \boldsymbol{e}_{p+2}, ..., \boldsymbol{e}_n$ in such a way that the forms $\Phi_{p+1}, \Phi_{p+2}, ...\Phi_{n-1}$ are identically zero, the form $\Phi_n$ is $A\,ds^2$, in other words such that we have

$$\omega_{i\alpha} = 0 \quad (\alpha = p+1, ..., n-1), \qquad \omega_{in} = A\omega_i.$$

Let us look for all the manifolds *embedded in the Euclidean space* that have this property. Denote now by $\alpha, \beta, ...$ the indices $p+1, p+2, ..., n-1$. the structure equations (9.26) whose left hand side is $d\omega_{in}$ gives $[dA\ \omega_i] = 0$. It follows from this last result that $A$ is a constant.

Two cases must be distinguished:

1° If $A = 0$, *the manifold is a plane manifold*, because the absolute differential of any normal vector is normal to the manifold.

2° If $A \neq 0$, the forms $[\omega_i\ \omega_{n\alpha}]$ are all zero: if therefore $p \geq 2$, the forms $\omega_{n\alpha}$ are zero and equations (9.26) whose left hand sides are $d\omega_{\alpha n}$ are identically satisfied. The point $O = M + \frac{1}{A}\boldsymbol{e}_n$ is then fixed, since its differential is zero because of the relations $\omega_{n\alpha} = 0$, $\omega_{ni} = -A\omega_i$. The manifold is then a locus of points equidistant from a fixed point $O$. On the other hand the $(p+1)$ plane containing the manifold $V$ and the point $O$ is fixed because the differentials of the point $M$, the vectors $\boldsymbol{e}_i$ and of the vector $\boldsymbol{e}_n$ are in this $p+1$ plane.

THEOREM 9.3 *The manifolds $V$ of the Euclidean space all of whose lines are a lines of curvature, are the plane manifolds and the p-dimensional hyperspheres.*

## IV. – Riemannian spaces that satisfy the axiom of the plane.

**211.** Let us give another application of the method of the moving frame by proving analytically the theorem already proved by geometric considerations (n° 179). Suppose that an $n$-dimensional Riemannian space has the property that every $p$ dimensional manifold $(2 \leq p \leq n-1)$ geodesic at a point is totally geodesic.

Attach to any point of a manifold $V_p$ a rectangular frame of reference as in the preceding section. We have, by moving along this manifold, $\omega_\alpha = 0$. The manifold is totally geodesic if the absolute differential of each of the vectors $\boldsymbol{e}_i$ is tangent to the manifold, in other words, if the forms $\omega_{i\alpha}$ are all zero (it is the same for

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[5] These manifolds are the manifolds with zero Gaussian torsion. *See*, on the subject of the concept of the Gaussian curvature of a manifold, and, more generally on the properties of manifolds embedded in a Riemannian space, E. Cartan, *La Géométrie des espaces de Riemann* (*Mém. Sc. math.*, IX, Chap. VI, p. 43-51).

all the forms $\Phi_\alpha$). Equations (9.26) then show that all the components $R_{i\alpha kh}$ must be zero ($ikh = 1, 2, ..., p; \alpha = p + 1, ..., n$). In particular if $i, j, k$ now denote three distinct indices taken from the sequence $1, 2, ..., n$, all the components of the form $R_{ijik}$ are zero: we can in fact always consider a manifold $V_p$ geodesic at a given point such that at this point the two vectors $\boldsymbol{e}_i, \boldsymbol{e}_k$ are tangent to it while the vector $\boldsymbol{e}_j$ is normal to it. This shows that *the mixed Riemannian curvature of two plane elements that have a common linear element and perpendicular to each other is zero.* We will show that this property entrains the isotropy of the space at each of its points.

**212.** In fact, by an infinitesimal rotation of the frame of reference attached to any point of the space, the components $R_{ijkh}$ will undergo a certain infinitesimal linear substitution. No, by a simple infinitesimal rotation by an angle $\alpha$ parallel to the biplane determined by the basis vectors $\boldsymbol{e}_\ell, \boldsymbol{e}_m$, the components $X_\ell$ of a vector undergoes the elementary variation $\alpha X_m$ and the component $X_m$ the elementary variation $-\alpha X_\ell$, with the other components not varying. By such a rotation, which we will denote by the symbol $\{\ell m\}$, the quantity $R_{ijkh}$, which transforms as the product $X_i Y_j Z_k T_h$ of four vectors, will undergo an elementary variation equal, up to a factor $\alpha$, to the sum of the components obtained by replacing, wherever it is found, the index $\ell$ by the index $m$, reduced by the sum of the components obtained by replacing, wherever it is found, the index $m$ by the index $\ell$.

Let us apply then the rotation $\{jk\}$ to the component $R_{ijik}$, which, by hypothesis, must be zero, whatever the choice of rectangular frames of reference; we will have

$$R_{ijij} = R_{ikik} \qquad \text{(do not sum).} \tag{9.28}$$

Now let us apply to the same component $R_{ijik}$ the rotation $\{i\ell\}$, it will lead to

$$R_{\ell jik} + R_{ij\ell k} = 0;$$

by cyclic permutation of the indices $i, j, k$, we will have

$$R_{\ell jki} = R_{\ell kij} = R_{\ell ijk},$$

But since the sum of the three parts of this double equality is zero, each of them is zero.

The only components of the Riemann-Christoffel tensor which are not zero are thus necessarily of the form $R_{ijij}$. But since according to (9.28) these components keep their numerical value by the change of any one of its two indices, so that they all have the same value $-K$. the space is thus isotropic at each of its points.

Q.E.D.

The proof shows that if all the $p$-dimensional manifolds geodesic at a point are totally geodesic for a particular value of $p$ at least equal to 2, it is the same for every other value of $p \geq 2$ (*Cf.* n° 115).

# 10 Riemann normal coordinates

### I. – Normal coordinates.

**213.** Consider a given point $O$ of a Riemannian space and imagine, which is always possible, a rectangular frame of reference $(R_0)$ that has this point as origin. Any point of the space, sufficiently close to $O$ is on a particular geodesic from $O$; let $\alpha_i$ be the direction cosines of its tangent at the origin and $s$ the arc length $OM$ measured on this geodesic. The *normal coordinates* of the point $M$ are the $n$ quantities $x^i$ defined by the relations

$$x^i = \alpha^i s. \tag{10.1}$$

Practically we can always, starting from any system of coordinates $(u^1, ..., u^n)$, deduce another such that, at the point $O$, the coordinates are all zero and that the coefficients $g_{ij}$ of the fundamental form are equal to 1 for $i = j$, and to 0 for $i \neq j$. It is sufficient to perform on the $u^i$ a suitable linear substitution with constant coefficients. Assume this done. We will arrive at normal coordinates by integrating the differential equations of the geodesics

$$\frac{d^2u^i}{ds^2} + \Gamma_k{}^i{}_h \frac{du^k}{ds}\frac{du^h}{ds} = 0,$$

with the initial conditions $u^i = 0$ for $s = 0$. We will then have

$$x^i = s\left(\frac{du^i}{ds}\right)_0.$$

**214.** Formulae (10.1) define, if we want, a representation of the Riemannian space on an Euclidean space in which the $x^i$ play the role of the classical rectangular coordinates. We see immediately that a geodesic from $O$ is represented by a straight line, and a geodesic manifold at $O$ by a plane manifold. The Euclidean space on which this representation is made will be called the *normal Euclidean space at the point O*.

It is easy to see that the normal Euclidean space osculates the Riemannian space at $O$. In fact, let the differential equations of the geodesics be, in normal coordinates

$$\frac{d^2u^i}{ds^2} + \Gamma_k{}^i{}_h \frac{du^k}{ds}\frac{du^h}{ds} = 0. \tag{10.2}$$

They are satisfied if we replace the $x^i$ by $\alpha^i s$, where the $\alpha^i$ are arbitrary constants; we have therefore, at all points of the geodesic considered,

$$\Gamma_{k\ h}^{\ i}\alpha^k\alpha^h = 0. \tag{10.3}$$

In particular, at the point $O$, the relation is valid whatever the constants $\alpha^i$. We have therefore

$$\Gamma_{k\ h}^{\ i} = 0,$$

which expresses precisely that the normal Euclidean space osculates at $O$ the Riemannian space.

**215.** We propose to calculate the form of $ds^2$ of the Riemannian space in the neighbourhood of the origin $O$. By putting

$$ds_0^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2,$$

the difference $ds^2 - ds_0^2$ must have all its coefficients of at least second degree with respect to the $x^i$. Let $\Phi(x, dx)$ be the collection of the terms of second degree in the expansion, assumed possible, of the form $ds^2 - ds_0^2$. When we move on a geodesic from $O$, we obviously have $ds^2 = ds_0^2$; also the $dx^i$ are proportional to $x^i$; we have then

$$\Phi(x, x) = 0.$$

Consequently, we must assume that the form $\Phi(x, dx)$ is homogeneous and of second degree with respect to the quantities $x^i\,dx^j - x^j\,dx^i$. A calculation will allow us to verify this later (n° 223).

## II. – The fundamental differential equations.

**216.** The frame of reference $(R_0)$ at the point $O$ being rectangular, let us attach to any point $M$, sufficiently close to $O$, the rectangular frame of reference $(R)$ obtained from it by parallel transport along the arc of the geodesic $OM$. We will say that the family of frames of reference $(R)$ is *adapted* to the normal coordinates considered, which are determined by the originating frame of reference $(R_0)$.

That said, we will calculate the forms $\omega^i = \omega_i, \omega_{ij} = -\omega_{ji}$ which define the infinitesimal translation and rotation that take us from the frame of reference at the point $M$ to the frame of reference at the infinitely close point $M'$. But, instead of expressing these forms by means of the normal coordinates $x^i$, we will express then first by means of the $n+1$ redundant coordinates $a^1, a^2, ..., a^n, t$ such that we have $x^i = a^i t$; the coordinates $a^i$ are constant along a geodesic from $O$ and are the direction parameters of this geodesic at $O$, the coordinate $t$ being zero at $O$. It is very clear that the forms $\omega_i, \omega_{ij}$ depend only on the products $a^i t = x^i$; we can, to express them by means of the $x^i$, replace $t$ everywhere by 1 and $a^i$ by $x^i$.

If we leave the coordinates $a^i$ fixed, by varying only $t$, the frame of reference $(R)$ undergoes a parallel transport and, for the same value of $dt$, the vectors $\overrightarrow{MM'}$ remain equivalent to themselves and consequently keep the same components with are clearly $\omega^i = a^i dt$; as for the forms $\omega_{ij}$, these are obviously zero under the same conditions.

Denote now by $\varpi^i, \varpi_{ij}$ the expressions that the forms $\omega^i, \omega_{ij}$ take when, with $t$ fixed, we vary the $a^i$; these are forms linear in $da^1, da^2, ..., da^n$ whose coefficients are functions of $t$ and the $a^i$.

The complete forms $\omega^i, \omega_{ij}$ are given by the formulae

$$\left.\begin{array}{rcl} \omega^i(t,a;dt,da) & = & a^i dt + \varpi^i(t,a;da), \\ \omega_{ij}(t,a;dt,da) & = & +\varpi_{ij}(t,a;da). \end{array}\right\} \tag{10.4}$$

**217.** We will seek to determine the forms $\varpi^i, \varpi_{ij}$, *considered as functions of* $t$, with the quantities $a^i, da^i$ playing the role of parameters, by a system of linear differential equations of the first order.

For this, we begin from the *equations of structure* [n$^{\text{os}}$ 201 and 202, formulae (**??**) and (**??**)]

$$\left.\begin{array}{rcl} d\omega^i & = & [\omega^k \ \omega_{ki}], \\ d\omega_{ij} & = & [\omega_{ik} \ \omega_{kj}] + \frac{1}{2} R_{ijkh}[\omega^k \ \omega^h]. \end{array}\right\} \tag{10.5}$$

By replacing $\omega^i$ and $\omega_{ij}$ by their expressions (10.4) and identifying in the expanded equations the terms that contain $dt$, we get

$$\left.\begin{array}{r} [da^i \ dt] + \left[dt \ \dfrac{\partial \varpi^i}{\partial t}\right] + d\varpi^i = [(a^k dt + \omega^k) \ \varpi_{ki}], \\ \left[dt \ \dfrac{\partial \varpi_{ij}}{\partial t}\right] + d\varpi_{ij} = [\varpi_{ik} \ \varpi_{kj}] + \dfrac{1}{2} R_{ijkh}[(a^k dt + \varpi^k) \ (a^h dt + \varpi^h)]; \end{array}\right\} \tag{10.6}$$

we have denoted by $d\varpi^i$, $d\varpi_{ij}$ the exterior derivatives of the forms $\varpi^i$, $\varpi_{ij}$, *regarding $t$ in them as a parameter*. By equating in them the collection of terms that contain $dt$ on the two sider of relations (10.6), we obtain the differential equations that we seek, which are fundamental

$$\left.\begin{array}{rcl} \dfrac{\partial \varpi^i}{\partial t} & = & da^i + a^k \varpi_{ki}, \\ \dfrac{\partial \varpi_{ij}}{\partial t} & = & \frac{1}{2} R_{ijkh}(a^k \varpi^h - a^h \varpi^k) = R_{ijkh} a^k \varpi^h. \end{array}\right\} \tag{10.7}$$

We have to integrate these differential equations. The initial conditions are obvious, because for $t = 0$, the frame of reference $(R)$ is, for all the values of $a^i$, the fixed frame of reference $(R_0)$ at the point $O$, so that we have, whatever the values of the $a^i$ and the $da^i$,

$$\varpi^i(0,a;da) = 0, \qquad \varpi_{ij}(0,a;da) = 0. \tag{10.8}$$

We get for the forms $\overline{\omega}^{\,i}$ the system of differential equations

$$\frac{\partial^2\overline{\omega}^{\,i}}{\partial t^2} = R_{kihj}a^k a^h \overline{\omega}^{\,j}, \tag{10.9}$$

with the initial conditions

$$\overline{\omega}^{\,i} = 0, \qquad \frac{\partial\overline{\omega}^{\,i}}{\partial t} = da^i, \qquad \text{for } t = 0. \tag{10.10}$$

**218.** From differential equations (10.9) we draw the following important consequence:

THEOREM 10.1 *If we know as a function of the normal coordinates with respect to an originating frame of reference* $(R_0)$*, the components of the Riemann-Christoffel tensor referred to the family of frames of reference adapted to these coordinates, the fundamental form, expressed by means of the normal coordinates considered, is uniquely determined.*

In fact, if we assume the components $R_{kihj}$ are known functions of the coordinates $t, a^1, a^2, ..., a^n$, equations (10.9) admit an uniquely determined solution if we take into consideration the initial conditions (10.10). On the other hand the forms $\omega^i$, according to (10.4), are deduced from the forms $\overline{\omega}^{\,i}$ by making $t = 1$ and $a^i = x^i$. The $ds^2$ of the space is therefore perfectly determined.

We can add an important corollary:

THEOREM 10.2 *Given two Riemannian spaces* $E$, $E'$ *with the same number* $n$ *of dimensions and in each of them two systems of normal coordinates respectively relative to two originating rectangular frames of reference* $(R_0), (R'_0)$*, if the components of the Riemann-Christoffel tensor, referred in each space to the family of frames of reference adapted to the system of coordinates, are the same functions of the normal coordinates, the two spaces are applicable one onto the other.*

In fact, the two fundamental forms have as coefficients the same functions of the normal coordinates. The application is *local*, because the theorem makes sense only in the neighbourhood of the point of origin such that for any point of this neighbourhood there passes one and one only geodesic from the point of origin and not outside the neighbourhood.

**219.** Consider the case of an *analytic* space. This means that we can choose a system of coordinates $u^i$ such that the components $g_{ij}$ of the fundamental tensor will be *analytic* functions of the coordinates. In this case, the normal coordinates $x^i$ relative to any originating frame of reference $(R_0)$ are analytic functions of the $u^i$ and the components $R_{ijkh}$ of the Riemannian curvature referred to the family of frames of reference adapted to these normal coordinates $x^i$ are also analytic functions of the $x^i$ and consequently of the coordinates $t$ and $a^i$. They are thus completely determined if we know for $t = 0$ their numerical values as well as those of their successive derivatives with respect to $t$. Now when we vary $t$ by leaving $a^1, a^2, ..., a^n$ fixed, the frame $(R)$ moving parallel to itself, the ordinary differential $dR_{ijkh}$, taken with respect to $t$, coincides with the absolute differential; we thus have at any point

$$\frac{\partial R_{ijkh}}{\partial t} = R_{ijkh|\ell} a^\ell, \qquad \frac{\partial^2 R_{ijkh}}{\partial t^2} = R_{ijkh|\ell m} a^\ell a^m.$$

We thus arrive at the following theorem:

THEOREM 10.3 *The $ds^2$ of an analytic Riemannian space expressed by means of normal coordinates with respect to a point $O$ is completely determined if we know at $O$ the numerical values of the components of the Riemann-Christoffel tensor and of all their successive covariant derivatives.*

To this theorem also corresponds a sufficient condition of application of two analytic Riemannian spaces with the same number of dimensions.

## III. – The $ds^2$ of spaces of constant curvature expressed in terms of normal coordinates

**220.** According to the theorems of n°, all Riemannian spaces with given constant curvature $K$ and of the same number of dimensions have the same fundamental form referred to any system of normal coordinates. All these spaces are thus applicable and in an infinity of ways, because once the rectangular frame of reference $(R_0)$ is chosen in one of these spaces, we can make it correspond to any rectangular frame in any of the other spaces. We recover also by this argument the property of free mobility of spaces with constant curvature.

To calculate effectively $ds^2$ in normal coordinates, we will integrate equations (10.9). We see first, according to the first equations (10.7) and the antisymmetry of the forms $\overline{\omega}_{ij}$, that the sum $a^i \omega_i$ admits as derivative with respect to $t$ the sum $a^i\, da^i$. We have therefore

$$a^i \omega_i = t a^i\, da^i;$$

by putting

$$(a^1)^2 + (a^2)^2 + \cdots + (a^n)^2 = \ell^2,$$

we have

$$a^i \overline{\omega}_i = t\ell\, d\ell. \tag{10.11}$$

Note now that equations (10.9) take the form

$$\frac{\partial^2 \overline{\omega}^i}{\partial t^2} = -K a^k (a^k \overline{\omega}^{\,i} - a^i \overline{\omega}^{\,k});$$

we deduce

$$\begin{aligned}\frac{\partial^2(a^i\overline{\omega}^{\,j}-a^j\overline{\omega}^{\,i})}{\partial t^2}&=-Ka^ka^i(a^k\overline{\omega}^{\,j}-a^j\overline{\omega}^{\,k})+Ka^ka^i(a^k\overline{\omega}^{\,i}-a^i\overline{\omega}^{\,k})\\&=-K\ell^2(a^i\overline{\omega}^{\,j}-a^j\overline{\omega}^{\,i});\end{aligned}$$

since the initial value of the derivative of $a^i\overline{\omega}^{\,j}-a^j\overline{\omega}^{\,i}$ with respect to $t$ is $a^i\,da^j - a^j\,da^i$, we deduce by an immediate calculation

$$a^i\overline{\omega}^{\,j}-a^j\overline{\omega}^{\,i}=\begin{cases}\dfrac{\sin(\ell\sqrt{K}\,t}{\ell\sqrt{K}}\,(a^i\,da^j-a^j\,da^i) & (K>0);\\[2ex] \dfrac{\sin(\ell\sqrt{-K}\,t}{\ell\sqrt{-K}}\,(a^i\,da^j-a^j\,da^i) & (K<0).\end{cases}\tag{10.12}$$

By summing the squares of the $\frac{n(n-1)}{2}$ equations (10.12), we get the relation

$$\begin{aligned}&\frac{\sin^2(\ell\sqrt{K}\,t}{K\ell^2}\sum_{(ij)}(a^i\,da^j-a^j\,da^i)^2\\&\quad=\ell^2[(\overline{\omega}^{\,1})^2+(\overline{\omega}^{\,2})^2+\cdots+(\overline{\omega}^{\,n})^2]-\ell^2[a^1\,da^1+\cdots+a^n\,da^n]^2.\end{aligned}\tag{10.13}$$

The sum is over pairwise combinations of the indices $i, j = 1, 2, ..., n$.

**221.** To get the $ds^2$ we sought in normal coordinates, it is sufficient to replace in formula (10.13) $t$ by 1 and $a^i$ by $x^i$ and to see what becomes of the sum $(\overline{\omega}^{\,1})^2+(\overline{\omega}^{\,2})^2+\cdots+(\overline{\omega}^{\,n})^2$. We thus get

$$ds^2=\frac{\left\{\begin{aligned}&x^1\,dx^1+x^2\,dx^2+\cdots+x^n\,dx^n\\&\quad+\frac{\sin^2\sqrt{K[(x^1)^2+\cdots+(x^n)^2]}}{K[(x^1)^2+\cdots+(x^n)^2]}\sum_{(ij)}(x^i\,dx^j-x^j\,dx^i)^2\end{aligned}\right\}}{(x^1)^2+(x^2)^2+\cdots+(x^n)^2}.\tag{10.14}$$

But we can also get the form of $ds^2$ in *polar coordinates* by subjecting $a^1, a^2, ..., a^n$ to the condition $\ell = 1$; $t$ then becomes the radius vector $OM$, which we will denote by $r$.

We deduce immediately from equations (10.11) and (10.9) that we have

$$\overline{\omega}^{\,i}=\frac{\sin(r\sqrt{K}\,)}{r\sqrt{K}}\,da^i,$$

from which

$$\omega^i=a\,dr+\frac{\sin(r\sqrt{K}\,)}{r\sqrt{K}}\,da^i,$$

and, by an easy calculation,

$$\begin{aligned}ds^2=dr^2+\frac{\sin^2(r\sqrt{K}\,)}{Kr^2}\,d\sigma^2\quad(K>0)\\ \text{or}\quad dr^2-\frac{\sinh^2(r\sqrt{-K}\,)}{Kr^2}\,d\sigma^2\quad(K<0),\end{aligned}\tag{10.15}$$

where we have denoted by $d\sigma^2$ the sum $(da^1)^2 + (da^2)^2 + \cdots + (da^n)^2$, the fundamental form of the hypersphere of radius 1 of the Euclidean space of $(n-1)$ dimensions.

**222.** We can moreover put formula (10.14) into another form, which we get by replacing $(x^1\,dx^1 + x^2\,dx^2 + \cdots + x^n\,dx^n)^2$ by

$$r^2\,ds_0^2 - \sum_{(ij)} (x^i\,dx^j - x^j\,dx^i)^2,$$

which gives[1]

$$ds^2 = ds_0^2 - \frac{Kr^2 - \sin^2(r\sqrt{K}\,)}{Kr^4} \sum_{(ij)} (x^i\,dx^j - x^j\,dx^i)^2, \tag{10.16}$$

where $ds_0^2$ denotes the fundamental form of the Euclidean space referred to rectangular coordinates $x^1, x^2, ..., x^n$, and where $r$ denotes, as in the preceding paragraph, the square root of $(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2$. We have moreover in the neighbourhood of the origin

$$\frac{Kr^2 - \sin^2(r\sqrt{K}\,)}{Kr^4} = \frac{1}{3}\,K - \frac{2}{45}\,K^2r^2 + \frac{2^7}{8!}\,K^3r^4 - \frac{2^9}{10!}\,K^4r^8 + \cdots\,,$$

and this formula applies whether $K$ is positive or negative.

Formula (10.16) confirms, for the case of spaces of constant curvature, the suggestion made in n° 215 about the form of the expression $ds^2 - ds_0^2$. It is this suggestion whose general validity we shall now prove.

## IV. – Properties of the fundamental form in normal coordinates.

**223.** We will prove the following two theorems:

THEOREM I. — *The form $\overline{\omega}^{\,i} - t\,da^i$ is a linear combination of the expressions $a^k\,da^h - a^h\,da^k$.*

THEOREM II. — *The form $ds^2 - ds_0^2$ is a quadratic form in the expressions $x^k\,dx^h - x^h\,dx^k$.*

We begin with theorem I. The fundamental differential equations (10.9) (n° 217) give

$$\frac{\partial^2}{\partial t^2}(\overline{\omega}^{\,i} - t\,da^i) = R_{kihj}a^k a^h \overline{\omega}^{\,j}. \tag{10.9}$$

Put *a priori*

$$\overline{\omega}^{\,i} - t\,da^i = A_{ikh}a^k\,da^h \qquad (A_{ikh} = -A_{ihk}).$$

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[1] This formula is due to E. Beltrami (*Annali di Mat.,* 2° serie. t. 2, 1869, p 24 [formula (20)].

Equations (10.9) are satisfied if the unknown functions $A_{ikh}$ satisfy the equations

$$\frac{\partial^2 A_{ikh}}{\partial t^2} = ta^r R_{rikh} + a^r a^s R_{risj} A_{jkh}, \tag{10.17}$$

with initial conditions $A_{ikh} = 0$, $\frac{\partial A_{ikh}}{\partial t} = 0$. These equations admit a definite solution, but it is necessary to verify that the functions obtained are antisymmetric with respect to the two last indices. Now this is obvious because the functions $A_{ikh} + A_{ihk}$, which are zero as well as their first derivatives for $t = 0$, satisfy a system of linear and *homogeneous* differential equations. The theorem is thus proved.

**224.** The $ds^2$ of the space is the sum of the squares of the expressions $\varpi^i$, when we replace $t$ by 1 and $a^i$ by $x^i$. Now the formula

$$\varpi^i = t\,da^i + A_{ikh} a^k\,da^h$$

shows that the proof of theorem II reduces to proving that the sum $A_{ikh} da^i$ is a linear combination of the expressions $a^r\,da^s - a^s\,da^r$. So put

$$A_{ikh} da^i = B_{khrs} a^r\,da^s \qquad (B_{khrs} = -B_{khsr} = -B_{hkrs}).$$

These relations will be compatible with equations (10.10) if the functions $B_{khrs}$, which are zero as well as their first derivatives for $t = 0$, satisfy the differential equations

$$\frac{\partial^2 B_{khrs}}{\partial t^2} = tR_{rskh} + a^t R_{rsij} A_{jkh}, \tag{10.18}$$

and it is clear that the solution of these equations which is compatible with the initial conditions is antisymmetric at the same time with respect to the indices $r$ and $s$ and with respect to the indices $k$ and $h$.

**225.** We will now look for a limited expansion of the forms $\omega^i$ as well as of the fundamental form. For this let us determine the expansions of the functions $A_{ikh}$ and $B_{khrs}$ up to terms in $t^4$, assuming of course that the components of the initial fundamental tensor admit continuous partial derivatives of sufficiently high order. A very easy calculation gives, according to (10.17) and (10.18),

$$\begin{aligned} A_{ikh} &= \frac{1}{6}\,t^3 a^r R_{rikh} + \frac{1}{12}\,t^4 a^r a^s R_{rikh|s}, \\ B_{khrs} &= \frac{1}{6}\,t^3 R_{rskh} + \frac{1}{12}\,t^4 a^u R_{rskh|u}. \end{aligned}$$

It is necessary naturally to give the components of the Riemann-Christoffel tensor and to their covariant derivatives the numerical values that they have at the origin.

We deduce the expansion

$$\varpi^i = t\,da^i + \frac{1}{6}\,t^3 a^r \left( R_{rikh} + \frac{1}{2}\,ta^s R_{rikh|s} \right) a^k\,da^h,$$

from which

$$\omega^i = dx^i + \frac{1}{6}\, x^r \left( R_{rikh} + \frac{1}{2}\, x^s R_{rikh|s} \right) x^k \, dx^h. \tag{10.19}$$

We will get $ds^2$ by calculating the sum of the squares of the forms $\overline{\omega}^{\,i}$, and then setting $t = 1$, $a^i = x^i$. We will get

$$ds^2 = ds_0^2 + G_{khrs} x^k x^r \, dx^h \, dx^s,$$

where $G_{khrs}$ is deduced from the expression $B_{khrs} + B_{rskh} + A_{ikh} A_{irs}$ by replacing $t$ by 1 and $a^i$ by $x^i$. The limited expansion of $G_{khrs}$ is

$$\frac{1}{3} \left( R_{rskh} + \frac{1}{2}\, x^u R_{rskh|u} \right),$$

from which the limited expansion[2]

$$ds^2 = ds_0^2 + \frac{1}{3} \left( R_{rskh} + \frac{1}{2}\, x^u R_{rskh|u} \right) x^r x^k \, dx^s \, dx^h, \tag{10.20}$$

or

$$ds^2 = ds_0^2 + \frac{1}{12} \left( R_{rskh} + \frac{1}{2}\, x^u R_{rskh|u} \right) (x^r \, dx^s - x^s \, dx^r)(x^k \, dx^h - x^h \, dx^k). \tag{10.20$'$}$$

**226.** We deduce from (10.20) the limited expansions of the components of the fundamental tensor *referred to normal coordinates*

$$g_{ij} = \delta_{ij} + \frac{1}{3} \left( R_{ikjh} + \frac{1}{2}\, x^\ell R_{ikjh|\ell} \right) x^k x^h \quad \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \tag{10.21}$$

as well as the Christoffel symbols of the first kind, which are in first approximation

$$\Gamma_{ikj} = -\frac{1}{3} (R_{ikjh} + R_{jkih}) x^h; \tag{10.22}$$

the symbols of the second kind $\Gamma_{i\ j}^{\ k}$, to the same approximation, have the same expressions because of the property of the $g^{ij}$ of being, to this approximation, equal to 1 for $i = j$ and to 0 for $i \neq j$.

We will now apply the limited expansions (10.19) and (10.20) to the study of the modifications borne by the Riemannian curvature of the space at lengths measured in the neighbourhood of the point $O$ in the Riemannian space and in the normal osculating Euclidean space at $O$.

## V. – Comparison of distances in Riemannian space and in the osculating normal Euclidean space.

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[2] This expansion, disregarding the term in parentheses $\frac{1}{2}\, x^u R_{rskh|u}$, is due to Riemann (*Gesamm. Werke,* Leipzig, 1876, p 261). It was proved for the first time by R. Dedekind, in his Note from *Gesamm. Werke* of Riemann (p 384-391).

**227.** We can interpret formula (10.20) geometrically. Consider in the normal Euclidean space with rectangular coordinates $x^i$, a very small parallelogram $OMM'P$ that has $O$ as one of its vertices; let $x^i$ be the coordinates of $M$, $dx^i$ thos of $P$ and consequently $x^i + dx^i$ those of $M'$. The parallelogram defines a bivector with components

$$p^{ij} = x^i\,dx^j - x^j\,dx^i.$$

Formula (10.20') states that the square $ds^2$ of the distance $MM'$, measured with the Riemannian metric, is equal to the square $ds_0^2$ of this same distance, measured with the normal Euclidean metric, increased by the quantity $\frac{1}{12}R_{rskh}p^{rs}p^{kh}$. Now, according to the definition itself of the Riemannian curvature $K$ of the space in the direction of the plane element of the parallelogram [n° 174, formula (7.19)], this quantity is equal to $-\frac{1}{3}K\,dS^2$, where $dS$ denotes the area of the parallelogram. We thus have the formula

$$ds^2 = ds_0^2 - \frac{1}{3}K\,dS^2.$$

Let $h$ be the distance from $O$ to $MM'$; we have

$$dS = h\,ds_0,$$

and, consequently,

$$ds^2 = ds_0^2\left(1 - \frac{1}{3}Kh^2\right),$$

or

$$ds = ds_0\left(1 - \frac{1}{6}Kh^2\right). \tag{10.23}$$

We see that if the curvature $K$ is positive, the representation in the normal Euclidean space increases the lengths of lines drawn in the neighbourhood of $O$. If $K$ is negative, lengths are on the contrary reduced.

**228.** We will give relation (10.23) a more rigorous and more precise proof.

Take through $O$, in the Riemannian space, a geodesic segment $OM$ of length $a$, and let $x^i$ be the normal coordinates of its endpoint. Then take through $M$ a geodesic segment $MP$ of length $b$ and suppose that the unit vector tangent at $M$ to this segment arises from parallel transport along $OM$ of a unit vector with origin $O$ and with components $\alpha^i$; finally, let $y^i$ be the coordinates of $P$. We will compare the length $b$ of the segment $MP$, as measured in the Riemannian space, with the length

$$b_0 = \sqrt{(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2}$$

as measured in the normal Euclidean space.

Let us evaluate first the components $X^i$, referred to the natural frame of reference with origin $M$ pertaining to the normal coordinates, of the unit vector which is the result of parallel transport along $OM$ of the vector $\alpha^i$ with origin $O$. The components $\omega^i$ of the vector $\alpha^i$ with respect to the rectangular frame of Reference $(R_0)$ are equal to $\alpha^i$; the components $\omega^i$ of the second vector with origin $M$, with respect to the rectangular frame of reference adapted to the system of normal coordinates are also equal to $\alpha^i$, but, according to (10.19), they are equal, up to infinitesimals of the third order, to[3]

$$\omega^i = X^i + \frac{1}{6} R_{rikh} x^r x^k X^h;$$

resulting in the formulae we seek

$$X^i = \alpha^i - \frac{1}{6} R_{rikh} x^r x^k \alpha^h. \tag{10.24}$$

Let us move now to the calculation of the coordinates of the point $P$. The geodesic $MP$ is provided by integration of the differential equations

$$\frac{d^2 y^i}{ds^2} + \Gamma_k{}^i{}_h \frac{dy^k}{ds}\frac{dy^h}{ds} = 0,$$

with the initial conditions $y^i = x^i$ for $s = 0$. We deduce

$$y^i = x^i + bX^i - \frac{1}{2} b^2 (\Gamma_k{}^i{}_h)_M X^k X^h - \frac{1}{6} b^3 \left(\frac{\partial \Gamma_k{}^i{}_h}{\partial y^r}\right)_M X^k X^h X^r,$$

by neglecting the terms of fourth order. Now let us replace the $X^i$ by their values (10.24) and note that up to infinitesimals of second order. we have, according to (10.22),

$$(\Gamma_k{}^i{}_h)_M = -\frac{1}{3}(R_{kihr} + R_{hikr}) x^r,$$

and that up to infinitesimals of the first order, we have

$$\left(\frac{\partial \Gamma_k{}^i{}_h}{\partial y^r}\right)_M = -\frac{1}{3}(R_{kihr} + R_{hikr}).$$

It follows that, up to infinitesimals of the fourth order,

$$\begin{aligned} y^i = x^i + b\alpha^i - \frac{1}{6} b x^r x^s R_{rish} \alpha^h + \frac{1}{6} b^2 \left(R_{kihr} + R_{hikr}\right) \alpha^k \alpha^h \\ + \frac{1}{18} b^3 \left(R_{kihr} + R_{hikr}\right) \alpha^k \alpha^h \alpha^r. \end{aligned} \tag{10.25}$$

we can deduce, up to fifth order,

$$\begin{aligned} b_0^2 = (y^1 - x^1)^2 + \cdots + (y^n - x^n)^2 = b^2 - \frac{1}{3} b^2 R_{rish} x^r x^s \alpha^i \alpha^h \\ + \frac{1}{3} b^3 x^r \left(R_{kihr} + R_{hikr}\right) \alpha^i \alpha^k \alpha^h \\ + \frac{1}{9} b^4 \alpha^i \left(R_{kihr} + R_{hikr}\right) \alpha^k \alpha^h \alpha^r. \end{aligned}$$

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[3] In fact in the formulae (10.19), the $dx^k$ can be regarded as the components *with respect to a natural frame of reference* of a vector whose components, with respect to the natural frame $(R)$, are equal to $\omega^i$.

But, because of the antisymmetry of the components $R_{ijkh}$ with respect to the two first or the two last indices, the two last sums of the right hand side are zero. As regards the second term of the right hand side, which can be written

$$-\frac{1}{12}\,b^2 R_{rish}(\alpha^r x^i - \alpha^i x^r)(\alpha^s x^h - \alpha^h x^s),$$

it is equal to $\frac{1}{3}\,b^2 K h^2$, where $K$ denotes the Riemannian curvature along the plane element which contains the directions which contains the directions $OM$ and $MP$ and $h$ the distance from $O$ to $MP$. We recover the formula

$$b_0^2 = b^2\left(1 + \frac{1}{3}\,Kh^2\right),$$

or

$$b = b_0\left(1 - \frac{1}{6}\,Kh^2\right), \tag{10.26}$$

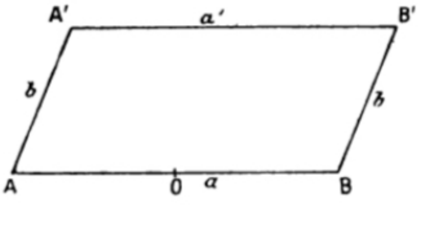
identical, up to notation, to formula (10.23).

Q.E.D.

## VI. – The parallelogramoid of Levi-Civita.

**229.** Levi-Civita calls the figure obtained by starting from a geodesic arc $AB$, and from another geodesic arc $AA'$, from this geodesic arc transported by parallelism to $BB'$ along the geodesic $AB$, and finally from the geodesic arc joining $A'B'$, a *parallelogramoid.* If we call $a$ the length of the arc $AB$, $b$ the common length of the arcs $AA'$ and $BB'$, and finally $a'$ the length of the arc $A'B'$, there exists a remarkable formula that gives the Riemannian curvature $K$ of the space in the direction of the parallelogramoid (assumed very small) by means of the lengths $a$ and $a'$ and the area $S$ of the parallelogramoid.

**Figure 1**



To find this formula, refer the figure to the normal Euclidean space at the centre $O$ of $AB$. Denote by $\alpha^i$ the direction parameters of the unit vector tangent to $AA'$ at the point $A$, once transported by parallelism from $A$ to $O$.

Formulae (10.25) give the normal coordinates $y^i$ of $B'$ by replacing the $x^i$ by the normal coordinates of $B$.

The normal coordinates $y^i$ of $A'$ are deduced by changing $x^i$ into $-x^i$.

The square of the Euclidean distance ${a'}^2_0$ of the two given points $A'$ and $B'$, by a calculation analogous to that of n° 228,

$$ {a'}^2_0 = a^2 + \frac{8}{3}\, b^2 R_{ikjh} x^i x^j x^k x^h , $$

by neglecting infinitesimals of fifth order.

Suppose

$$ x^1 = \frac{a}{2}, \qquad x^2 = 0, \qquad x^3 = 0 \qquad ...; $$
$$ \alpha^1 = \cos\varphi, \qquad \alpha^2 = \sin\varphi, \qquad x^3 = 0 \qquad ...; $$

we will have

$$ {a'}^2_0 = a^2 \left(1 - \frac{2}{3}\, K b^2 \sin^2\varphi\right), $$

and, taking into account formula (10.26),

$$ {a'}^2 = a^2 \left(1 - \frac{2}{3}\, K b^2 \sin^2\varphi\right)\left(1 - \frac{1}{3}\, K h^2\right), $$

where $h$ obviously denotes $b\sin\varphi$. Consequently

$$ {a'}^2 = a^2 \left(1 - K b^2 \sin^2\varphi\right) = a^2 - K S^2 . $$

We arrive finally at the formula of Levi-Civita[4]

$$ K = \frac{a^2 - {a'}^2}{S^2} . \tag{10.27} $$

## VII. – Geodesic triangles.

**230.** The use of normal coordinates allows us to complete a previously proved theorem (n° on the sum of the angles f a geodesic triangle.

Consider in the Riemannian space a very small geodesic triangle $ABC$, and represent it in the normal Euclidean space at $C$ (Figure 2). Calling the Riemannian lengths of the three sides $a, b, c$; the Euclidean distance of the vertices taken pairwise $a_0, b_0, c_0$, we have

$$ a = a_0, \qquad b = b_0, $$

and, according to (10.26),

$$ c = c_0 \left(1 - \frac{1}{6}\, K h^2\right), $$

where $h$ denotes the height from $C$. Now we have

$$ c_0^2 = a^2 + b^2 - 2ab\cos C, $$

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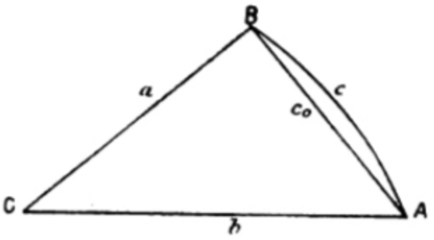
[4] T. Levi-Civita, *Nozione di parallelismo in una varietà qualunque* (*Rend. Cic. mat. Palermo*, t. 42, 1917, p 291).

**Figure 2**



from which

$$c^2 = a^2 + b^2 - 2ab\cos C - \frac{1}{3}\,Kh^2c^2,$$
$$c^2 = a^2 + b^2 - 2ab\cos C - \frac{2}{3}\,KSab\sin C,$$

by denoting the area of the triangle by $S$. this formula can moreover be written as

$$c^2 = a^2 + b^2 - 2ab\cos\left(C - \frac{KS}{3}\right). \tag{10.28}$$

It leads to the following theorem, which is classical in the theory of surfaces[5]:

*If we construct in an Euclidean plane the rectilinear triangle having the same sides as the geodesic triangle, the angles $A_0, B_0, C_0$ of this triangle are obtained by subtracting from the angles $A, B, C$ of the geodesic triangle the common value $\frac{1}{3}KS$, where $K$ is the Riemannian curvature of the space in the direction of the plane element of the triangle.*

**231.** The formulae

$$\left.\begin{aligned} A_0 &= A - \tfrac{1}{3}\,KS,\\ B_0 &= B - \tfrac{1}{3}\,KS,\\ C_0 &= C - \tfrac{1}{3}\,KS \end{aligned}\right\} \tag{10.29}$$

give, by addition,

$$\pi = A + B + C - KS$$

from which

$$K = \frac{A + B + C - \pi}{S}, \tag{10.30}$$

a formula that we have already proved (n° 106).

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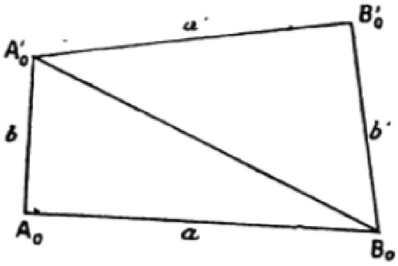
[5] Darboux, *Théorie des surfaces*, t. III, Book VI, Chap.VI.

**232.** We could deduce from the preceding result another proof of the theorem of Levi-Civita on the parallelogramoid, by decomposing this parallelogramoid into two geodesic triangles. We will limit ourselves to proving, by the preceding, a formula similar to that of Levi-Civita and due to F. Severi.

Consider a very small arc $AB = a$ of a geodesic; raise at $A$ a perpendicular geodesic arc $AA'$, of length $b$; finally drop from $A'$ a geodesic $A'B'$ perpendicular to the geodesic from $B$ which results from parallel transport of the geodesic $AA'$ along the arc $AB$. We thus obtain a geodesic quadrilateral whose three angles are right angles, namely the angles at $A$, at $B$ and at $B'$.[6] Let $a'$ and $b'$ be the lengths of the arcs $A'B'$ and $BB'$.

Take the diagonal $A'B$ and construct in an Euclidean plane the two rectilinear triangles $A_0B_0A_0'$, $B_0B_0'A_0'$ that have the same sides as the two corresponding geodesic triangles (Figure 3). Denoting by $S$ the area of the quadrilateral, we have

**Figure 3**



$$\widehat{A}_0 = \frac{\pi}{2} - \frac{KS}{6},$$
$$\widehat{B}_0' = \frac{\pi}{2} - \frac{KS}{6}.$$

Evaluation of the angle $\widehat{A_0B_0B_0'}$ is less immediate, because the sum of the angles $\widehat{ABA'}$ and $\widehat{A'BB'}$ is not rigorously equal to the angle $\widehat{ABB'} = \frac{\pi}{2}$, since the three geodesics $BA$, $BA'$, $BB'$ are not necessarily tangent at $B$ to the same plane element. We will show however that to the degree of approximation considered, everything happens as if it were so, and that we have

$$\widehat{A_0B_0B_0'} = \frac{\pi}{2} - \frac{KS}{3}.$$

Consider in fact the normal Euclidean space at $B$; take as the $x^1$ axis the direction $BA$ and for the $x^2$ axis the perpendicular direction $BB'$.

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[6] This is *Lambert's parallelogram* of non-Euclidean geometry.

The normal coordinates of $A'$ are, according to (10.25),

$$x^1 = a - \frac{1}{3}\,Kab^2,$$
$$x^2 = b + \frac{1}{6}\,Ka^2b,$$

where the other coordinates are of third order.

The relations

$$\cos\widehat{ABA'} = \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + \cdots}}\ ,$$
$$\cos\widehat{B'BA'} = \frac{x^2}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + \cdots}}\ ,$$

giving, by neglecting under the square root sign the terms of sixth order,

$$\cos\widehat{ABA'} = \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}}\ , \qquad \cos\widehat{B'BA'} = \frac{x^2}{\sqrt{(x^1)^2 + (x^2)^2}}\ ,$$

from which we deduce that the sum of the angles $\widehat{ABA'}$, and $\widehat{B'BA'}$ is equal to $\frac{\pi}{2}$, up to infinitesimals of the fourth order.

This point having been made, project in Figure 3 the contour $A_0A'_0B'_0B_0$ onto $A_0B_0$. If we note that the angle of the two directions $A_0B_0$ and $A'_0B'_0$ is equal to $\frac{KS}{2}$, we have

$$a = a' + b\,\frac{KS}{6} + b'\,\frac{KS}{3} = a' + b\,\frac{KS}{2}$$

up to infinitesimals of the fourth order. The two relations

$$a - a' = b\,\frac{KS}{2},$$
$$a + a' = 2a,$$

exact up to infinitesimals of the fourth and of the second order respectively, give by multiplcation

$$a^2 - {a'}^2 = KS^2,$$

exact up to infinitesimals of the fifth order.

We can deduce the formula of Severi[7]

$$K = \frac{a^2 - {a'}^2}{S^2}, \tag{10.31}$$

identical to that of Levi-Civita.

We will have, because of symmetry,

$$K = \frac{{b'}^2 - b^2}{S^2}.$$

---

[7] F. Severi, *Sulla curvatura delle superficie e varietà* (*Rend. Circ. mat. Palermo*, 42, 1917, p. 227-253). To tell the truth, the author confines himself to the case of spaces of two dimensions.

## VIII. – Circles, spheres, hyperspheres.

**233.** We will point out new formulas which involve this time the curvature of the space in a plane direction of any number of directions.

First draw, on a geodesic surface at $O$, a circumference with centre $O$, defined by carrying, on the various geodesics from $O$, a constant (very small) length $R$.

We can always suppose that the geodesic surface is defined, in normal coordinates, by

$$x^3 = x^4 = \cdots = x^n = 0.$$

We have then on this surface,

$$ds^2 = (dx^1)^2 + (dx^2)^2 - \frac{1}{3}\,K(x^1\,dx^2 - x^2\,dx^1)^2,$$

where $K$ denotes the curvature of the space at $O$ in the direction of the geodesic surface considered.

The area $\mathcal{A}$ of the circle of radius $R$ is given by the double integral

$$\iint \sqrt{g}\,dx^1\,dx^2.$$

Now we have

$$g = \begin{vmatrix} 1 - \frac{1}{3}\,K(x^1)^2 & \frac{1}{3}\,Kx^1x^2 \\ \frac{1}{3}\,Kx^1x^2 & 1 - \frac{1}{3}\,K(x^2)^2 \end{vmatrix} = 1 - \frac{1}{3}\,Kr^2,$$

where we denote by $r$ the distance of the running point to the point $O$. We have therefore, by using polar coordinates,

$$\mathcal{A} = \iint \left(1 - \frac{1}{3}\,Kr^2\right) r\,dr\,d\theta = \pi R^2 - \frac{1}{12}\,K\pi r^4. \tag{10.32}$$

Let $\mathcal{A}_0$ be the area of an Euclidean circle of radius $R$. We have

$$\mathcal{A} = \mathcal{A}_0\left(1 - \frac{1}{12}\,KR^2\right),$$

or again

$$\frac{\mathcal{A}_0 - \mathcal{A}}{\mathcal{A}_0 R^2} = \frac{K}{12}. \tag{10.33}$$

This formula allows us to define the Riemannian curvature $K$ by means of the area $\mathcal{A}$ of a circle of radius $R$. This area is smaller than in the Euclidean plane if the curvature is positive, larger if the curvature is negative.

The length $\mathcal{C}$ of the circumference is easily deduced from formula (10.32). In fact, $\mathcal{A}$ is a function of $R$ whose derivative is precisely equal to $\mathcal{C}$. We have then

$$\mathcal{C} = \mathcal{C}_0 - \frac{1}{3}\,K\pi R^3 = \mathcal{C}_0\left(1 - \frac{1}{6}\,KR^2\right),$$

from which

$$\frac{\mathcal{C}_0 - \mathcal{C}}{\mathcal{C}_0 R^2} = \frac{K}{6}. \tag{10.34}$$

**234.** Take now a geodesic three-dimensional manifold at $O$, which we can always suppose defined by the equations

$$x^4 = \cdots = x^n = 0.$$

We have, on this manifold,

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{1}{3}[R_{2323}(x^2\,dx^3 - x^3\,dx^2)^2 + \cdots].$$

The volume $V$ of a sphere with centre $O$ and of radius $R$ is given by the triple integral

$$\iiint \sqrt{g}\;dx^1\,dx^2\,dx^3$$

over, in the normal Euclidean space, the Euclidean sphere with centre $O$ and of the same radius. Now we have, by calling $\gamma_{ij}$ the coefficients of the form $ds^2 - ds_0^2$,

$$g = 1 + \gamma_{11} + \gamma_{22} + \gamma_{33} = 1 + \Phi(x^1, x^2, x^3),$$

where $\Phi$ is an homogeneous polynomial of the second degree.

In the integral

$$\iiint \sqrt{g}\;dx^1\,dx^2\,dx^3 = V_0 + \frac{1}{2}\iiint \Phi(x^1, x^2, x^3)\;dx^1\,dx^2\,dx^3,$$

it will be sufficient to consider the square terms of $\Phi$, the others obviously giving zero integrals. On the other hand, by reason of symmetry, we have

$$\begin{aligned}
&\iiint (x^1)^2\;dx^1\,dx^2\,dx^3\\
&= \iiint (x^2)^2\;dx^1\,dx^2\,dx^3\\
&= \iiint (x^3)^2\;dx^1\,dx^2\,dx^3 = \frac{1}{3}\iiint r^2\;dx^1\,dx^2\,dx^3 = \frac{4}{15}\pi R^5.
\end{aligned}$$

We therefore need only calculate the sum of the coefficients of the square terms in $\Phi$, which sum is equal to

$$\frac{2}{3}(R_{2323} + R_{3131} + R_{1212}).$$

We thus have finally

$$V = V_0 + \frac{4}{45}(R_{2323} + R_{3131} + R_{1212})\pi R^3.$$

Now the quantity $-(R_{2323} + R_{3131} + R_{1212})$ represents (n° 193) the curvature

$K$ (per unit volume) of the space in the direction of the plane element of three dimensions considered. We thus have

$$V = V_0 \left(1 - \frac{K}{15} R^2\right),$$

or

$$\frac{V_0 - V}{V_0 R^2} = \frac{K}{15}. \tag{10.35}$$

We see that the change undergone by the expression of the volume of a (very small) sphere on moving from the Euclidean space to a Riemannian space depends only on the curvature of the space in the direction of the plane element of three dimensions which contains the sphere.

On passing from the volume to the surface, we find immediately, by differentiation,

$$\frac{S_0 - S}{S_0 R^2} = \frac{K}{9}. \tag{10.36}$$

**235.** The preceding formulae generalise without difficulty for the volume and the surface of an hypersphere of radius $R$ drawn in a geodesic manifold $V_p$ at $O$, that we can suppose defined by the equations

$$x^{p+1} = \cdots = x^n = 0.$$

We will have here

$$V = \iiint \left[1 + \frac{1}{2}(\gamma_{11} + \gamma_{22} + \cdots + \gamma_{pp})\right] dx^1\, dx^2 ... dx^p,$$

where the integral extends, in the normal Euclidean space, to an hypersphere with centre $O$ and of radius $R$.

We need consider only the square terms in the $\gamma_{ii}$. We have on the other hand

$$\int (x^1)^2\ dx^1\, dx^2 ... dx^p = \frac{1}{p} \int r^2\ dx^1\, dx^2 ... dx^p.$$

Let

$$V_0 = \alpha r^p,$$

be the expression for the volume of an Euclidean hypersphere of radius $r$, where $\alpha$ is a suitably chosen constant; its surface is

$$S_0 = p\alpha r^{p-1}.$$

We have

$$\int r^2\ dx^1\, dx^2 ... dx^p = \int r^2 S_0(r)\, dr = p\alpha \int r^{p+1}\, dr = \frac{p}{p+2}\, \alpha r^{p+2} = \frac{p}{p+2}\, V_0 r^2.$$

The sum of the coefficients of the square terms in $\gamma_{11} + \gamma_{22} + \cdots + \gamma_{nn}$ is

$$\frac{2}{3} \sum_{(ij)} R_{ijij} = -\frac{2}{3}\, K,$$

where $K$ denotes the curvature at $O$ in the direction of the manifold $V_p$.

We have therefore

$$V = V_0 + \frac{1}{3(p+2)} K V_0 R^2,$$

from which

$$\frac{V_0 - V}{V_0 R^2} = \frac{K}{3(p+2)}. \tag{10.37}$$

Finally the formula

$$V = \alpha R^p - \frac{1}{3(p+2)}\, K \alpha R^{p+2}$$

gives, by differentiation,

$$S = p\alpha R^{p-1} - \frac{1}{3}\, K \alpha R^{p+1} = S_0 \left(1 - \frac{K}{3p}\, R^2\right),$$

from which[8]

$$\frac{S_0 - S}{S_0 R^2} = \frac{K}{3p}. \tag{10.38}$$

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[8] The general formulae established in this section were given for the first time by H. Vermeil (*Gött. Nachr.*, 1917, p 334-344).

# 11 Symmetry and Parallel Transport. Symmetric Spaces

## I. – Symmetry and Parallel Transport.

**236.** Consider the point transformation, defined in the neighbourhood of a point $O$, which passes from a point $M$ to a point $M'$ symmetric to $M$ with respect to $O$ on the geodesic $MO$ extended beyond $O$; this means that the length of the arc $OM'$ is equal to the length $OM$. This transformation, which we will call a *symmetry with respect to* $O$, is not in general an isometry. It also pairs any vector at $M$ with a specific vector at the symmetric point $M'$; if the first is identified with the velocity of a moving point, the second will be the velocity of a moving point constantly symmetric with the first.

In normal coordinates $x^i$ with origin $O$, the symmetry is translated, for points, by the equations

$$(x^i)' = -x^i;$$

for the vectors, referred to the natural frames of reference associated with normal coordinates, by the equations

$$(X^i)' = -X^i.$$

Consider two points $M$, $M'$ symmetric with respect to $O$ and very close to $O$, as well as two symmetric vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ at $M$ and $M'$. The vector $\boldsymbol{x}''$, with origin $M'$, equal and opposite to $\boldsymbol{x}'$, has the same components as the vector $\boldsymbol{x}$; consequently since the quantities ${\Gamma_i}^k{}_j$ are zero at $O$, it is the result of parallel transport of the vector $\boldsymbol{x}$ from $M$ to $M'$. We thus have the following new construction for parallel transport.

*To transport by parallelism a vector* $\boldsymbol{x}$*, its origin moving from* $M$ *to an infinitely close point* $M'$*, it is sufficient to construct the symmetric* $\boldsymbol{x}'$ *of* $\boldsymbol{x}$ *with respect to the midpoint* $O$ *of the geodesic arc* $MM'$*, then the vector* $\boldsymbol{x}''$ *that is equal and opposite to* $\boldsymbol{x}'$*.*

We will refer to this construction by the name of *transport by symmetry.*

**237.** The preceding construction naturally only gives a result that is exact up to an infinitesimal of order greater than the first, by taking as principal infinitesimal the distance $MM'$. We will show that if the *parallel transport is* *performed along the arc of the geodesic $MM'$, the transport by symmetry gives a result that is exact up to infinitesimals of the third order.*

Note for this that if we use the rectangular frames of reference $(R)$ *adapted* to the normal coordinates considered (n° 216), parallel transport along a geodesic that passes through $O$ does not change the components of this vector with respect to the frames $(R)$; now the components of a vector are nothing other than the forms $\omega^i(x; dx)$ where we give the differentials the numerical values that have the components $X^i$ of the vector referred to the *natural* frame of reference of the same origin associated with the normal coordinates. That said, to perform the parallel transport by symmetry with respect to $O$ comes down to passing from the quantities $\omega^i(x; dx)$ to quantities $\omega^i(-x; dx)$; since parallel transport along the geodesic $MM'$ conserves the numerical values of the forms $\omega^i$, we are brought back to comparing the quantities $\omega^i(x; dx)$ and $\omega^i(-x; dx)$. Now the limited expansion obtained in Chapter X, n° 225, formula (10.19), gives

$$\omega^i(x; dx) - \omega^i(-x; dx) = \frac{1}{6}\, R_{rikh|s} x^r x^k x^s dx^h, \tag{11.1}$$

which is clearly, if $dx^h$ is finite, an infinitesimal quantity of third order.

**238.** Let us look for the conditions that the given Riemannian space must satisfy so that transport by symmetry gives a construction of parallel transport along the geodesic which is exact to infinitesimals of the fourth order. We have to express that the cubic forms

$$F_{ij} = R_{rikj|s} x^r x^k x^s \tag{11.2}$$

are identically zero. This will translate into a certain number of linear relations with constant coefficients between the components of the derived Riemann-Christoffel tensor. It is clear that given the geometric significance of these relations, they are independent of the choice of frame of reference. We will use this remark at the same time as the Bianchi identities of n° 191 [formula (8.13)].

Note for this that the components $R_{ijkh|\ell}$ transform under a change of frame of reference as the products $X_i Y_j Z_k T_h U_\ell$ of the components of five arbitrary vectors. Consider then a simple infinitesimal rotation of the frame of reference parallel to the biplane determined by the basis vectors $\boldsymbol{e}_r, \boldsymbol{e}_s$.

If $\epsilon$ is the infinitesimal angle of rotation, the component $X_r$ of a vector undergoes an elementary increase $\epsilon X_s$, and the components $X_s$ an elementary increase $-\epsilon X_r$, with the other components unchanged. It follows (n° 212) that the elementary increase undergone by the components $R_{ijkh|\ell}$ will be, up to a factor $\epsilon$, the sum of the components obtained by replacing, wherever found, the index $r$ by $s$, reduced by the sum of the components obtained by replacing, wherever found, the index $s$ by $r$. We will denote the rotation considered by the symbol $\{rs\}$.

**239.** Let us apply the preceding considerations to the coefficient of $(x^i)^3$ in the form $F_{ij}$; since this coefficient must be zero, we will have[1]

$$R_{ijij|i} = 0. \tag{11.3}$$

The rotation $\{jk\}$ will allow us to deduce from this equality the following

$$R_{ijik|i} = 0, \tag{11.4}$$

and similarly the rotation $\{ik\}$, applied to (11.3), will give

$$2R_{ijkj|i} + R_{ijij|k} = 0. \tag{11.5}$$

The Bianchi identity

$$R_{ijij|k} + R_{ijjk|i} - R_{ijik|j} = 0$$

allows (11.5) to be written in the form

$$R_{ijik|j} - 3R_{ijjk|i} = 0,$$

from which, by exchanging the two indices $i$ and $j$,

$$R_{ijjk|i} - 3R_{ijik|j} = 0.$$

From these two last relations as well as from (11.5) follow the equalities

$$R_{ijik|j} = 0, \qquad R_{ijij|k} = 0, \qquad R_{ijjk|i} = 0. \tag{11.6}$$

The rotation $\{j\ell\}$ applied to the first equality (11.6), gives finally

$$R_{i\ell ik|j} + R_{ijik|\ell} = 0,$$

from which, by cyclic permutation of the indices $j, k, \ell$ we get two new relations that show that the sum of any two of the three components $R_{ijik|\ell}, R_{iki\ell|j}, R_{i\ell ij|k}$ is zero. They are therefore zero:

$$R_{ijik|\ell} = 0. \tag{11.7}$$

We deduce from (11.7), by applying of the Bianchi identities

$$R_{ijik|\ell} + R_{ijk\ell|i} - R_{iji\ell|k} = 0,$$

the new equality

$$R_{ijk\ell|i} = 0. \tag{11.8}$$

Equalities (11.3) and (11.8) show that the only components which can be different from zero are those whose five indices are distinct.

Now the rotation $\{im\}$ applied to (11.8) give the new equality

$$R_{k\ell mj|i} + R_{k\ell ij|m} = 0,$$

---

[1] In equality (11.3) and those that follow, do not sum over indices that appear more than once, since these indices have only individual values. In all these equalities the indices denoted by distinct letters are assumed distinct.

from which

$$R_{k\ell ij|m} = R_{k\ell jm|i} = R_{k\ell mi|j};$$

But the sum of these three last components is zero, so we have, whatever are the indices $i, j, k, \ell, m$,

$$R_{ijk\ell|m} = 0. \tag{11.9}$$

All the components of the derived tensor of the Riemann-Christoffel tensor are thus zero.

THEOREM 11.1 *For transport by symmetry to realise parallel transport along an infinitesimal arc of a geodesic up to infinitesimals of order greater than three, it is necessary and sufficient that the derivative of the Riemann-Christoffel tensor be zero.*

We will note that the vanishing of the derived tensor of the Riemannian curvature means geometrically that *parallel transport conserves Riemannian curvature.*

The given proof shows that this property is a consequence of the single equation (11.3), which expresses that *the Riemannian curvature of the space in the direction of a plane element has its absolute derivative zero in any direction on this plane element.*

**240.** We will now prove the following theorem:

THEOREM 11.2 *If the derived tensor of the Riemann-Christoffel tensor is zero, transport by symmetry realises rigorously parallel transport along an arc of a geodesic; furthermore the vanishing of this tensor is the necessary and sufficient condition for symmetry with respect to a point to be an isometry.*

In fact, parallel transport conserves the Riemannian curvature, the components $R_{ijkh}$ of the Riemann-Christoffel tensor, referred to a system of rectangular frames of reference $(R)$ adapted to a system of normal coordinates with origin $O$, are absolute constants. Let us then take up again the fundamental differential equations that define the forms $\overline{\omega}^{\,i}(t, a; da)$ and $\overline{\omega}_{ij}$ [Chapter X, n° 217, formulae (10.7)]:

$$\left.\begin{array}{rcl} \dfrac{\partial \overline{\omega}^{\,i}}{\partial t} &=& da^i + a^k \overline{\omega}_{ki}\,, \\ \dfrac{\partial \overline{\omega}_{ij}}{\partial t} &=& R_{ijkh} a^k \overline{\omega}^h\,. \end{array}\right\} \tag{11.10}$$

We will deduce the relations

$$\overline{\omega}^{\,i}(t, a; da) = \overline{\omega}^{\,i}(t, -a; da). \tag{11.11}$$

Put in fact

$$\overline{\omega}^{\,i}(t,a;da) - \overline{\omega}^{\,i}(t,-a;da) = \varpi^i,$$
$$\overline{\omega}_{ij}(t,a;da) + \overline{\omega}_{ij}(t,-a;da) = \varpi_{ij}.$$

A quick calculation gives

$$\frac{\partial \varpi^i}{\partial t} = a^k \varpi_{ki}\,, \qquad\qquad \frac{\partial \varpi_{ij}}{\partial t} = R_{ijkh} a^k \varpi^h\,.$$

This is a system of linear homogeneous differential equations with constant coefficients; since the initial values of the unknown functions are zero for $t = 0$, these functions are all zero, hence follows the relation (11.11) to be proved.

Bt replacing now in this relation $t$ by 1, $a^k$ by $x^k$ and $da^k$ by $dx^k$, we obtain the identities

$$\omega^i(x;dx) = \omega^i(-x;dx). \tag{11.12}$$

These prove:

1°. that transport by symmetry along the arc of a geodesic realise rigorously parallel transport along this arc;
2°. that symmetry with respect to $O$ leaves invariant the fundamental form, in other words it is an isometry.

**241.** It remains to prove that if symmetry with respect to a point is an isometry, parallel transport conserves Riemannian curvature. This is obvious geometrically. In fact, begin from a plane element with origin $M$ defined by the bivector formed from two vectors $\boldsymbol{x}, \boldsymbol{y}$; the symmetry with respect to $O$ transforms these vectors into two vectors $\boldsymbol{x}', \boldsymbol{y}'$ with origin $M'$. Since the symmetry is assumed isometric, it conserves the Riemannian curvature, which is deduced uniquely from the fundamental form. The Riemannian curvature along the plane element $[\boldsymbol{x}\boldsymbol{y}]$ is the same as alonf the plane element $[\boldsymbol{x}'\boldsymbol{y}']$; but these two plane elements are deduced the one from the other by parallel transport, up to infinitesimals of the third order; parallel transport thus conserves the Riemannian curvature up to infinitesimals of the third order, and consequently the absolute derivative of the Riemannian curvature is zero, and parallel transport conserves exactly the Riemannian curvature.

Q.E.D.

The theorem of n° 240 is thus completely proved.

## II. – Symmetric Riemannian Spaces.

**242.** The Riemannian spaces that we will call *symmetric* can be defined by two properties that we have just proved to be equivalent:

1°. A symmetry with respect to any point is an isometry;

2°. Parallel transport conserves Riemannian curvature.

We will prove that the search for symmetric spaces reduces to a problem of algebra.

Introduce at a point $O$ of the space normal coordinates corresponding to a rectangular frame of reference $(R_0)$ with origin $O$. The components $R_{ijkh}$ of the Riemann-Christoffel tensor, referred to frames of reference *adapted* to this system of coordinates, are absolute constants that satisfy the classical symmetry relations

$$\left.\begin{array}{l} R_{ijkh} = -R_{jikh} = -R_{ijhk}, \\ R_{ijkh} + R_{ikhj} + R_{ihjk} = 0. \end{array}\right\} \tag{11.13}$$

On the other hand expressing that the absolute differential of each of these components is zero, we get the relations

$$R_{rjkh}\omega_{ir} + R_{irkh}\omega_{jr} + R_{ijrh}\omega_{kr} + R_{ijkr}\omega_{hr} = 0,$$

from which, by supposing introduced the redundant coordinates $t, a^i$, and noting that $\omega_{ij} = \overline{\omega}_{ij}$, we have, by applying the fundamental differential equations (11.10),

$$R_{rjkh}R_{ir\ell m} + R_{irkh}R_{jr\ell m} + R_{ijrh}R_{kr\ell m} + R_{ijkr}R_{hr\ell m} = 0 \tag{11.14}$$
$$(i, j, k, h, \ell, m = 1, 2, ..., n).$$

We will show that, conversely, *to any system of constants satisfying the quadratic relations (11.13) and (11.14), there corresponds a symmetric Riemannian space, and this space is locally applicable onto itself an infinity of ways.*

**243.** Let us integrate in fact the fundamental differential equations

$$\left.\begin{array}{rcl} \dfrac{\partial \overline{\omega}^{\,i}}{\partial t} &=& da^i + a^k \overline{\omega}_{ki}\,, \\ \dfrac{\partial \overline{\omega}_{ij}}{\partial t} &=& R_{ijkh} a^k \overline{\omega}^h\,, \end{array}\right\} \tag{11.10}$$

with the initial conditions $\overline{\omega}^{\,i} = 0, \overline{\omega}_{ij} = 0$. This is a system of linear differential equations *with constant coefficients* which provide for the forms $\overline{\omega}^{\,i}, \overline{\omega}_{ij}$ linear expressions in $da^1, da^2, ..., da^n$ whose coefficients are *integral* functions in $t, a^1, a^2, ..., a^n$. If we put $t = 1$ and we replace $a^i$ by $x^i$, we obtain the forms $\omega^i(x; dx), \omega_{ij}(x; dx)$; the determinant of the coefficients of $dx^1, dx^2, ..., dx^n$ in $\omega^1, \omega^2, ..., \omega^n$ is different from zero for very small values of the $x^i$ (equal to 1 at point $O$).

But it remains to prove that the forms $\omega^i, \omega_{ij}$ thus obtained satisfy the equations of structure

$$\left.\begin{array}{rcl} d\omega^i &=& [\omega^k\ \omega_{ki}], \\ d\omega_{ij} &=& [\omega_{ik}\ \omega_{kj}] + \frac{1}{2}\, R_{ijkh}[\omega^k \omega^h]; \end{array}\right\} \tag{11.15}$$

it is sufficient moreover to prove that these relations are satisfied by the forms $\varpi^i(t, a; da), \varpi_{ij}(t, a; da)$, where the exterior derivatives $d\varpi^i$ and $d\varpi_{ij}$ are calculated as if $t$ were a constant.

Put

$$\left.\begin{aligned} d\varpi^i &= [\varpi^k\, \varpi_{ki}] + \varepsilon^i, \\ d\varpi_{ij} &= [\varpi_{ik}\, \varpi_{kj}] + \tfrac{1}{2}\, R_{ijkh}[\varpi^k \varpi^h] + \varepsilon_{ij}, \end{aligned}\right\} \tag{11.16}$$

where we denote by $\varepsilon^i$ and $\varepsilon_{ij}$ exterior quadratic differential forms in $da^1, da^2, ..., da^n$, obviously zero for $t = 0$, because for $t = 0$ the forms $\varpi^i$ and $\varpi_{ij}$ themselves are identically zero.

We will establish a system of linear and homogeneous differential equations that must be satisfied by the forms $\varepsilon^i$ and $\varepsilon_{ij}$. Once this is done, it becomes clear that forms that are zero for $t = 0$ are identically zero, that which proves relations (11.15).

We arrive at the system of differential equations sought by deriving equations (11.16) with respect to $t$. It is sufficient for this to note that if $\varpi$ is a differential form whose coefficients depend on a parameter $t$, the operations of differentiation with respect to $t$ and exterior differentiation are permutable, in other words that we have[2]

$$\frac{\partial}{\partial t}(d\varpi) = d\frac{\partial \varpi}{\partial t}. \tag{11.17}$$

Apply this relation to the form $\varpi^i$; we obtain, *by taking into account relations (11.13)*,

$$\frac{\partial \epsilon^i}{\partial t} = a^k \epsilon_{ki}\,. \tag{11.18}$$

The calculation of $d\frac{\partial \varpi_{ij}}{\partial t}$ and of $\frac{\partial}{\partial t}(d\varpi_{ij})$ gives consequently

$$d\frac{\partial \varpi_{ij}}{\partial t} = R_{ijkh}[da^k\, \varpi^k] + R_{ijkh} a^k \varpi^\ell \varpi_{\ell h} + R_{ijkh} a^k \epsilon^h,$$

$$\begin{aligned} \frac{\partial}{\partial t}(d\varpi_{ij}) = R_{ikh\ell} a^h [\varpi^\ell\, \varpi_{kj}] & \\ - R_{kjh\ell} a^h [\varpi^\ell \varpi_{ik}] &+ R_{ijkh}[(da^k + a^\ell \varpi_{\ell k})\, \varpi^h] + \frac{\partial \epsilon_{ij}}{\partial t}\,. \end{aligned}$$

Comparison of these two formulas gives

$$\frac{\partial \epsilon_{ij}}{\partial t} = R_{ijkh} a^k \epsilon^h + a^h [\varpi^\ell (R_{ijhk} \varpi_{\ell k} + R_{kjh\ell} \varpi_{ik} + R_{ikh\ell} \varpi_{jk} + R_{ijk\ell} \varpi_{hk})].$$

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[2] This relation is easy to prove. If for example $\varpi = \frac{1}{2}\, a_{ij}[du^i\ du^j]$, we have

$$d\frac{\partial \varpi}{\partial t} = \frac{1}{2} \frac{\partial^2 a_{ij}}{\partial t \partial u^k}[du^k\ du^i\ du^j],$$

$$\frac{\partial d\varpi}{\partial t} = \frac{\partial}{\partial t}\{\frac{1}{2}\, \frac{\partial a_{ij}}{\partial u^k}[du^k\ du^i\ du^j] = \frac{1}{2}\, \frac{\partial^2 a_{ij}}{\partial t \partial u^k}[du^k\ du^i\ du^j].$$

But the form

$$R_{ijhk}\overline{\omega}_{\ell k} + R_{kjh\ell}\overline{\omega}_{ik} + R_{ikh\ell}\overline{\omega}_{jk} + R_{ijk\ell}\overline{\omega}_{hk}\,, \tag{11.19}$$

zero for $t = 0$, admits with respect to $t$ a derivative equal to

$$a^r\overline{\omega}^{\,s}(R_{ijkh}R_{\ell krs} + R_{kjh\ell}R_{ikrs} + R_{ikh\ell}R_{jkrs} + R_{ijk\ell}R_{hkrs}),$$

and which consequently is zero by taking into account the quadratic relations (11.14), where it is sufficient to change $k$ into $h$, $h$ into $\ell$, $r$ into $k$, $\ell$ into $r$, and $m$ into $s$. We thus have

$$\frac{\partial \epsilon_{ij}}{\partial t} = R_{ijkh}a^k\epsilon^h. \tag{11.20}$$

Equations (11.18) and (11.20) are linear and homogeneous, which proves that the structure equations (11.15) are satisfied. The space obtained on the other hand is symmetric because the vanishing of the form (11.19) proves the vanishing of the absolute differential of the components $R_{ijk\ell}$ of the Riemann-Christoffel tensor.

**244.** *Note.* — Equations (11.14) state that a certain 6-index tensor is zero, namely the tensor whose covariant components referred to any Cartesian frame of reference are

$$H_{ijkh\ell m} = R_{rjkh}R_i{}^r{}_{\ell m} + R_{irkh}R_j{}^r{}_{\ell m} + R_{ijrh}R_k{}^r{}_{\ell m} + R_{ijkr}R_h{}^r{}_{\ell m}.$$

A necessary condition for a Riemannian space to be symmetric is thus that the preceding tensor be identically zero. But *this condition s not sufficient*, as the example of Riemannian spaces in two dimensions shows, for which the tensor is always zero. It expresses in reality that the twice derived tensor of Riemann-Christoffel has its components $R_{ijkh|\ell m}$ symmetric with respect to the two last indices.

## III. – Rigid Displacements of a Symmetric Space.

**245.** Spaces with constant Riemannian curvature are obviously symmetric spaces and these are the simplest. The reasoning just presented emphasises a number of properties that symmetric spaces have in common with spaces of constant curvature. These properties arise from the very simple remark that the expression for the fundamental form as a function of normal coordinates with respect to a point $O$ will be found to be identical to itself whenever we take an originating rectangular frame of reference $(R_0)$ with respect to which the components $R_{ijkh}$ of the Riemann-Christoffel tensor have the same constant values. This could arise in two ways:

1°. By replacing the rectangular frame of reference $(R_0)$ by any of the frames of reference which are obtained by parallel transport along a geodesic through $O$.

2°. By retaining the origin $O$ of the frame of reference $(R_0)$, but imposing on this frame of reference a suitable rotation about $O$. Now the infinitesimal variation that $R_{ijkh}$ undergoes as the result of an infinitesimal rotation of angle $\varepsilon$ parallel to the biplane $[\boldsymbol{e}_r\boldsymbol{e}_s]$ was evaluated in n° 238. It follows that the infinitesimal variation undergone as the result of an infinitesimal rotation defined by the bivector with components $a_{ij}$ is

$$R_{rjkh}a_{\ell r} + R_{irkh}a_{jr} + R_{ijrh}a_{kr} + R_{ijkr}a_{hr}\,. \tag{11.21}$$

Consequently, the most general infinitesimal rotation $(a_{ij})$ that the frame of reference $(R_0)$ can undergo without changing the components of the Riemannian curvature is given by the system of equations

$$R_{rjkh}a_{\ell r} + R_{irkh}a_{jr} + R_{ijrh}a_{kr} + R_{ijkr}a_{hr} = 0. \tag{11.22}$$

If these equations in $\frac{n(n-1)}{2}$ unknowns $a_{ij} = -a_{ji}$ are in number $\frac{n(n-1)}{2} - \rho$ independent ones, the symmetric space admits a connected group of rigid rotations about $O$ that depends on $\rho$ parameters. This group is called the *isotropy group* of the point $O$.

It follows from this that that the rectangular frames of reference that can be chosen as originating frames of reference of a system of normal coordinates providing an identical fundamental form depends on $n + \rho$ parameters.

Consequently, *the space admits a connected group of $n + \rho$ parameters of rigid displacements.* If the space is of constant curvature, there is no equation (11.22), the number $\rho$ is equal to $\frac{n(n-1)}{2}$ and the space has completely free mobility.

**246.** Among the rigid displacements of a symmetric space, we point out those that arise from two successive symmetries with respect to two points $A, B$; these are the transvections of É. Cartan; in such a displacement each point of the geodesic $AB$ undergoes along this geodesic a displacement equal to double the distance $AB$. The geodesic $AB$ is the *basis geodesic* of the transvection. In the case of the Euclidean space, the transvections are translations; they admit an infinity of basis geodesics all parallel to each other.

In spaces of non-zero constant curvature, a transvection admits only one basis geodesic.

## IV. – Irreducible Symmetric Spaces.

**247.** The concept of an irreducible symmetric space rests on the concept of the *topological product* of two spaces.

Given two spaces $E_1, E_2$ respectively of dimensions $n_1$ and $n_2$, we call a space of dimension $n_1 + n_2$ where each point is defined as the ordered set $(M_1, M_2)$ of a point of $E_1$ and a point of $E_2$ the *topological product* $E$ of these two spaces.

If the spaces $E_1$ and $E_2$ are Riemannian, the space $E$ is by definition the

Riemannian space whose fundamental form is the sum of the fundamental forms of $E_1, E_2$. In other words, if we consider two infinitely close points $(M_1, M_2)$, $(M_1', M_2')$ of $E$, the square of the distance of these two points is by definition the sum of the squares of the distance of the two points $M_1, M_1'$ of $E_1$ and of the square of the distance of the two points $M_2, M_2'$ of $E_2$.

If we choose in $E_1$ a system of coordinates $u^i$ $(i = 1, 2, ..., n_1$, and in $E_2$ a system of coordinates $v^\alpha$ $(\alpha = 1, 2, ..., n_2$, any point of $E$ will be defined by the $n_1 + n_2$ coordinates $u^i, v^\alpha$. The point $M_1$ of $E_1$ can be called the projection of the point $(M_1, M_2)$ of $E$ onto the space $E_1$ and the point $M_2$ its projection onto the space $E_2$. If the fundamental forms of $E_1$ and $E_2$ are respectively

$$g_{ij}(u)du^i du^j, \qquad \gamma_{\alpha\beta} dv^\alpha dv^\beta,$$

the fundamental form of $E$ is the sum

$$g_{ij}(u)du^i du^j + \gamma_{\alpha\beta} dv^\alpha dv^\beta.$$

**248.** If we consider the Euclidean space osculating at a point $M_1$ of $E_1$ and the Euclidean space osculating at a point $M_2$ of $E_2$, the Euclidean space, the topological product of these two Euclidean spaces, osculates the space $E$ at the point $(M_1, M_2)$ where $M_1$ and $M_2$ are the projections onto $E_1$ and $E_2$. It follows that the absolute differential of a variable vector of $E$ admits as projections onto $E_1$ or onto $E_2$ the absolute differential of the projection of the vector. If along a line $(C)$ of $E$ the absolute differential of the unit tangent vector is zero, the line $(C)$ is a geodesic: consequently the projection $(C_1)$ of $C$ onto $E_1$ will be a geodesic of $E_1$ and its projection $(C_2)$ onto $E_2$ will be a geodesic of $E_2$. Further two equal arcs of $(C)$ project as two equal arcs of $C_1$. We deduce easily that if the symmetry with respect to a point $(O_1, O_2)$ of $E$ is an isometry, it is the same in the space $E_1$ for the symmetry with respect to the point $O_1$ and in the space $E_2$ for the symmetry with respect to the point $O_2$, and conversely.

THEOREM 11.3 *For the topological product of two Riemannian spaces to be symmetric, it is necessary and sufficient that each of these two spaces be symmetric.*

**249.** DEFINITION. — *A symmetric space is said to be irreducible or irreducible according as it can or cannot be considered as the topological product of two other Riemannian spaces.*

A space of non-zero constant curvature is an irreducible symmetric space. In fact, suppose it reducible and choose in each component space a system of normal coordinates, namely

for the first $n_1$ coordinates $x^i$ $(i = 1, 2, ..., n_1)$,
for the second $n_2$ coordinates $x^\alpha$ $(\alpha = n_1 + 1, n_1 + 2, ..., n_1 + n_1)$.

It is obvious that the forms $\omega_{i\alpha}$ are zero and consequently, according to (11.10), the forms $\Omega_{i\alpha}$; now this is impossible because we have

$$\Omega_{i\alpha} = -K[\omega^i\ \omega^\alpha] \qquad (K \neq 0).$$

**250.** We will now prove the following theorem, which reduces the search for symmetric spaces to that of symmetric spaces of constant Riemannian curvature of the second kind.

THEOREM 11.4 *Any irreducible symmetric space is of constant curvature of the second kind.*

Recall that a space of constant curvature of the second kind is a space whose curvature is the same in all $n-1$ dimensional directions (n° 200), or also whose contracted Ricci tensor is the product by a constant of the fundamental tensor.

Attach to a point $O$ of the space a rectangular frame of reference $(R_0)$ such that the components of the Ricci tensor are all zero, except $R_{11}, R_{22}, ..., R_{nn}$.

By expressing that the absolute differentials of the components of the tensor are zero, we will have, for example by reasoning on $R_{12}$,

$$R_{i2}\omega_{1i} + R_{1i}\omega_{2i} = 0$$

or

$$(R_{22} - R_{11})\omega_{12} = 0.$$

The form $\omega_{12}$ is thus zero if $R_{11} \neq R_{22}$.

Suppose that the $n$ components $R_{11}, R_{22}, ..., R_{nn}$ are not equal to each other, that for example $R_{11} = R_{22} = R_{pp}$, with the next components $R_{p+1,p+1}, ..., R_{nn}$ being distinct from the first ones. Denoting by the Latin letters $i, j, ...$, the first $p$ indices and by the Greek letters $\alpha, \beta, ...$, the other $n-p$, we have

$$\omega_{i\alpha} = 0 \qquad (i = 1, ..., p; \alpha = p+1, ..., n).$$

Consider then the system of normal coordinates determined by the originating frame of reference $(R_0)$. The fundamental differential equations (11.10) show that the components $R_{i\alpha kh}$ of the Riemann-Christoffel tensor are all zero, whether the indices $k, h$ are Latin or Greek; we deduce easily that the only components of this tensor which are not zero are those whose indices are exclusively Latin or exclusively Greek.[3] But then equations 11.10) can be divided into two groups, the first in the equations that contain only Latin indices, the second in the equations that contain only Greek indices. The quadratic form

$$(\omega^1)^2 + (\omega^2)^2 + \cdots + (\omega^p)^2$$

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[3] We have in fact

$$R_{ij\alpha\beta} = R_{i\alpha j\beta} - R_{i\beta j\alpha} = 0.$$

then contains only the variables $x^1, x^2, ..., x^p$ and their differentials, and the quadratic form

$$(\omega^{p+1})^2 + (\omega^{p+2})^2 + \cdots + (\omega^n)^2$$

only the variables $x^{p+1}, ..., x^n$ and their differentials. The given space is thus not irreducible, contrary to hypothesis.

Since the components $R_{11}, R_{22}, ..., R_{nn}$ are equal to each other, the space is of constant curvature of the second kind.

*Note.* — The converse of the preceding theorem is not exact: a symmetric space of constant curvature of the second kind is not necessarily irreducible, as is show by the simple example of Euclidean spaces.

**251.** We will not pursue the study of symmetric spaces. They were completely determined by É. Cartan; [4] the Hermitian, elliptic and hyperbolic spaces of G. Fubini and É. Study were part of it. The irreducible spaces have a non-zero scalar Riemannian curvature; if it is positive, the Riemannian curvature of the space is everywhere positive or zero; if it is negative, the Riemannian curvature is everywhere negative or zero.

The complete theory of symmetric spaces is intimately tied to the theory of finite and continuous groups of transformations of S. Lie, and more particularly to that of simple groups.

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[4] E. Cartan, *Sur une classes remarquable d'espaces de Riemann (Bull. Soc. Math. France*, t. 54, 1926, p. 214-264; t. 55, 1927, p. 114-134); *Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple (Ann. Éc. Norm.*, t. 41, 1927, p. 345-467); *Groupes simples clos et ouverts et géométrie riemannienne (J. Math. pures appl.*, t. 8, 1929, p. 1-33); *La theorie des groupes finis et continus et I'Analysis situs (Mémorial Sc. Math.*, t. XLII, 1930); *Les espaces riemanniens symétriques (Verh. Int. Math. Kongresses, Zürlch*, I, 1932, p. 153-161); *Sur les domaines bornés homogènes de l'espace de n variables complexes (Abh. Math. Seminar Hamburg*, t. 11, 1935, p. 116-162).

# 12 Groups of rigid displacements in a Riemannian space.

## I. — General.

**252.** We have already spoken about displacements of a space with constant curvature, of a symmetric space. Recall that a rigid displacement, or more simply a displacement, in a Riemannian space is a point transformation which conserve the distance of any two infinitely close points, or again which leaves invariant the fundamental form of the space. A displacement conserves the arc length of any curve, the area of any portion of a surface, etc. We know also that it conserves the Riemannian curvature at a point according to a given plane direction, etc.

We will deal more particularly in this chapter with continuous groups of displacements. A (connected) continuous family of displacements forms a group if, at the same time as a displacement, it contains the inverse displacement and if, at the same time as two displacements, it contains the resultant displacement, whatever the order in which we perform these displacements.[1] All continuous groups of displacements depends on a certain number of parameters (the order of the group). This order is at most equal to $\frac{n(n+1)}{2}$, the last case is realised only if the space has constant curvature. We have seen above (n$^{\text{o}}$ 245) how we could determine the order of the largest group of displacements of a symmetric space.

**253.** We can propose on the subject of groups two principal problems, namely

**Problem I.** — *To determine the groups of transformations capable of representing analytically a group of displacements of a Riemannian space, and to determine for each of these groups the corresponding Riemannian spaces.*

**Problem II.** — *Given a Riemannian space, determine all its rigid displacements.*

We will deal in this Chapter exclusively with the first problem: the second, which is intimately related to the theory of the application of two given Riemannian spaces, will be treated in the following Chapter.

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[1] Any continuous family of displacements forms a group if it is not contained in a larger continuous family.

**254.** The groups that we consider will be assumed to satisfy certain analytical conditions that characterise the groups of which Lie has constructed the theory (Lie groups). These conditions are the following:

*The coordinates* $(u^i)'$ *of the transformed point of the point* $u^i$ *by the transformations of the group admit continuous partial derivatives of two first orders with respect to the coordinates* $u^i$ *and the parameters* $a^k$ *of the group* [the $(u^i)'$ are twice continuously differentiable functions of the $u^i$ and the $a^k$].

We will confine ourselves consequently only to changes of coordinates for which the new coordinates are twice continuously differentiable functions of the old.

The components $g_{ij}$ of the fundamental tensor are also assumed twice continuously differentiable; it can also happen that the Christoffel symbols and the components of the Riemannian curvature are assumed to admit continuous partial derivatives up to a certain order, which entail hypotheses and consequences for the $g_{ij}$. We will thus suppose in principle that we will never be stopped by the existence of the derivatives that arise in our calculations.[2]

## II. — Transitive and intransitive groups; trajectories.

**255.** A group $G$ of rigid displacements is said to be *transitive* if there exists at least one displacement taking an arbitrarily given point $M$ to any other arbitrarily given point $M'$. The group is said to be intransitive otherwise.

The group will be said to be *simply transitive* if the displacement which takes a point $M$ into another point $M'$ is unique, or *multiply transitive*[3] otherwise.

We call the manifold which is the locus of transformed points of a given point by the displacements of the group the *trajectory* of the group $G$ of rigid displacements. Any trajectory $\Sigma$ of the group can be regarded as arising from the displacements of the group applied to any of its points.

Considered as a Riemannian space whose fundamental form is that which is induced by its presence in the given space; any trajectory $\Sigma$ admits for itself the group of displacements $G$ which, from this point of view, is transitive. Let $p < n$ be the number of dimensions of these trajectories. If the group $G$ is of order $p$, it acts on $\Sigma$ as a simply transitive group; if it is of order greater than $p$, as a multiply transitive group. In the special case $p = 1$ where the trajectories are lines, the group $G$ is necessarily of one parameter, because the largest connected group of rigid displacements of a line is given by the equation $u' = u + a$, where

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[2] We will leave aside Riemannian spaces whose $ds^2$ is indefinite. Let us point out that in his thesis *Sur les variétés à trois dimensions* (1891), E. Cotton has determined the *complex* three dimensional analytic spaces that admit a continuous group of displacements.

[3] This term must not be taken in the sense that it usually has in the theory of the group of permutation of $n$ letters, where le group is called, for example, *doubly transitive* if it contains an operation allowing us to pass from two arbitrarily given letters to any two other letters.

$u$ denotes the curvilinear abscissa reckoned from a fixed point.

**256.** If the group of displacements $G$ is intransitive, the given space is generated by the family of the trajectories $\Sigma$ of the group; through each point of the space there passes one and only one trajectory and these trajectories depend on $n-p$ parameters, if we assume that the trajectories are $p$-dimensional. These $n-p$ parameters can be considered as forming $n-p$ of the coordinates of any point of the space. To individualise then the points of a trajectory, it will be necessary to introduce $p$ other coordinates.

## III. — Frames of reference adapted to a group of displacements.

**257.** We can apply the method of the moving frame by attaching to each point of the space a Cartesian frame of reference *or even a whole family* of Cartesian frames of reference in the following way.

Given a trajectory $\Sigma$, we will attach to a particular point $A$ of $\Sigma$ a particular Cartesian frame of reference $(R_A)$ and we will consider all the frames of reference which are deduced from $(R_A)$ by the rigid displacements of the group. This makes sense, because any point transformation of the space, at the same time that it transforms a point $M$ to a point $M'$, transforms in a well defined way any vector at $M$ into a vector at $M'$.[4] If the transformation is a rigid displacement, it conserves the length of vectors, the scalar product of two vectors, etc.

Consequently the transform of a frame of reference $(R_M)$ with origin $M$ gives a frame of reference $(R_{M'})$ with origin $M'$ whose basis vectors have the same mutual scalar products as the basis vectors of $(R_M)$: stated differently, *the frames of reference* $(R_M)$ *and* $(R_{M'})$ *are equal.*

Any family of frames obtained by the preceding method will be said to be *adapted* to the group of displacements considered. The preceding remarks lead us immediately to

**Theorem I.** — *The components* $g_{ij}$ *of the fundamental tensor referred to a family of frames of reference adapted to a group of displacements* $G$ *of the space, are invariant under this group.* If the group is transitive, these are absolute constants; if the group is intransitive, they depend only on the $n-p$ parameters which individualise the trajectories of the group.

**258.** We have another similar theorem relating to the forms $\omega^i, {\omega_i}^j$ which define the relative position of two infinitely close frames of reference. In fact, let $(R_M)$ and $R_{M'})$ be two infinitely close frames of reference of the family attached

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[4] The components of a vector, referred to the natural frame of reference associated to the chosen coordinates, transforms as the differentials of the coordinates under the action of the point transformation considered.

to two infinitely close points $M$, $M'$; a displacement of the group transforms them into two other infinitely close frames of reference $(R_N)$, $(R_{N'})$. The figure formed by $(R_N)$, $(R_{N'})$ is obviously equal to the figure formed by $(R_M)$, $(R_{M'})$;[5] we thus have

**Theorem II.** — *The forms $\omega^i, \omega_i{}^j$ which define the structure of the space referred to a family of frames of reference adapted to a group of displacements $G$ of the space are invariant under this group.*

Let us add one last theorem relating to the case where the trajectories are transformed in a simply transitive way by the group $G$. To each point of the space is the attached only one frame of reference and the forms $\omega_i{}^j$ are specific linear combinations od the forms $\omega^i$

$$\omega_i{}^j = \gamma_i{}^j{}_k\, \omega^k. \tag{12.1}$$

**Theorem III.** — *The coefficients $\gamma_i{}^j{}_k$ (generalised Christoffel symbols) are invariant under the group of displacements if the order of this group is equal to the number of dimensions of the trajectories of the group.*

This theorem is in fact a direct consequence of relations (12.1) and of the invariance of the forms $\omega^i, \omega_i{}^j$.

The same theorem applies to $\gamma_{ijk}$ as well as to the components of the Riemannian curvature (the Riemann-Christoffel tensor).

In certain problems it may be convenient to use rectangular frames of reference, but there are others where it is preferable to allow ourselves greater latitude in the choice of frames of reference adapted to the group.

## IV. — Riemannian spaces that admit a simply transitive group of displacements.

**259.** According to the relations developed in the preceding sections, we understand, and this will be proved later (Section X), that the search for Riemannian spaces that admit a group of displacements essentially comes down to searching for those which are transformed transitively by this group. Also it is with this case that we will begin, by assuming first that the group is simply transitive.

Use a system of frames of reference adapted to the group $G$, with the forms $\omega^i, \omega_i{}^j$ invariant under the group, where the first $n$ are linearly independent. We

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[5] The components $\omega^i$, with respect to the frame of reference $(R_M)$, of the infinitesimal vector $\overrightarrow{MM'}$ are equal to the components $(\omega^i)'$, with respect to the frame of reference $(R_N)$, of the infinitesimal vector $\overrightarrow{NN'}$; furthermore the quantities $\omega_i{}^j$ which determine the components with respect to $(R_M)$ of the basis vectors of $(R_{M'})$ are equal to the analogous quantities $(\omega_i{}^j)'$.

will search for what conditions these forms must satisfy for the space to admit a simply transitive group of displacements.

The displacements of the group $G$, if it exists, will be give by the integration of the total differential equations

$$\omega^i(u'; du') = \omega^i(u; du), \tag{12.2}$$

where the $(u^i)'$ are unknown functions of the $u^i$. Since a solution must exist such that to given values of the $u^i$ correspond given values of the $(u^i)'$, the system is completely integrable.

We can find directly the necessary conditions for it to be so. Form the exterior differentials $d\omega^i$ of the forms $\omega^i$, which we will express as exterior quadratic forms in $\omega^1, \omega^2, ..., \omega^n$ :

$$d\omega^i = \frac{1}{2} c_{kh}{}^i [\omega^k \; \omega^h] \qquad (c_{kh}{}^i = -c_{hk}{}^i). \tag{12.3}$$

Since the group $G$ leaves invariant the forms $\omega^i$ and consequently their exterior derivatives $d\omega^i$, it will leave invariant the coefficients $c_{kh}{}^i$; since the group $G$ is transitive, its coefficients are thus constants.

**Theorem I.** — *The forms $\omega^i$ with respect to a system of frames of reference adapted to a simply transitive group of displacements satisfy relations (12.3) where the coefficients $c_{kh}{}^i$ are constants.*

**260.** Theorem I admits an important complement. We will prove that

**Theorem II.** — *Any simply transitive group of transformations in $n$ variables can be considered as a simply transitive group of displacements for an infinity of $n$-dimensional Riemannian spaces.*[6]

In fact let

$$(u^i)' = f^i(u; a) \qquad (i = 1, 2, ..., n) \tag{12.4}$$

be the finite equations of the transformations of the group, where we assume the $f^i$ are twice continuously differentiable functions of the arguments $u^1, u^2, ..., u^n$; $a^1, a^2, ..., a^n$. Regard the $u^i$ as the coordinates of a point of a space of $n$ dimensions. Let $O$ be a fixed point with coordinates $(U^i)_0$ and let $M$ be any point with coordinates $u^i$; there exists a transformation $T_u$ of the group taking us from the point $M$ to the point $O$ and the parameters $a^k$ of this transformation are obtained by solving equations (12.4) where we replace their left hand sides by the constants $(U^i)_0$. Now let $M'$ be a point with coordinates $(u^i)'$ and let $T_b$ be the transformation of $G$ that takes us from the point $M'$ to $O$, with parameters $b^h$ which we obtain by a method similar to the preceding. We can pass from

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[6] This theorem solves, with regard to simply transitive groups, the first part of Problem I, or rather reduces it to the search for simply transitive groups, which comes only from the theory of groups.

$M$ to $M'$ by performing first the transformation $T_a$ which takes $M$ to $O$, then the transformation $T_b^{-1}$, inverse to $T_b$, which takes $O$ to $M'$; let $T_c = T_b^{-1} T_a$ be the resultant transformation. Now let $(u^i + du^i)$ be a point $M_1$ infinitely close to $M$ and let $M_1'$ be the point infinitely close to $M'$ which we obtain from $M_1$ by the transformation $T_c$; finally let $O_1$ be the point infinitely close to $O$, with coordinates $(U^i)_0 + (dU^i)_0$, which we deduce from $M_1$ by the transformation $T_a$: it will be deduced from $M_1'$ by the transformation $T_b$. We will thus have the relations

$$(dU^i)_0 = \frac{\partial f^i(u;a)}{\partial u^k}\, du^k = \frac{\partial f^i(u';b)}{\partial (u^k)'}\, d(u^k)'.$$

Replace in the derivatives $\frac{\partial f^i(u;a)}{\partial u^k}$ the parameters $a^1, a^2, ..., a^n$ by their values as functions of the $u^i$; do the same for the derivatives $\frac{\partial f^i(u';b)}{\partial (u^k)'}$. We arrive in thus way at relations of the form

$$\omega^i(u; du) = \omega^i(u'; du'), \tag{12.5}$$

whence the following Lemma which is a fundamental theorem of the theory of groups.

**Lemma.** — *Any simply transitive group can be defined as the set of transformations which leave invariant $n$ linearly independent differential forms $\omega^i(u; du)$.*[7]

**261.** The preceding Lemma proves Theorem II. In fact the group considered that leaves invariant the forms $\omega^i$, leaves invariant all the differential quadratic forms

$$g_{ij}\omega^i\omega^j$$

with constant coefficients $g_{ij}$. On the other hand the form $g_{ij}\omega^i\omega^j$ defines a Riemannian space of $n$ dimensions subject only to the condition that the coefficients $g_{ij}$ are chosen in such a way as to make the form positive definite.

It is important to note that, if we vary the choice of the values of the constants $g_{ij}$, the Riemannian spaces obtained are no longer in general applicable to each other. There is nevertheless a very special case where we obtain one and the same Riemannian space, this is where the constants ${c_{kh}}^i$ are zero. The forms $\omega^i$ are then exact differentials, that we can suppose to be $du^i$, and the corresponding fundamental forms have constant coefficients. All the spaces obtained are locally Euclidean, and the group $G$ is then the group of translations.

**262.** The constants ${c_{kh}}^i$ are called the *constants of structure* of the group $G$. They define in fact the *structure* or the *law of composition* of the transformations of the group, in the sense that, if we consider two simply transitive groups with

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[7] The transformations which leave invariant the forms $\omega^i$ cannot depend on more than $n$ parameters, because equations (12.5) can admit only one solution at most that pairs up with the given values of the $u^i$ the given values of the $(u^i)'$. *See* Note V, p. 367.

the same constants of structure, these two groups are *similar*, that is to say they can be considered as providing the same geometric transformations in a space where we have performed a change of coordinates. Consider in fact $n$ linearly independent differential forms $\varpi^i(v, dv)$ that satisfy the same relations (12.3)

$$d\varpi^i = \frac{1}{2}\, c_{kh}{}^i [\varpi^k\ \varpi^h].$$

The equations

$$\varpi^i(v; dv) = \omega^i(u; du) \qquad (i = 1, 2, ..., n), \tag{12.6}$$

where the variables $v^i$ are regarded as unknown functions of the variables $u^i$, form a completely integrable system. We have in fact, by putting $\varpi^i - \omega^i = \theta^i$, the identities

$$d\theta^i = \frac{1}{2}\, c_{kh}{}^i [\varpi^k\ \varpi^h - \omega^k \omega^h] = \frac{1}{2}\, c_{kh}{}^i [\theta^k\ \varpi^h - \theta^h \omega^k]$$

or

$$d\theta^i \equiv 0 \qquad (\mathrm{mod}\ \theta^1, \theta^2, ..., \theta^n),$$

which express the complete integrability of the system (Note V).

Any solution of equations (12.6) can be regarded as defining a change of coordinates in the space of the $u^i$ and by this change of coordinates the set of transformations which leave fixed the forms $\omega^i$ becomes the set of transformations which leave fixed the forms $\varpi^i$. We thus have the same group operating in the same space, but with two different analytic representations.

**263.** We add that any Riemannian space admitting a simply transitive group of displacements $G$ can be regarded as *the representative space* of the group, in the sense that any transformation of the group can be represented by the point $M$ into which the transformation takes an original point $O$, chosen once for all. The product of the transformation represented by the point $M$ with the transformation represented by the point $A$ (the first transformation being performed first) is represented by the point $M'$ to which is brought the point $M$ by the displacement which takes $O$ to $A$, which can be expressed symbolically by the relation

$$T_{M'} = T_A T_M\ .$$

This relation thus defines at the same time, if we regard $A$ as a fixed point, an operation (displacement) performed on the points $M$ of the space and an operation performed on the transformations $T_M$ of the group.

**264.** It s clear that we can, without changing a simply transitive group $G$, replace the forms $\omega^i$ by $n$ other forms, independent linear combinations with constant coefficients of the first: geometrically this comes down, in any of the Riemannian spaces admitting $G$ as a group of displacements, to changing the frame of reference applied to a particular point $O$ into any other frame of reference. The change considered modifies the value of the components ${c_{kh}}^i$. These constants here play the role of the components of a tensor of three indices, antisymmetric with respect to the two first indices. *It is this tensor*, considered as a whole, *which defines the structure of the group*, and not only its components considered in isolation.

**265.** We can ask whether there exist simply transitive groups with arbitrary constants of structure ${c_{kh}}^i$. It is not so. The exterior derivative of equations (12.3) give in fact

$${c_{kh}}^i[d\omega^k\ \omega^h] = 0 \qquad \text{or} \qquad {c_{kh}}^m {c_{m\ell}}^i[\omega^k\ \omega^h\ \omega^\ell] = 0.$$

By grouping together the terms in $[\omega^k\ \omega^h\ \omega^\ell]$, we obtain the quadratic relations

$${c_{kh}}^m {c_{m\ell}}^i + {c_{h\ell}}^m {c_{mk}}^i + {c_{\ell k}}^m {c_{mh}}^i = 0 \qquad (i, k, h, \ell = 1, 2, ..., n). \tag{12.7}$$

These relations are classical in the theory of groups.[8] We will see in the following Section (n° 268) that conversely if these relations are satisfied, there exists a system of $n$ linearly independent differential forms $\omega^i(u; du)$ that satisfy relations (12.3).

## V. — Canonical coordinates in a Riemannian space that admit a simply transitive group of rigid displacements.

**266.** The *canonical coordinates* that we will introduce are analogues of the normal coordinates of Riemann, but are distinct from them. The are closely related to the canonical parameters of the theory of groups of Lie.

Suppose attached to different points of the space a system of frames of reference adapted to the group. Let $O$ be a point of the space and denote by $a^1, a^2, ..., a^n$ the direction parameters, with respect to the frame of reference $(R_0)$ attached at $O$ of a direction from this point.[9] Consider the continuous family of points $M$, and consequently of frames $(R_M)$ defined by the differential equations

$$\omega^i(u; du) = a^i\, dt. \tag{12.8}$$

where $t$ is an independent variable, with the initial conditions $u^i = (u^i)_0$ for $t = 0$ [where the $(u^i)_0$ are the coordinates of the point $O$]. The point $M$ will describe a curve $(C)$ starting from $O$ and whose tangent at each point will have the direction parameters $a^1, a^2, ..., a^n$ with respect to the frame of reference $(R_M)$; the line element tangent at $M$ to the curve $(C)$ will thus result from the line element tangent at $O$ by the displacement of the group $G$ which takes $O$ to $M$.

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[8] *See* for example É. Cartan, *La théorie des groupes finis et continus et la Géométrie différentielle*, Chapter XIII, p. 233-241.

[9] The parameters $a^1, a^2, ..., a^n$ introduced here *must not be confused* with the parameters of the group, introduced in n° 251 and in equations (12.4).

All these displacements, which bring the point $O$ to the different points $M$ of the curve $(C)$, form a group. Consider in fact the figure formed by the nfinitely close frames of reference $(R_M)$, $(R_{M_1})$, where $M$ and $M_1$ are two points of the curve $(C)$ with parameters $t, t_\varepsilon$, and the figure formed by the frames of reference $(R_N)$, $(R_{N_1})$, where $N$ and $N_1$ are two points of the curve $(C)$ with parameters $t', t' + \varepsilon$ with the same infinitesimal $\varepsilon$.These two figures are equal, because of the equality one to the other of the forms $\omega^i, {\omega_i}^j$ relative to their two figures. The rigid displacement which takes $M$ to $N$ thus takes $M_1$ to $N_1$ and, step by step, the point with parameter $t + h$ to the point with parameter $t' + h$. This displacement is moreover that which takes $O$ to the point of $(C)$ with parameter $t' - t$; it slides the curve on itself. The equation which defines the transformations of the one parameter group, which which leaves fixed the forms $a^i \, dt$, that is to say the form $dt$, is simply $t' = t +$ constant. We see that the total group $G$ admits an infinity of one parameter subgroups each of which corresponds to a choice of direction parameters $a^i$. The curves $(C)$ are the *trajectories* of these subgroups.

We can add that the curves $(C)$ are not any; each of them has all its curvatures constant. The curves $(C)$ develop onto the Euclidean space tangent at one of their points, as lines or circumferences if $n = 2$, and as lines, circumferences or circular helixes if $n = 3$.

It is clear that the point of the curve $(C)$ defined by the $n + 1$ redundant coordinates $a^1, a^2, ..., a^n, t$, the differential equations (12.15) can be written as

$$\omega^i(u; du) = d(a^i t).$$

The quantities $x^i = a^i t$ are the *canonical parameters* of rigid displacement which takes the point $O$ to the point $M$ with parameter $t$ on the curve $(C)$ of parameters $a^1, a^2, ..., a^n$. These are the *canonical parameters* of the point $M$ of the Riemannian space.

Any point of the space sufficiently close to $O$ admits well defined canonical coordinates, but it is not certain that it will be so for every point of the space,[10] and we can give examples where, in fact, it is not.

**267.** We can seek to determine what are the expressions for the forms $\omega^i$ when we introduce canonical coordinates; these expressions moreover do not depend on the point $O$ chosen as origin.

Introduce again the redundant coordinates $a^i, t$; moving along a curve $(C)$ $(da^i = 0)$, we have $\omega^i = a^i \, dt$. Put

$$\omega^i = a^i \, dt + \varpi^i(t, a^i; da^i); \tag{12.9}$$

the forms $\varpi^i$ are zero for $t = 0$ because if we fix $t$ and the fixed value of $t$ is zero, the frame of reference $(R_0)$ does not vary and the forms $\omega^i$ vanish. Lets us

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[10] In other words, it is not certain that all points of the space can be deduced from $O$ by a transformation of a one parameter subgroup of the total group.

highlight then, in equations (12.3) of n° 259,

$$d\omega^i = \frac{1}{2}\, c_{kh}{}^i [\omega^k\ \omega^h],$$

where we replace the forms $\omega^i$ by their expressions (12.9), those forms that contain $dt$. We find

$$[da^i\ dt + \left[dt\ \frac{\partial \varpi^{\,i}}{\partial t}\right] = c_{kh}{}^i\, a^k [dt\ \varpi^{\,h}]$$

from which

$$\frac{\partial \varpi^{\,i}}{\partial t} = da^i + c_{kh}{}^i\, a^k \omega^{\,h}. \tag{12.10}$$

We thus have, to determine the forms $\varpi^{\,i}$, zero for $t = 0$, a system of ordinary differential equations with constant coefficients (the $a^k$ and $da^k$ are in fact to be regarded as constant parameters). The forms $\varpi^{\,i}$ will thus be linear combinations of the $da^k$ whose coefficients will be *entire functions* of $t, a^1, a^2, ..., a^n$. This proves in passing the theorem of n° 262 according to which two simply transitive groups that have the same structure constants are similar.

If we replace in the forms $\varpi^{\,i}$ the variable $t$ by 1 and the arguments $a^1, a^2, ..., a^n$ by $x^1, x^2, ..., x^n$, we will obtain the forms $\omega^i(x; dx)$ expressed by means of canonical coordinates. The coefficients of the differentials $dx^k$ in these forms are *entire functions* of $x^1, x^2, ..., x^n$. *A Riemannian space that admits a simply transitive group of rigid displacements thus becomes an analytic space when we introduce canonical coordinates.* In truth we can affirm it only for a sufficiently small region $\mathcal{V}_O$ surrounding the origin $O$, in which the determinant of the coefficients of the $dx^k$ in the forms $\omega^i$ is different from zero. But we prove that we can cover all the space by neighbourhoods overlapping one another, in each of which we can introduce coordinates such that the fundamental form becomes analytic. To appreciate the guiding idea of the proof, it is sufficient to note that if $A$ is a point of the neighbourhood $\mathcal{V}_O$, close to the boundary of $\mathcal{V}_O$ for example, we need only apply to $\mathcal{V}_O$ the rigid displacement $T$ which takes $O$ to $A$; we will obtain a certain neighbourhood $\mathcal{V}_A$ of $A$, in which we will introduce the following new system of coordinates: we will attribute to any point $M$ of $\mathcal{V}_A$ as new coordinates the canonical coordinate of that point of $\mathcal{V}_O$ which the displacement $T$ has taken to $M$.

**268.** We claimed in n° 265 that given a system of constants $c_{kh}{}^i = -c_{hk}{}^i$ that satisfy the quadratic relations (12.7), there exist forms $\omega^i$ that satisfy relations (12.3). We can now prove this theorem by showing that the forms $\omega^i(x; dx)$ constructed by means of the solution forms $\varpi^{\,i}(t, a; da)$ of equations ([12.10) satisfy the equations

$$d\omega^i = \frac{1}{2}\, c_{kh}{}^i [\omega^k\ \omega^h],$$

where we regard $t$ as a constant parameter, the $a^i$ as variables, and the $da^i$ as their differentials. It is sufficient to apply the reasoning used for a similar theorem in the theory of symmetric spaces (Chapter XI, n° 243). We leave the reader to do the proof, which moreover is here easier than that of n° 243.

**269.** We prove in the theory of groups a theorem which comes down to the following and of which the preceding number proves only a small part.

**Theorem.** — *Given a system of constants* ${c_{kh}}^i = -{c_{hk}}^i$ *that satisfies the quadratic relations (12.7), there exists a simply connected, normal, analytic Riemannian space that admits a simply transitive group of rigid displacements whose* ${c_{kh}}^i$ *are the structure constants.*

In general, according to a previous remark, there exist an infinity, all homeomorphic with each other. There can exist also normal, analytic Riemannian spaces that are not simply connected. These are problems regarding the topology of Lie groups.[11]

## VI. — Canonical coordinates and normal coordinates.

**270.** There is a great similarity between canonical coordinates and the normal coordinates of Riemann, but in general there is no identity. We can propose to look into which case canonical coordinates are normal coordinates. Obviously it is necessary and sufficient for this that the trajectories $(C)$ of the one parameter subgroups of the given group be geodesics of the space. Now if we move along a trajectory with parameters $a^i$, de have

$$\frac{Da^i \boldsymbol{e}_i}{dt} = a^i {\gamma_i}^k{}_h a^h \boldsymbol{e}_k;$$

for this trajectory to be a geodesique, it is necessary and sufficient that the quadratic form ${\gamma_i}^k{}_h a^i a^h$ be idetically zero, that is to say that the coefficients ${\gamma_i}^k{}_h$ (and consequently $\gamma_{ikh}$) be antisymmetric with respect to the two outer indices. Now we have

$${\gamma_k}^i{}_h - {\gamma_h}^i{}_k = {c_{kh}}^i\,,$$

from which we get

$${\gamma_k}^i{}_h = \frac{1}{2}\,{c_{kh}}^i\,, \qquad \gamma_{kih} = \frac{1}{2}\,c_{khi}\,.$$

Since the forms $\omega_{ij}$ are antisymmetric with respect to their two indices, it follows that the covariant components $c_{khi}$, already antisymmetric with respect

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[11] *See* É. Cartan, *La théorie des groupes finis et continus et l'Analysis situs* (*Mém. Sc. Math.*, XLII, 1930), as well as, by the same author, *La topologie des groupes de Lie* (*Exposés de Géométrie*, VIII, Hermann, 1936).

to the first two indices, are also with respect to the two outer indices, from which we deduce easily that the tensor ${c_{kh}}^i$ is a trivector. We thus have in this case

$$\omega_i{}^j = \frac{1}{2}\, c_{ik}{}^j \omega^k \, . \tag{12.11}$$

Conversely if the tensor ${c_{kh}}^i$ is a trivector, since the difference $\gamma_{kih} - \gamma_{hik} = c_{khi}$ changes sign by exchange of the indices $k$ and $i$, and on the other hand since $\gamma_{kih}$ is antisymmetric with respect to the two first indices, we will have

$$\gamma_{kih} - \gamma_{hik} + \gamma_{ikh} - \gamma_{hki} = 0,$$

or

$$\gamma_{ihk} + \gamma_{khi} = 0;$$

there is therefore antisymmetry with respect to the two outer indices, from which

**Theorem.** — *For the canonical coordinates of a space that admits a simply transitive group of displacements to be the normal coordinates of Riemann, it is necessary and sufficient that the tensor* ${c_{kh}}^i$ *be a trivector.*

**271.** The condition found involves at the same time the structure constants ${c_{kh}}^i$ of the space and the components $g_{ij}$ of the fundamental tensor with respect to the system of frames of reference adapted to the group. Given the group, to have a Riemannian space admitting this group as the group of displacements and such that the canonical coordinates are at the same time normal coordinates, it is necessary and sufficient that we can find constants $g_{ij}$ that can be the coefficients of a positive definite quadratic form and that satisfy the relations

$$g_{im} c_{kj}{}^m + g_{jm} c_{ki}{}^m = 0 \qquad (i, j, k = 1, 2, .., n). \tag{12.12}$$

This last condition hugely limits the choice of group.

**272.** We know that the normal Euclidean space at a point $O$ of a Riemannian space osculates this space at this point. So it is the same for the canonical Euclidean space, if the given space admits a simply transitive group of displacements such that the canonical coordinates are the normal coordinates. *The converse is true.*

In fact integration of equations (12.10) of n° 267 gives easily a limited expansion of the forms $\omega^i$, namely

$$\omega^i = dx^i + \frac{1}{2}\, c_{kh}{}^i x^k \, dx^h \, ,$$

from which we deduce a limited expansion corresponding to the fundamental form

$$ds^2 = ds_0^2 + \frac{1}{4}(g_{jm} c_{ki}{}^m + g_{im} c_{kj}{}^m) x^k \, dx^i \, dx^j \, ,$$

where $ds_0^2$ denotes the fundamental form of the Euclidean space with coordinates $x^i$,

$$ds_0^2 g_{ij} dx^i\, dx^j\ .$$

We see that the condition for the canonical space at $O$ to osculate the Riemannian space is the condition already found (12.12) for the canonical coordinates to be normal coordinates.

**Theorem II.** — *For the canonical coordinates of a space that admits a simply transitive group of displacements to be normal coordinates, it is necessary and sufficient that the canonical Euclidean space at a point osculate the Riemannian space at this point.*

**273.** Finally we will add a new property of the spaces that interest us.

**Theorem III.** — *Any Riemannian space that admits a simply transitive group of displacements such that the canonical coordinates at a point are normal coordinates is symmetric.*

It suffices to prove that the derived tensor of the Riemann-Christoffel tensor is zero. Find first the components of this tensor. We have, according to (12.11),

$$\omega_i{}^j = \frac{1}{2}\, c_{ik}{}^j \omega^k\ , \tag{12.13}$$

from which

$$\omega_{ij} = -\frac{1}{2}\, c_{ijk} \omega^k\ .$$

We get, by taking into account formulae (9.7) of n° 202 and the quadratic relations (12.7) (n° 265),

$$\Omega_{ij} = d\omega_{ij} + \omega_{ik}\omega_j{}^k = -\frac{1}{8}\, c_{ijm} c_{kh}{}^m [\omega^k\ \omega^h],$$

from which

$$R_{ijkh} = -\frac{1}{4}\, c_{ij}{}^m c_{khm}\ .$$

If we then form the absolute differential of $R_{ijkh}$ we get, by noting that the components $R_{ijkh}$ are themselves constants,

$$DR_{ijkh} = R_{mjkh}\omega_i{}^m + R_{imkh}\omega_j{}^m + R_{ijmh}\omega_k{}^m + R_{ijkm}\omega_h{}^m;$$

by replacing the $\omega_i{}^j$ by their values (12.11) and taking into account relations (12.7), we establish that the absolute differential considered is zero.

**274.** The Riemannian spaces that admit a simply transitive group of displacements and whose canonical coordinates are at the same time normal form a particular, but important, class of symmetric spaces. We can prove that the group of displacements considered $G$ forms part of a larger group of displacements whose order can reach twice the order of the group $G$.

Recall (n° 263) that if the point $M'$ is obtained from the point $M$ by the displacement $T_A$ which takes $O$ into $A$, we have the relation

$$T_{M'} = T_A T_M. \tag{12.14}$$

That said, the trajectories $(C)$ of the one-parameter subgroups of $G$ are here geodesics, so that the displacement $T_M$ which takes $O$ to $M$ and the displacement $T_{M'}$ which takes $O$ to the symmetric point $M'$ of $M$ with respect to $O$ are inverses of one another: the relation

$$T_{M'} = (T_M)^{-1} \tag{12.15}$$

thus defines the symmetry with respect to $O$, an isometric transformation.

That said, start from any point $M$ and perform successively the following isometric transformations :

**1°** Symmetry with respect to $O$ taking $M$ to $P$;
**2°** displacement $T_A^{-1}$ taking $A$ to $O$ and $P$ to $Q$;
**3°** symmetry with respect to $O$ taking $Q$ to $M'$.

An easy calculation gives, by taking into account (12.15),

$$T_{M'} = T_M T_A. \tag{12.16}$$

Transormation (12.16) obviously defines a new group of displacements $G^*$, distinct in general from $G$. Consequently by combining the displacements of $G$ and those of $G^*$, we obtain a new group $\Gamma$ defined by the relations

$$T_{M'} = T_A T_M T_B; \tag{12.17}$$

this group $\Gamma$, which depends on two arbitrary points $A$ and $B$, is in general of order $2n$.

Note that the symmetry with respect o the point $A$ is defined by

$$T_{M'} = T_A T_M^{-1} T_A. \tag{12.18}$$

This relation expresses in fact that the two transformations $T_A^{-1} T_{M'}$ and $T_A^{-1} T_M$ are inverses of one another, that is to say that the points $P$ and $P'$ which are obtained from $M$ and $M'$ by the displacements which take $A$ to $O$ are symmetric with respect to $O$.

Finally the *transvections* (n° 246) along a geodesic resulting from two successive symmetries with respect to $O$ and with respect to a point $A$ of this geodesic are given by the relation

$$T_{M'} = T_A T_M T_A. \tag{12.19}$$

We can show that, apart from Euclidean space, for which the groups $G$ and $G^*$ merge, the group $\Gamma$ is of order $2n$ if the space is irreducible; it s then an example of a closed simple group.

## VII. — Isogonal parallelism connected to a simply transitive group of displacements.

**275.** Return to the general case of a space that admits a simply transitive group of displacements $G$. We can define in this space an *absolute equivalence* of vectors, said to be *absolute* because, to determine the equivalence of two vectors with different origins $M$ and $N$, it is not necessary, as in the case of the equivalence of Levi-Civita, to fix a path joining the first point to the second.

The absolute equivalence of two vectors in question is defined by the condition that these two vectors have respectively the same components with respect to frames of reference that have these points as origin and are adapted to the group $G$. Analytically, if we use the canonical coordinates $x^i$, any vector, which can be regarded can be regarded as joining the point of origin $M\,(x^i)$ and an infinitely close point $(x^i+dx^i)$, has as components with respect to the corresponding frame of reference $(R_M)$ adapted to the group the quantities $\omega^i(x;dx)$. *Two vectors are thus equivalent if for these two vectors the forms $\omega^i(x;dx)$ have the same values.*

If we consider a point $A$ infinitely close to the point $O$ of origin of the canonical coordinates, a vector $(x,dx)$ joining two infinitely close points $M$, $M'$ will be equivalent to the vector $\overrightarrow{OA}$ if the displacement which takes $O$ into $M$ and consequently $(R_O)$ into $(R_M)$, takes $A$ into $M'$, which translates into the formula

$$T_{M'} = T_M T_A \quad \text{or} \quad T_M^{-1} T_{M'} = T_A. \tag{12.20}$$

**276.** Since the frames of reference adapted to the group are all equal between them, two equipollent vectors having the same components with respect to two of these frames have the same length, and also the scalar product of two vectors with the same origin is equal to that of two vectors which are equivalent and have another common origin. If therefore two vectors with origin $A$ intersect a a certain angle, equivalent vectors at any other point $M$ will intersect at the same angle: we are dealing with an *isogonal equivalence*.

**277.** Consider the field of vectors equivalent to a given vector: the trajectories of this field form a congruence of curves; the one that starts at the origin is one of the curves $(C)$, say $(C_O)$, defined in n° 266 and which slide on themselves by the displacements of a one parameter subgroup $g$ of $G$; the one that starts from any point $A$ is deduced from $(C_O)$ by the displacement $T_A$ which takes $O$ into $A$, and we pass from $O$ to a running point of this trajectory by performing first a displacement of $g$, then the displacement $T_A$, so that the trajectory is the locus of the points $M$ defined by the relation

$$T_M = T_A\ g. \tag{12.21}$$

As proof we see that if $M$ and $M'$ are two infinitely close points of the line thus obtained, the displacement $T_M^{-1}T_{M'}$ belongs to $g$ and can be represented by $T_{O'}$ where $O'$ denotes a point infinitely close to $O$ on $(C_O)$.

All trajectories of the field considered can be regarded as parallels; this is an *absolute* and *isogonal parallelism*; any vector tangent at any point of one of these trajectories is equivalent to a vector tangent to another of these trajectories, given arbitrarily, at any point of this trajectory.

**278.** If the canonical coordinates of the space are normal, the space is symmetric and the curves $(C)$ are geodesics; there exists therefore an isogonal absolute parallelism of geodesics related to the group $G$. But there exists a second related to the second group of simply transitive displacements $G^*$ which the space admits (n° 274).

We will say that an infinitesimal vector $\overrightarrow{MM'}$ is equivalent of the second kind to the infinitesimal vector $\overrightarrow{OA}$ if there is a relation, similar to (12.20),

$$T_{M'} = T_A T_M \qquad \text{or} \qquad T_{M'}T_M^{-1} = T_A. \tag{20*}$$

Two vectors equivalent of the second kind to $\overrightarrow{OA}$ are equivalent with each other.

All the properties proved for equivalence with respect to the group $G$ extend to equivalence with respect to the group $G^*$. It is the same for isogonal absolute parallelism of geodesics. The geodesic parallel of the second kind to the geodesic $OA$ is the locus of points $M$ defined by the relation

$$T_M = g\, T_A. \tag{21*}$$

**279.** The is a simple relation between the two equivalences.

**Theorem.** — *if two vectors are equivalent of the first [second] kind, their symmetries with respect to a point of the space are equivalent of the second [first] kind.*

Relation (20*) can in fact be written as

$$T_{M'}^{-1} = T_M^{-1} T_A^{-1}\,,$$

and it states that the vector that has as origin the point symmetric to $M$ with respect to $O$ and as endpoint the point symmetric to $M'$ with respect to $O$ is equivalent of the first kind to the vector symmetric to $\overrightarrow{OA}$ with respect to $O$. The theorem is still true if we perform the symmetry with respect to any point of the space.

From this it follows immediately that *in canonical coordinates* two vectors are equivalent of the second kind if the terms $\omega^i(-x; dx)$ have the same values for the two vectors.[12]

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[12] The $x^i$ are normal coordinates, but the $\omega^i$ are forms with respect to the frames of reference *adapted to the group G* which must not be confused with the *adapted* frames of

**280.** We call the elements of the group $G$ *translations of the first kind* and the elements of the group $G^*$ *translations of the second kind.* Consider the subgroup $g$ of translations of the first kind that slide on itself a geodesic $OA$ from $O$. Applied to a point $M$ of the space, they will transform it into a variable point on a geodesic of the second kind parallel to $OA$, according to the formula (20*); applied to any geodesic, they transform it into a family of geodesics of the first kind which are all parallel to each other. There are similar properties for the subgroups $g^*$ of translations of the second kind.

**Theorem.** — *All one parameter subgroups of translations of the first [ second ] kind slide on themselves all geodesics belonging to the same congruence of parallel geodesics of the second [ first ] kind, and it transforms any geodesic in a continuous set of parallel geodesics of the first [ second ] kind into the given geodesic.*

This theorem shows the analogy that the translations which have just been defined have with the translations of elementary geometry.

Another simple property is that an infinitesimal translation makes all points of the space undergo displacements of the same length; the infinitesimal vectors described by the points of the space are in fact all equivalent to each other, whether of the first or of the second kind.

**281.** A famous particular case is provided by the three dimensional space with constant positive curvature (spherical space or elliptical space). It corresponds to a simply transitive group of three parameters whose structure tensor ${c_{kh}}^i$ is a trivector. When the Riemannian space which this group leaves invariant is referred to rectangular frames of reference, all the structure constants reduce to one only $c_{123} = c$, and we have

$$\left.\begin{array}{lcl} d\omega_1 &=& c[\omega_2\,\omega_3], \\ d\omega_2 &=& c[\omega_3\,\omega_1], \\ d\omega_3 &=& c[\omega_1\,\omega_2], \end{array}\right\} \tag{12.22}$$

with, [formula (12.13) of n° 273],

$$\begin{array}{lll} \omega_{23} = -\dfrac{1}{2}\,c\omega_1, & \omega_{31} = -\dfrac{1}{2}\,c\omega_2, & \omega_{12} = -\dfrac{1}{2}\,c\omega_3, \\ \Omega_{23} = -\dfrac{1}{4}\,c^2[\omega_2\,\omega_3], & \Omega_{31} = -\dfrac{1}{4}\,c^2[\omega_3\,\omega_1], & \Omega_{12} = -\dfrac{1}{4}\,c^2[\omega_1\,\omega_2. \end{array}$$

The space is thus with constant positive curvature equal to $\frac{1}{4}\,c^2$. Take $c = 2$, which is no loss of generality. We then know (n° 140) that the $ds^2$ of the space can be put into the form

$$ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2, \tag{12.23}$$

reference ce adapted to the *adapted to the group $G$*, which must not be confused with the frames of reference *adapted to the normal coordinates.*

where we assume that the coordinates $x_i$, not canonical, are related by the relation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

Equations (12.22) and (12.23) are satisfied by taking

$$\left.\begin{array}{rcl} \omega_1 & = & x_0\,dx_1 - x_1\,dx_0 + x_2\,dx_3 - x_3\,dx_2, \\ \omega_2 & = & x_0\,dx_2 - x_2\,dx_0 + x_3\,dx_1 - x_1\,dx_3, \\ \omega_3 & = & x_0\,dx_3 - x_3\,dx_0 + x_1\,dx_2 - x_2\,dx_1, \end{array}\right\} \tag{12.24}$$

as is shown by a very simple calculation, where we take into account the relation

$$x_0\,dx_0 + x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3 = 0.$$

The geodesics are defined by equations of first degree in $x_0, x_1, x_2, x_3$. Take for the point $O$ the point $x_0 = 1, x_1 = x_2 = x_3 = 0$; the symmetric of the point $(x_0, x_1, x_2, x_3)$ is the point $(x_0, -x_1, -x_2, -x_3)$; in fact the two points are aligned with the point $O$ and they are at the same distance from the point $O$, since $ds^2$ does not change when $x_1, x_2, x_3$ change sign. It follows that if two vectors equivalent of the first type are characterised by equality of the forms $\omega_1, \omega_2, \omega_3$, two vectors equivalent of the second type are characterised by the equality of the forms

$$\varpi_1 = x_0\,dx_1 - x_1\,dx_0 - x_2\,dx_3 + x_3\,dx_2,$$
$$\varpi_2 = x_0\,dx_2 - x_2\,dx_0 - x_3\,dx_1 + x_1\,dx_3,$$
$$\varpi_3 = x_0\,dx_3 - x_3\,dx_0 - x_1\,dx_2 + x_2\,dx_1,$$

where the sum of the squares is the $ds^2$ of the space.

If we place ourselves in the elliptic space, that is to say in the projective space of three dimensions referred to homogeneous coordinates $x_0, x_1, x_2, x_3$, two finite vectors bounded by the points $x_i, y_i$, are equivalent of the first kind if, for these two vectors, the Plücker coordinates

$$p_{01} + p_{23}, \quad p_{02} + p_{31}, \quad p_{03} + p_{12}$$

are the same; they are equivalent of the second kind if it is the components

$$p_{01} - p_{23}, \quad p_{02} - p_{31}, \quad p_{03} - p_{12}$$

which are the same.

**282.** Let us represent a point $(x^i)$ by the quaternion

$$\boldsymbol{X} = x_0 + x_1 i + x_2 j + x_3 k;$$

The vector quaternion $i\omega_1 + j\omega_2 + x_3\omega_3$ is none other than $-\boldsymbol{X}\,d\overline{\boldsymbol{X}}$, where we denote by $\overline{\boldsymbol{X}}$ the conjugate quaternion $x_0 - x_1 i - x_2 j - x_3 k$. The vector quaternion $i\varpi_1 + j\varpi_2 + x_3\varpi_3$ is none other than $\overline{\boldsymbol{X}}\,d\boldsymbol{X}$.[13]

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[13] Recall that the units $i, j, k$ obey the multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k.$$

The translations of the first kind are defined by the formulae

$$\boldsymbol{X}' = \boldsymbol{X}\boldsymbol{A}, \tag{12.25}$$

where $\boldsymbol{A}$ is an unit quaternion ($a_0^2+a_1^2+a_2^2+a_3^2 = 1$): in fact they leave invariant the forms $\omega_i$, because

$$\boldsymbol{X}'\, d\overline{\boldsymbol{X}}\,' = \boldsymbol{X}\boldsymbol{A}\overline{\boldsymbol{A}}\, d\overline{\boldsymbol{X}} = \boldsymbol{X}\, d\overline{\boldsymbol{X}},$$

because of the assumption that $\boldsymbol{A}\overline{\boldsymbol{A}} = 1$.

The translations of the second kind are defined by the formulae

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}, \tag{12.26}$$

where $\boldsymbol{A}$ is an unit quaternion: in fact they leave invariant the forms $\overline{\omega}_i$, because we have

$$\overline{\boldsymbol{X}}'\, d\boldsymbol{X}' = \overline{\boldsymbol{X}}\; \overline{\boldsymbol{A}}\boldsymbol{A}\, d\boldsymbol{X} = \overline{\boldsymbol{X}}\, d\boldsymbol{X}.$$

These translations were considered for the first time by Clifford, who is also the first to have conceived the two absolute parallelisms that carry his name.[14]

*Note.* — The existence of these absolute parallelisms does not generalise to spaces with positive constant curvature of higher dimensions; nevertheless in the elliptic space of seven dimensions, there exists and infinity of isogonal absolute parallelisms forming two disting families, but they are not related to the existence of a simply transitive group of displacements in this space. It is, with the representation spaces of closed simple groups, the only irreducible Riemannian space (n$^{\text{o}}$ 249) admitting isogonal absolute parallelisms.[15]

## VIII. — Riemannian spaces that admit a multiply transitive group of displacements.

**283.** If an $n$-dimensional Riemannian space admits a transitive group $G$ with $r > n$ parameters, the displacements which leave fixed a point $O$ generate a group $g$ with $r - n$ parameters, which is the *group of rigid rotations* around $O$, or the *isotropy group*, or also the *stability group* of $O$. If we use a system of rectangular frames of reference adapted to the group, the stability group $g$ is translated analytically by the group of orthogonal substitutions which the components of a vector with origin $O$ undergo through the effect of the rotations of $g$. The connected part of this group which contains the identity rotation is generated by the infinitesimal rotations each of which is defined a bivector $\xi_{ij}$, where these

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[14] *Papers* (London, 1882, p. 181, 236, 378, 402). For more details, see É. Cartan, *Leçons sur la Géométrie projective complexe* (Gauthier-Villars, 1931), 2$^{\text{o}}$ partie, Chap. IV, Section VI.

[15] É. Cartan and J. A. Schouten, *On the Riemannian Geometries admitting an absolute parallelism* (*Proc. Amsterdam*, t. 29, 1926, p. 933-946).

bivectors are required to satisfy $\frac{n(n+1)}{2} - r$ independent linear relations with constant coefficients

$$A_{\alpha ij}\xi_{ij} = 0 \qquad \left(\alpha = 1, 2, ..., \frac{n(n+1)}{2} - r\right). \tag{12.27}$$

The frames of reference attached to different points of the space depend on $r$ parameters of which $n$ are the coordinates $u^1, 2, ..., u^n$ of the origin of the frame of reference, where the $r-n$ others $v^1, v^2, ..., v^{r-n}$ are used to individualise the frames of reference with the same origin. The forms $\omega_1, \omega_2, ..., \omega_n$ are linear in $du^1, du^2, ..., du^n$, since they are zero when the point $(u^i)$ remains fixed; as for the forms $\omega_{ij}$, they are linear with respect to the $du^i$ and to the $dv^i$, but the forms $A_{\alpha ij}$ depend only on the differentials $du^i$, since when the point $(u^i)$ remains fixed, the forms $\omega_{ij}$ that define the elementary rotation undergone by the frame of reference are none other than the components $\xi_{ij}$ of a bivector that satisfies relation (12.27).

There exist therefore between the $\frac{n(n+1)}{2}$ forms $\omega_i, \omega_{ij}$ linear relations in number $\frac{n(n+1)}{2} - r$ of the form

$$A_{\alpha ij} = C_{\alpha k}\omega_k; \tag{12.28}$$

the coefficients $C_{\alpha k}$ are constants since any displacement of $g$ that leaves invariant the forms $\omega_i, \omega_{ij}$ leaves invariant the coefficients $C_{\alpha k}$ which are therefore constants. Those are the only linear relations between these forms.

**284.** It can happen that the infinitesimal rotations of the stability group of $O$ leave invariant a certain number of vectors from $O$; suppose that we have $\nu$ linearly independent ones that generate a linear manifold of dimension $\nu$.

The geodesics from $O$ tangent to this manifold are invariants for the group of rigid rotations $g$ and each of their points is also invariant under $g$. These geodesics generate a manifold $V_\nu$ geodesic at $O$. The stability group of any point $M$ of this manifold is the group $g$ itself; the manifold $V_\nu$ is thus geodesic at each of its points; it is consequently a totally geodesic manifold.

**Theorem.** — *The geodesics tangent at a point to vectors invariant under stability the group of this point generate a totally geodesic manifold.*

These totally geodesic manifolds depend on $n - \nu$ parameters since one and only one passes through any point of the space, and they are transformed among themselves by the group of displacements $G$.

It can moreover happen that the group $g$ does not leave any vector invariant; the manifolds $V_\nu$ reduce then to a point.

**285.** The constants $C_{\alpha k}$ which enter into the right hand sides of formulae (12.28) are not arbitrary. In fact let $(R_M)$ and $(R_{M'})$ be two infinitely close frames of reference of the system of frames of reference adapted to the group.

Perform on each of them a rotation around its origin, where these two rotations are defined respectively relative to $(R_M)$ and $(R_{M'})$ by a bivector with the same components $\xi_{Ij}$; let $(\overline{R}_M)$ and $(\overline{R}_{M'})$ be the frames of reference that we deduce. The figure formed by $(R_{M'})$ and $(\overline{R}_{M'})$ is equal to the figure formed by $(R_M)$ and $(\overline{R}_M)$; it is thus the same geometric infinitesimal displacement which carries both $(R_M)$ into $(R_{M'})$ and $(\overline{R}_M)$ into $(\overline{R}_{M'})$. This displacement is represented analytically with respect to $(R_M)$ by the set consisting of the vector $\omega_i$ and the bivector $\omega_{ij}$; by the change of the system of reference which takes us from $(R_M)$ to $(\overline{R}_M)$, the components $\omega_i, \omega_{ij}$ undergo the variations

$$\delta\omega_i = \xi_{ik}\omega_k, \qquad \delta\omega_{ij} = \xi_{ik}\omega_{kj} + \xi_{jk}\omega_{ik};$$

since relations (12.28) need to be kept, we will have

$$A_{\alpha ij}(\xi_{ik}\omega_{kj} + \xi_{jk}\omega_{ik}) = C_{\alpha h}\xi_{hk}\omega_k. \tag{12.29}$$

*Relations (12.29) will have to be consequences of equations (12.28) whenever the $\xi_{ij}$ satisfy relations (12.27).* These conditions limit the choice of the constants $C_{\alpha k}$. They would moreover be found automatically by application of the general method which will be indicated in the following paragraph.

**286.** We can express the $\omega_{ij}$ as linear combinations with constant coefficients of the $\omega^i$ and of $r-n$ of the forms $\omega_{ij}$, which we will denote by $\varpi_1, \varpi_2, ..., \varpi_{r-n}$.[16]

The exterior differentials $d\omega_i$ are *known* exterior quadratic forms of these $r$ forms in which the $\varpi_\alpha$ enter with degree one at most. By taking the exterior derivative of the equations written in this way, we will have relations between the $C_{\alpha k}$, including the ones discussed in the preceding paragraph, and we will also have conditions that the forms $d\varpi_\alpha$ must satisfy. By expressing them in the most general way possible as quadratic forms with constant coefficients of the $\omega_i$ and the $\varpi_\alpha$ and by finding the exterior derivative of the equations thus obtained, we can obtain new conditions that all the constants successively introduced must satisfy. This done, we will have arrived at a system of $r$ forms $\theta^1, \theta^2, ..., \theta^r$ ($\theta^i = \omega_i$ for $i \leq n$, $\theta^{n+\alpha} = \varpi_\alpha$ for $n+1 \leq \alpha \leq r$) that satisfy the relations

$$d\theta^i = \frac{1}{2}\, c_{kh}{}^i[\theta^k\ \theta^h], \tag{12.30}$$

with constant coefficients.

**287.** The coefficients $c_{kh}{}^i$ necessarily satisfy equations (12.7) of n° 265, since exterior differentiation of equations (12.30) gives, by hypothesis, relations which are all consequences of relations (12.30) themselves. There exists therefore a

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[16] More generally, and this could be convenient in certain cases, we can express the $\frac{n(n-1)}{2}$ forms $\omega_{ij}$ as linear combinations of the $r-n$ forms $\varpi_\alpha$ and the $\omega_h$ on condition that we take for the $\varpi_\alpha$ forms linear with respect to the $\omega_{ij}$ and the $\omega_i$, such that, by limiting them to terms in $\omega_{ij}$, they form, with the left hand sides of the equations (12.28), a system of $\frac{n(n-1)}{2}$ independent forms.

group simply transitive $G$ of order $r$ which admits the ${c_{kh}}^i$ as constants of structure. We will prove that we can find $n$ functions $x^i$ independent of the $r$ variables transformed by $G$ which are transformed among themselves by $G$, and moreover that there exists an infinity of quadratic differential forms, constructed from the $x^i$ and the $dx^i$, which are invariant under $G$.

For this we will begin from the remark that among equations (12.30) we find the equations

$$d\omega_i = [\omega_k \ \omega_{ki}],$$

where the $\frac{n(n-1)}{2}$ forms $\omega_{ij}$ are antisymmetric and where the $\omega_i$ are linearly independent.

We will prove first the following theorem, important in its own right.

**Theorem.** — *Given n linearly independent differential forms constructed from $r > n$ variables $u!, u^2, ..., u^n$ and their differentials, and that satisfy the relations*

$$d\omega_i = [\omega_k \ \omega_{ki}], \tag{12.31}$$

*where the $\omega_{ij} = -\omega_{ji}$ are also constructed from the same variables and their differentials, we can find n independent functions $x^1, x^2, ..., x^n$ of $u^1, u^2, ..., u^r$ such that the quadratic form $(\omega_1)^2 + (\omega_2)^2 + \cdots + (\omega_n)^2$ depend only on the functions $x^i$ and their differentials.*

In fact first the equations $\omega_1 = \omega_2 = \cdots = \omega_n$ form a completely integrable system since the $d\omega_i$ are congruent to zero (mod $\omega_1, \omega_2, \cdots, \omega_n$) (Note V, p 367). Let $x^1, x^2, ..., x^n$ be a system of $n$ independent first integrals of this system, and denote by $y^1, ..., y^{r-n}$ a system of $r - n$ other functions that form with the $x^i$ a set of $r$ independent functions.

The $\omega_i$ are linear in $dx^1, dx^2, ..., dx^n$; we thus have

$$(\omega_1)^2 + (\omega_2)^2 + \cdots + (\omega_n)^2 = g_{ij} dx^i dx^j,$$

the $g_{ij}$ may depend on the $x^i$ and the $y^j$. Now put

$$\omega_{ij} = \lambda_{ijk} dy^k + \cdots \qquad (\lambda_{ijk} = -\lambda_{jik}),$$

where the unwritten terms are linear in $dx^1, dx^2, ..., dx^n$. Equations (12.31) give, by writing only the terms that contain the differentials $dy^k$,

$$\left[dy^h \ \frac{\partial \omega_i}{\partial y^h}\right] = \lambda_{kih}[\omega_k \ dy^h],$$

from which

$$\frac{\partial \omega_i}{\partial y^h} = \lambda_{ikh}\omega_k,$$

from which finally

$$\omega_i \ \frac{\partial \omega_i}{\partial y^h} = \lambda_{ikh}\omega_i\omega_k = 0;$$

the form $g_{ij}dx^i dx^j$ is thus independent of the variables $y^k$.

Q.E.D.

We have proved the existence of a group $G$ of order $r$ in $n$ variables $x^1, x^2, ..., x^n$ and the existence of an $n$-dimensional Riemannian space that admits the group $G$ as a group of displacements; furthermore the group of rigid rotations around a point the is given group $g$.

**288.** There exists an infinity of Riemannian spaces admitting the group $G$ as the group of displacements with the stability group $g$ imposed in advance. In fact the group $g$, which acts on the forms $\omega_1, \omega_2, ..., \omega_n$ as an orthogonal group, can leave invariant a certain number $\nu$ of independent linear combinations of these forms, which we can suppose to be $\omega_{n-\nu+1}, ..., \omega_n$.[17] It can also be that it leaves invariant several forms quadratic in $\omega_1, \omega_2, ..., \omega_{n-\nu}$; let $\varphi_1, \varphi_2, ..., \varphi_h$ be these forms which we suppose linearly independent. That said, it is evident that the group $G$ will be the group of displacements for the Riemannian spaces of the fundamental form

$$\alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_h\varphi_h + \alpha_{ij}\omega_{n-\nu+i}\omega_{n-\nu+j} \qquad (i, j = 1, 2, ..., n-\nu),$$

with constant coefficients $\alpha_1, \alpha_2, ..., \alpha_h, \alpha_{ij}$ subject only to the condition that the fundamental form be positive definite. Furthermore for all these spaces the stability group of a point will be defined analytically by the same group $g$ of orthogonal substitutions.

Let us be satisfied with taking for $n = 4, r = 5$ the following example. Let

$$d\omega_1 = -[\omega_2\ \varpi], \quad d\omega_2 = [\omega_1\ \varpi], \quad d\omega_3 = -[\omega_4\ \varpi], \quad d\omega_4 = [\omega_3\ \varpi], \quad d\varpi = 0,$$

which corresponds to

$$\omega_{12} = \omega_{31} = \varpi, \qquad \omega_{13} = \omega_{14} = \omega_{23} = \omega_{24} = 0.$$

The infinitesimal transformations that the four forms $\omega_i$ undergo by an infinitesimal transformation of $g$ are

$$\delta\omega_1 = \omega_2, \qquad \delta\omega_2 = -\omega_1, \qquad \delta\omega_3 = \omega_4, \qquad \delta\omega_4 = -\omega_3;$$

there does not exist any form linear in $\omega_1, \omega_2, \omega_3, \omega_4$ which is invariant by $g$; the invariant quadratic forms are

$$\omega_1^2 + \omega_2^2, \qquad \omega_3^2 + \omega_4^2, \qquad \omega_1\omega_3 + \omega_2\omega_4, \qquad \omega_1\omega_4 - \omega_2\omega_3$$

and all their linear combinations. Therefore there exists an infinity of Riemannian spaces that admit the group $G$ as the group of displacements and their fundamental forms are

$$A(\omega_1^2 + \omega_2^2) + B(\omega_3^2 + \omega_4^2) + 2C(\omega_1\omega_3 + \omega_2\omega_4) + 2D(\omega_1\omega_4 - \omega_2\omega_3)$$

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[17] The vectors $\boldsymbol{e}_{n-\nu+1}, ..., \boldsymbol{e}_n$ and their linear combinations are at each point the vectors invariant under the stability group of this point; the number $\nu$ is thus the same as in n° 284.

where the constant coefficients $A, B, C, D$ satisfy the conditions

$$A > 0, \qquad AB - C^2 - D^2 > 0.$$

The equations of the group, by a convenient choice of variables, are

$$\begin{aligned}(x^1)' &= x^1 \cos c - x^2 \sin c + a^1,\\ (x^2)' &= x^1 \sin c + x^2 \cos c + a^2,\\ (x^3)' &= x^3 \cos c - x^4 \sin c + a^3,\\ (x^4)' &= x^3 \sin c + x^4 \cos c + a^4,\end{aligned}$$

and we could replace, in the fundamental forms indicated, $\omega_1, \omega_2, \omega_3, \omega_4$ by $dx^1, dx^2, dx^3, dx^4$. All the spaces obtained are locally Euclidean.

**289.** In the end we have solved in a certain sense Problem I with regard to Riemannian spaces that admit a transitive group. The solution of this problem has been reduced to several successive problems:

1° *To determine all orthogonal groups $g$ in $n$ variables.* — This problem was the subject of an important Memoir by S. Medici,[18] who solved it completely up to $n = 6$; we can now have the complete solution after the research of É. Cartan on the linear representations of simple groups.

2° *Given the orthogonal group $g$, to determine all the groups that admit $g$ as the stability group of a point.* The considerations of the previous paragraphs give, if not the groups themselves, at least their structure, from which the equations of these groups can be deduced by integration of ordinary differential equations.

3° *Given a group $G$, to determine all the Riemannian spaces admitting it as the group of displacements.* — This problem is also completely solved: if we confine ourselves to looking for the equations of structure of the space, they follow directly from the equations of structure of the group, which basically do not differ. If the group is effectively known, the Riemannian spaces are also deduced without integration.

**290.** Let us add finally, without proof, that the Riemannian spaces that admit a transitive group of displacements are *analytic spaces* by a convenient choice of coordinates, and that *to any given structure of the group $G$ and to any possible fundamental form constructed with the symbols $\omega_i$ corresponds a simply connected, normal, analytic Riemannian space*, with all spaces that admit the same expression of the fundamental form are locally applicable onto it. The different simply connected, normal, analytic spaces which correspond to the same group $G$, but not necessarily to the same expression of the fundamental form,

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[18] *Ann. Scuola norm. sup. Pisa*, t. 10, 1908, (160 pages).

are homeomorphic to each other.

## IX. — Three dimensional spaces that admit a multiply transitive group of displacements.

**291.** We will clarify the generalities of the preceding section by the study of three dimensional Riemannian space that admit a group of multiply transitive group of displacements.

The first problem to resolve is the search for connected orthogonal groups in three variables. The case where we take the three parameter group of all the rotations of ordinary space gives the spaces of constant curvature. Simply note in this case that the group of displacements of such a space is the same for the spaces whose constant curvature is different, but of the same sign, because at the same time as one $ds^2$ the group leaves invariant all those that we obtain by multiplying it by a positive factor. We thus have three possible groups, in fact distinct, that correspond respectively to three classes of spaces with constant positive, zero and negative curvature.

**292.** Any subgroup of the group of rotations of ordinary space has only one parameter and is the group of rotations around a fixed axis. To this subgroup thus corresponds the class of three dimensional spaces that admit a four parameter transitive group of displacements. these are the spaces that we will now determine.[19]

The rectangular frames of reference adapted to the group will be naturally all those which at a point $M$ of the space have their basis vector $\boldsymbol{e}_3$ on the axis of rigid rotations around this point. Relations (12.27) of n° 283 are then

$$\xi_{13} = \xi_{23} = 0, \tag{12.32}$$

and equations (12.28) become

$$\left.\begin{array}{rcl} \omega_{13} & = & a\omega_1 + b\omega_2 + c\omega_3, \\ \omega_{23} & = & a'\omega_1 + b'\omega_2 + c'\omega_3. \end{array}\right\} \tag{12.33}$$

If we apply the method indicated in n° 285, we see that, for an infinitesimal rotation of the group $g$, the components $\omega_1, \omega_2, \omega_3, \omega_{13}, \omega_{23}$ undergo the variations (we have suppressed the factor $\xi_{12}$)

$$\delta\omega_1 = \omega_2, \qquad \delta\omega_2 = \omega_1, \qquad \delta\omega_3 = 0,$$
$$\delta\omega_{13} = \omega_{23}, \qquad \delta\omega_{23} = -\omega_{13};$$

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[19] This problem was studied by G. Ricci, *Comptes rendus*, t. 127, 1898, p. 344-346; L. Bianchi, *Memor. Soc. ital. Scienze*, 3° série, t. 11, 1898, p. 267-352; G. Ricci, *Memor. Soc. ital. Scienze*, 3° série, t. 12, 1902, p. 69-92; C. Rimini, *Ann. Scuola norm. sup. Pisa*, t. 9, 1904, (57 pages).

we must have consequently

$$\omega_{23} = a\omega_2 - b\omega_1, \qquad -\omega_{13} = a'\omega_2 - b'\omega_1,$$

from which

$$c = c', \qquad a' = -b, \qquad b' = a,$$

and consequently

$$\left.\begin{array}{rcl}\omega_{13} & = & a\omega_1 + b\omega_2, \\ \omega_{23} & = & -b\omega_1 + a\omega_2.\end{array}\right\} \tag{12.34}$$

The fact that the coefficients $c$ and $c'$are zero expresses a general result proved geometrically in n° 284, namely that the trajectories of the vectors $\boldsymbol{e}_3$, invariant under $g$, are geodesics.

**293.** Following now the directions of n° 286, form the structure equations, by putting $\omega_{12} = \varpi$:

$$\left.\begin{array}{rcl}d\omega_1 & = & -[\omega_2\ \varpi] + a[\omega_1\ \omega_3] + b[\omega_2\ \omega_3], \\ d\omega_2 & = & [\omega_1\ \varpi] - b[\omega_1\ \omega_3] + a[\omega_2\ \omega_3], \\ d\omega_3 & = & 2b[\omega_1\ \omega_2].\end{array}\right\} \tag{12.35}$$

The exterior derivative gives

$$[\omega_2\ d\varpi] = 0, \qquad [\omega_1\ d\varpi] = 0, \qquad ab[\omega_1\ \omega_2\ \omega_3] = 0,$$

from which we deduce

$$d\varpi = c[\omega_1\ \omega_2], \tag{12.36}$$

by introducing a new constant, and the exterior derivative of (12.36) gives

$$ac[\omega_1\ \omega_2\ \omega_3] = 0.$$

Finally there remain three constants $a, b, c$, such that the two products $ab$ and $ac$ are not zero.

We must thus distinguish two cases according as $a$ is different from zero or not.

**294.** First case, $a \neq 0,\ b = c = 0$. — We thus have the equations of structure

$$\left.\begin{array}{rcl}d\omega_1 & = & -[\omega_2\ \varpi] + a[\omega_1\ \omega_3], \\ d\omega_2 & = & [\omega_1\ \varpi] + a[\omega_2\ \omega_3], \\ d\omega_3 & = & 0, \\ d\varpi & = & 0,\end{array}\right\} \tag{12.37}$$

with

$$\omega_{13} = a\omega_1, \qquad \omega_{23} = a\omega_2.$$

*We can moreover*, without changing the group, *reduce $a$ to one* by taking $a\omega^3$

for a new form $\omega_3$. The only linear combination of $\omega_1, \omega_2, \omega_3$ invariant under the subgroup $g$ is $\omega_3$, and the only quadratic form in $\omega_1, \omega_2$ invariant under $g$ is $\omega_1^2 + \omega_2^2$.

The most general fundamental form of the Riemannian spaces that admit $G$ as the group of displacements is thus

$$ds^2 = A(\omega_1^2 + \omega_2^2) + R\omega_3^2 \qquad (A > 0, B > 0).$$

To obtain an effective representation of the group $G$, note that by assuming that $a = 1$, equations (12.37) can be written as

$$\begin{aligned} &d(\omega_1 + i\omega_2) = -[(\omega_1 + i\omega_2)(\omega_3 + i\varpi)], \\ &d(\omega_3 + i\varpi) = 0. \end{aligned}$$

We have a possible solution of the problem by putting

$$\omega_3 + i\varpi = \frac{du + idv}{u + iv}, \qquad \omega_1 + i\omega_2 = \frac{dx + idy}{u + iv};$$

we deduce

$$\omega_1^2 + i\omega_2^2 = \frac{dx^2 + dy^2}{u^2 + v^2}, \qquad \omega_3 = \frac{u\,du + v\,dv}{u^2 + v^2}, \qquad \varpi = \frac{u\,dv - v\,du}{u^2 + v^2}.$$

Finally by putting

$$u = z\cos\theta, \qquad y = z\sin\theta,$$

we find

$$\omega_3 = \frac{dz}{z}, \qquad \varpi = d\theta.$$

The most general Riemannian space that admits the group $G$ as a group of displacements is thus defined by the fundamental form

$$ds^2 = \frac{A(dx^2 + dy^2) + Bdz^2}{z^2} \qquad (A > 0, B > 0). \tag{12.38}$$

As for the group itself, we obtain it easily in the form

$$\left.\begin{array}{rcl} x' &=& k(x\cos c - y\sin c) + x_0, \\ y' &=& k(x\sin c + y\cos c) + y_0, \\ z' &=& kz, \end{array}\right\} \tag{12.39}$$

with the four arbitrary constants $x_0, y_0, k > 0$, and $c$.

By interpreting $x, y, z$ as the rectangular coordinates of a point, it is the group of similitudes which leave invariant the plane $z = 0$.

The fundamental form (12.38) has negative constant curvature equal to $-1/B$; it is, by replacing $x$ and $y$ by $\sqrt{\frac{B}{A}}\,x$ and $\sqrt{\frac{B}{A}}\,y$, the anallagmatic form (6.30) of n° 150, where we have taken the $x, y$-plane as the absolute; The space is thus the non-Euclidean space of Lobachevsky, and its group of displacements is that which leaves fixed the bundle of horospheres tangent to one another at a point of the absolute (here the point at infinity), or again that which leaves invariant the bundle of non-Euclidean lines parallel to one another in the sense of Lobachevsky, which are the orthogonal trajectories of these horospheres.

We find again the property of the space of being with negative constant curvature with no need of finding and effective representation of the group. In fact if we calculate the forms $\Omega_{ij}$

$$\Omega_{ij} = d\omega_{ij} + [\omega_{ik}\ \omega_{jk}],$$

we get immediately

$$\Omega_{12} = a^2[\omega_1\ \omega_2], \qquad \Omega_{13} = a^2[\omega_1\ \omega_3], \qquad \Omega_{23} = a^2[\omega_2\ \omega_3].$$

**295. Second case.** — In the second case, which remains for us to examine, we must set $a = 0$. By taking then $\varpi - b\omega_3$ as a new form $\varpi$, we find, according to equations (12.35) and (12.36),

$$\left.\begin{aligned} d\omega_1 &= -\ [\omega_2\ \varpi], \\ d\omega_2 &= \quad [\omega_1\ \varpi], \\ d\omega_3 &= \ 2b[\omega_1\ \omega_2], \\ d\varpi &= -c[\omega_1\ \omega_2], \end{aligned}\right\} \tag{12.40}$$

where the coefficient $c$ is not the same as in formula (12.36).

We see moreover, by using the theorem of n° 287, that the form $\omega_1^2 + \omega_2^2$ is the $ds^2$ of a surface and that this surface is of constant curvature $c$. We can furthermore reduce $b$ and $c$ to fixed numerical values.

In the first place we can reduce $c$ to one of the values 1, 0 or $-1$. As regards the coefficient $b$, if it is not zero, we can also reduce it to 1, which gives six essentially distinct cases (six distinct groups).

The Riemannian curvature of the space is easily determined. We have

$$\omega_{12} = \varpi + b\omega_3, \qquad \omega_{13} = b\omega_2, \qquad \omega_{23} = -b\omega_1,$$

where

$$\begin{aligned} \Omega_{12} &= d\omega_{12} + [\omega_{13}\ \omega_{23}] = (3b^2 - c)[\omega_1\ \omega_2], \\ \Omega_{13} &= d\omega_{13} - [\omega_{12}\ \omega_{23}] = -b^2[\omega_1\ \omega_3], \\ \Omega_{23} &= d\omega_{23} + [\omega_{12}\ \omega_{13}] = -b^2[\omega_2\ \omega_3]. \end{aligned}$$

The equation of the Riemann quadric (n° 170) is then

$$b^2(X^2 + Y^2) + (c - 3b^2)Z^2 = 1.$$

If the Riemannian curvature $c$ of $ds^2$ represented by $\omega_1^2+\omega_2^2$ is less than $3b^2$, this quadric is an hyperboloid of revolution with one sheet whose asymptotic cone has its angle at the vertex greater than 120° if $c$ is negative, equal to 120° if $c$ is zero, less than 120° if $c$ is positive. If $c = 3b^2$, we have a cylinder of revolution; finally if $c > 3b^2$, we have an ellipsoid of revolution first elongated, then flattened, with the intermediated state where the quadric is a sphere corresponding to $c = 4b^2$ (a space with positive constant curvature).

These results are exact if $b$ is not zero. For $b = 0$ the Riemannian curvature is zero for all plane elements tangent to the vector $\boldsymbol{e}_3$; for all the other plane elements, it has the sign of $c$.

**296.** As for the effective realisation of the fundamental form, it is easy. We can take

$$\omega_1^2 + \omega_2^2 = A\,\frac{dx^2 + dy^2}{\left[1 + \dfrac{K}{4}(x^2 + y^2)\right]^2} \qquad (A > 0, K = 1, 0 \text{ or } -1),$$

$$\omega_3 = dz + B\,\frac{x\,dy - y\,dx}{\left[1 + \dfrac{K}{4}(x^2 + y^2)\right]^2} \qquad (K = 1, 0 \text{ or } -1).$$

The constants $b$ and $c$ of formulae (12.40) are related to the constants $A, B$ and $K$ by the relations

$$b = \frac{B}{A}, \qquad c = \frac{K}{A},$$

and we have the fundamental form

$$ds^2 = \frac{A(dx^2 + dy^2) + \left\{\left[1 + \dfrac{K}{4}(x^2 + y^2)\right] dz + B(x\,dy - y\,dx)\right\}^2}{\left[1 + \dfrac{K}{4}(x^2 + y^2)\right]^2} \tag{12.41}$$

$$(K = 1, 0 \text{ or } -1);$$

the equation of the Riemann quadric becomes

$$B^2(X^2 + Y^2) + (KA - 3B^2)Z^2 = A^2. \tag{12.42}$$

The space is with constant curvature in two cases, namely
$1^{\text{o}}$ $K = 0, B = 0$ (Euclidean space); $2^{\text{o}}$ $K = 1, A = 4B^2$ (spherical space).

**297.** It is interesting to study the form of the simply connected normal space in the different cases which can present themselves.

Suppose first $B = 0$, which means that geodesics tangent at each of their points to the axis of rigid rotations around this point form a congruence of normals to the surfaces $z =$ constant. In this case the space is the topological product of the straight line and of a Riemannian space of two dimensions with constant curvature, in other words these spaces are homeomorphic to the topological product of the Euclidean plane and the straight line, that is to say to the Euclidean space, if $K = -1$ or 0; they are homeomorphic to the topological product of the sphere with the straight line if $K = 1$.

If $B$ is not zero and $K = 0$ or $-1$, the result is the same. It is no longer so if $K = 1$; we will see that, in this case, the space is homeomorphic to the spherical space. In fact since the coefficient $b$ of equations (12.40) is not zero and the form $\varpi + \frac{c}{2b}\omega_3$ is an exact differential $d\rho$, we can take advantage of it by extracting from the family of frames of reference adapted to the group a restricted family such that there is only one frame of reference attached to a point of the space: it is sufficient to force the coordinates of a point and the parameter $v$ on which depend the frames of reference attached to this point to satisfy the relation $\rho = 0$; this is possible because $d\rho$ not being a linear combination of $\omega_1, \omega_2, \omega_3$, the function $\rho$ depends effectively on the parameter $v$. That said equations (12.40) will become

$$d\omega_1 = \frac{c}{2b}[\omega_2\ \omega_3], \qquad d\omega_2 = \frac{c}{2b}[\omega_3\ \omega_1], \qquad d\omega_3 = 2b[\omega_1\ \omega_2].$$

So putting

$$\omega_1 = \frac{2}{\sqrt{c}}\ \varpi_1, \qquad \omega_2 = \frac{2}{\sqrt{c}}\ \varpi_2, \qquad \omega_1 = \frac{4b}{c}\ \varpi_3,$$

or again, which comes to the same thing (do not forget that $K = 1$),

$$\omega_1 = 2\sqrt{A}\ \varpi_1, \qquad \omega_2 = 2\sqrt{A}\ \varpi_2, \qquad \omega_1 = 4b\ \varpi_3,$$

we get the formulae

$$d\varpi_1 = 2[\varpi_2\ \varpi_3], \qquad d\varpi_2 = 2[\varpi_3\ \varpi_1], \qquad d\varpi_3 = 2[\varpi_1\ \varpi_2].$$

These are formulae (12.22) of n° 281 which here define the spherical space of curvature 1. *The Riemannian spaces (12.41) for which $K = 1, B \neq 0$, are thus homeomorphic to the spherical space.* By referring to n° 281, we see that we can substitute for the form (12.41) for $ds^2$ of the spaces considered the form

$$\begin{aligned} ds^2 &= 4A(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) \\ &\quad + (16B^2 - 4A)(x_0\,dx_3 - x_3\,dx_0 + x_1\,dx_2 - x_2\,dx_1)^2, \end{aligned} \tag{12.43}$$

where the coordinates $x_i$ are related by the relation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

**Theorem.** — *The simply connected normal Riemannian spaces of three dimensions that admit a four parameter group of displacements are homeomorphic to Euclidean space, or to the topological product of a sphere and a straight line, or to the spherical space. All of these spaces whose Riemannian curvature is everywhere positive are homeomorphic to the spherical space, but the converse is not true.*

**298.** All the spaces obtained in the two cases that can arise admit a transitive group of displacements of four parameters, but this group is not always the largest group of displacements of the space, since some of the spaces are of constant curvature. So we have basically solved the problem of determining which are, in a space of constant curvature, the groups of displacements of four parameters. The first case (n$^o$ 294) gave us the spaces of negative constant curvature, where the four parameter group is that which leaves invariant a bundle of parallel straight lines in the sense of Lobachevsky. The second case gave us the Euclidean space and the spaces of positive constant curvature. For the Euclidean space, the four parameter group is that which leaves invariant a bundle of parallel straight lines. We leave to the reader the task of showing that, in the elliptic space, this is the group that leaves invariant a bundle of Clifford parallels (n$^o$ 281), since this follows very easily from the form (12.43) of the $ds^2$ that corresponds to spaces homeomorphic to the spherical space. *We can give a common geometric definition to four parameter groups of displacements of spaces with positive, zero or negative curvature*, provided that we give the expression “straight parallels” the sense of Lobachevsky, the common sense or the sense of Clifford according as the curvature is negative, zero or positive.

## X. — General intransitive groups of displacements.

**299.** We propose to show in this Section that the search for groups of transformations capable of being interpreted as intransitive groups of displacements in a Riemannian space comes down to that of transitive groups with which we dealt in the preceding sections. For this it is sufficient to show that if, in a Riemannian space of $N$ dimensions, the trajectories of an intransitive group are $n$-dimensional, we can always choose a system of coordinates so that the first $n$ coordinates are transformed among themselves by the group, while the $h = N-n$ others are invariant under this group.

Denote by $u^1, u^2, ..., u^h$ the parameters that individualise the trajectories. Start from a particular trajectory $\Sigma_O$ and from a particular point $O$ of this trajectory. Consider the geodesic manifold at $O$ formed from the geodesics normal to $\Sigma_O$ issuing from $O$; this is a manifold $V_h$ of $h$ dimensions. Take $h$ vectors tangent at $O$ to this manifold such that the coordinates of a point $M$ infinitely close to $O$ in $V_h$, considered with respect to these $h$ vectors taken as basis vectors of a Cartesian frame of reference tangent to $V_h$, are $du^1, du^2, ..., dh^h$. Under any displacement of the space that leaves fixed the point $O$, the components $du^1, du^2, ..., dh^h$ remain fixed; consequently the $h$ basis vectors considered remain all fixed, and as they are linearly independent, we arrive at

**Theorem.** — *Any vector at $O$ and normal to the trajectory $\Sigma_O$ is invariant under any displacement of the group $G$ that leaves fixed the point $O$.*

**300.** Any displacement that takes the point $O$ of $\Sigma_O$ into a point $A$ of $\Sigma_O$ will take the manifold $V_h$ through $O$ into another manifold $V_h$ through $A$ and which will be the locus of geodesics normal to $\Sigma_0$ and passing through $A$. All these manifolds $V_h$ fill all the space, at least in a sufficiently small neighbourhood of

$\Sigma_O$, and allows an identification of the different points of the space. For this it is sufficient to introduce in $\Sigma_O$ a system of coordinates $x^1, x^2, ..., x^n$ and to assign by convention to points of the manifold $V_h$ through the point $A$ of $\Sigma_O$, first the coordinates $x^1, x^2, ..., x^n$ and then the coordinates $u^1, u^2, ..., u^h$ of the trajectory on which the point considered is situated.

This said, the coordinates $x^1, x^2, ..., x^n$ of any point of the space are transformed to themselves by the group $G$ as it operates on the points of $\Sigma_O$.

**Theorem.** — *Given a Riemannian space of $n+h$ dimensions transformed by a group of displacements whose trajectories have n dimensions, we can always choose a system of coordinates $x^1, x^2, ..., x^n, u^1, u^2, ..., u^h$ such that the first n coordinates are transformed transitively by themselves, while the last h coordinates are invariants.*[20]

**301.** It follows from the preceding theorem that the search for groups capable of being regarded as groups of displacements of a Riemannian space comes down to the search of groups capable of being regarded as transitive groups of displacements. To the equations of such a group, assumed of $n$ variables, it is sufficient to add equations in any number $h$, that express that $h$ new variables are invariant under the group.

The first part of Problem I is thus solved if it is solved for the transitive groups of displacements.

The search for Riemannian spaces that admit a group of displacements capable of being represented analytically by a given group is also virtually solved. In fact let $G$ be a transitive group of displacements of a space of $n$ dimensions, and $g$ the corresponding group of stability. The group $g$, which acts on the forms $\omega_1, \omega_2, ..., \omega_n$ as orthogonal group (n° 288), leaves invariant $\nu$ independent linear combinations with constant coefficients of the $\omega_i$, which we can suppose are $\omega_{n-\nu+1}, ..., \omega_n$, and moreover $\ell$ independent quadratic forms $\varphi_1, \varphi_2, ..., \varphi_\ell$ in $\omega_1, \omega_2, ..., \omega_{n-\nu}$ with constant coefficients. That said, *the most general expression of the fundamental form invariant under the group $G$ is*

$$ds^2 = A_1(u)\varphi_1(\omega) + \cdots + A_\ell(u)\varphi_\ell(\omega) + \Psi(\omega_{n-\nu+1}, ..., \omega_n, du^1, du^2, ..., du^h),$$

*where $\Psi$ is a quadratic form whose coefficients depend only on $u^1, u^2, ..., u^h$.*

Naturally, it is necessary to suppose that $A_1, A_2, ..., A_\ell$ and the coefficients

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[20] *Cf.* an important memoire of G. Fubini (*Annali di Mat.*, 3° série, t. **8**, 1903, p. 39-81), where the referencing of the points of the space is similar, but different.

In the theory of the structure of groups, the property in question of groups of displacements of Riemannian spaces is expressed by saying that these groups do not have *essential invariants.*

As an example of a group admitting essential invariants, we cite the group in two variables $x, y$ and two parameters $a, b$, defined by the equations

$$x' = x + ay + b, \qquad y' = y.$$

of the quadratic form $\Psi$ are functions of $u^1, u^2, ..., u^h$ *such that* $ds^2$ *is positive definite.*

**302.** The manifolds $V_h(x)$, loci of the points whose $n$ first coordinates $x$ are fixed, are geodesic manifolds at the point where they cut orthogonally the trajectory $\Sigma_O$. But they are not in general totally geodesic, nor do they intersect orthogonally the other trajectories $\Sigma$.

If we consider on the contrary the manifold $V_{\nu+h}$ generated by the geodesics issuing from the point $O$ of $\Sigma_O$ whose tangent vectors at $O$ are invariant under the *group of stability* of $O$, that is to say by the subgroup $g$ of rigid rotations around $O$, this manifold, according to the reasoning presented in n° 284, is totally geodesic; it will thus be at each of its points generated by the geodesics that pass through this point and invariant under the group of rigid rotations around this point, which is the same as the group $g$; it will thus contain all the geodesics normal at this point to the trajectory $\Sigma$ that contains it; the manifold $V_{\nu+h}$ will thus cut orthogonally the trajectory $\Sigma$.

**Theorem.** — *If through a point of the space we construct the manifold* $V_{\nu+h}$ *generated by the geodesics invariant under the group of rotations around this point, this manifold is totally geodesic and cuts orthogonally all the trajectories of the group of displacements.*

The theorem of n° 284 is a particular case of the preceding theorem.

**303.** In the case where the trajectories are transformed individually by a simply transitive group, we have $\nu = n$ and the manifold $V_{\nu+h}$ merges with the entire space; the manifolds $V_h$ are not in general totally geodesic.

If on the contrary $\nu = 0$, that is to say if the group $g$ of rigid rotations around a point does not leave invariant any vector tangent at this point to the trajectory which contains it, $V_{\nu+h}$ merges with $V_h$ and we have the

**Theorem.** — *If the group of stability of a particular point of the space does not leave invariant any direction tangent at this point to the trajectory* $\Sigma$ *which contains it, the manifolds* $V_h$ *generated by the geodesics normal at this point to a particular trajectory are totally geodesic and cut orthogonally all the trajectories.*

## XI. — Groups of displacements whose trajectories are lines or surfaces.

**304.** If the trajectories of a group of displacements are one-dimensional, we know (n° 235) that the group has one parameter. Here $n = 1$ with one only form $\omega_1$, necessarily an exact differential, and that we can suppose to be $dx^1$. The most general form of $ds^2$ admitting a one-parameter group of displacements is

$$ds^2 = g_{ij}dx^i dx^j,$$

where the coefficients $g_{ij}$ are independent of $x^1$.

**305.** If the trajectories of the group are two-dimensional, the group has 2 or 3 parameters.

If the group $G$ has two parameters, it is simply transitive and can be defined as the set of transformations that leave invariant the two forms $\omega^1, \omega^2$ constructed with two variables $x^1$ and $x^2$ and their differentials and satisfying two structure relations

$$d\omega^1 = A[\omega^1\ \omega^2], \qquad d\omega^2 = B[\omega^1\ \omega^2].$$

Two cases are possible:

1° $A = B = 0$. — In this case $\omega^1$ and $\omega^2$ are exact differentials, that we can suppose to be $dx^1$ and $dx^2$, the group being

$$(x^1)' = x^1 + a, \qquad (x^2)' = x^2 + b;$$

it is Abelian.

*The most general corresponding fundamental form is*

$$ds^2 = g_{ij}dx^i dx^j,$$

*where the coefficients* $g_{ij}$ *are functions of* $h = n-2$ *coordinates* $x^3, x^4, ..., x^n$. The trajectories are surfaces of zero Riemannian curvature.

2° $A \neq 0$. — By taking $A\omega^2 - B\omega^1$ as a new form $\omega^2$, where the first form is kept, we come back to the case $A = 1, B = 0$. We can take

$$\omega^1 = \frac{dx^1}{x^2}, \qquad \omega^2 = \frac{dx^2}{x^2},$$

with the group

$$(x^1)' = ax^1 + b, \qquad (x^2)' = ax^2.$$

*The corresponding most general fundamental form is a quadratic form in* $\frac{dx^1}{x^2}, \frac{dx^2}{x^2}, dx^3, dx^4, ..., dx^n$ *with independent coefficients in* $x^1, x^2$. The trajectories are surfaces with negative constant Riemannian curvature.[21] The group is not abelian.

**306.** Suppose now the trajectories are two dimensional, but the group has three parameters. The trajectories are then surfaces with constant Riemannian curvature. The group is the same as that of a surface with constant curvature

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[21] This curvature can be variable from one trajectory to another.

equal to 1, $-1$ or 0 (n° 291). That said, *the spaces we seek will have their* $ds^2$ *of the form*

$$ds^2 = A(u)\,d\sigma^2 + g_{ij}du^i du^j \qquad (i,j = 3,4,...,n), \tag{12.44}$$

*where* $d\sigma^2$ *denotes the fundamental form of a surface with constant curvature 1, 0 or* $-1$*, and the* $g_{ij}$ *dpend only on* $u^3, u^4, ..., u^n$

The manifolds $V_h$ are totally geodesic and cut orthogonally the trajectories of the group, as this moreover follows immediately from formula (12.44).

**307.** ***Note I.*** — The results obtained in this Section give us all the Riemannian spaces of three dimensions that admit an intransitive group of displacements; the one which has three parameters corresponds to the fundamental form (12.44), which can be written here as

$$du^2 + A(u)d\sigma^2. \tag{12.44$'$}$$

***Note II.*** — All the Riemannian spaces that admit as a transitive group of displacements the group

$$x' = ax + b, \qquad y' = ay \tag{12.45}$$

admit a larger group, since these spaces are of constant curvature. But, if we complete group (12.45) by the addition of $h$ invariant variables, the Riemannian spaces admitting the group thus obtained as group of displacements, necessarily intransitive, do not in general admit a larger group. Suppose for example $h = 1$ and

$$ds^2 = A(z)\,\frac{dx^2+dy^2}{y^2} + 2B(z)\,\frac{dx\,dz}{y} + 2C(z)\,\frac{dy\,dz}{y} + D(z)\,dz^2\,;$$

any group of displacements of such a space leaves invariant the form $\frac{dx^2+dy^2}{y^2}$, but also, if the functions $B(z)$ and $C(z)$ are not both zero, one at least of the forms $\frac{dx}{y}$, $\frac{dy}{y}$, and consequently also the other; the group can thus not be of more than two parameters.

If on the contrary a multiply transitive group of displacements $G$ is such that any Riemannian space $E$ that admits this group admits a larger group of displacements $G'$, and *if the subgroup* $g$ *of* $G$ *of rotations around a point of the space does not leave invariant any vector at this point* ($\nu = 0$), any space $E'$ which admits $G$ as an intransitive group of displacements will admit also the larger group of displacements $G'$; this is because the fundamental form of this space will be, invariant under $G$ up to a factor, the sum of the fundamental form of $E$ and of a form constructed with the parameters $u^i$ of the trajectories and their differentials.

# 13 Applicable Riemannian spaces. Rigid Displacements of a Given Space.

## I. — Applicable Riemannian spaces.

**308.** Recall that two Riemannian spaces with the same number of dimensions are said to be *isometric* or *applicable* if there exists between these two spaces a point correspondence that conserves the fundamental form, that is to say such that the distance of two infinitely close points of one of the spaces is equal to the distance of two infinitely close corresponding points of the other space. Such a correspondence conserves also the length of arcs of a curve, the Riemannian curvature at a given point in a given plane direction, etc.

We have already shown (n° 220) that two spaces of the same constant Riemannian curvature are applicable;[1] we have thus found the conditions of applicability of two symmetric spaces: if we consider at a point, indeed any, of the first space the Riemann form

$$R_{ijkh}p^{ij}p^{kh},$$

and at a point, also any, of the second space the similar Riemann form

$$\overline{R}_{ijkh}\overline{p}^{\,ij}\overline{p}^{\,kh},$$

it is necessary and sufficient that we can pass from the first of these scalar tensors to the second by a suitable change of Cartesian system of reference.

The importance of the Riemannian curvature in the conditions of applicability also comes out of the necessary and sufficient conditions set out in Chapter X (n° 218), but the application of these conditions requires that we have beforehand determined the geodesics of the two spaces. Let theorem of n° 219 gives another statement, valid only in the case of analytic Riemannian spaces; it involves only one point of the space, but it requires knowledge at this point of the components $R_{ijkh}$ *and of all their successive covariant derivatives.*

**309.** Before dealing, in all generality, with the problem of recognising if two spaces with the same number of dimensions are applicable, we will first deal with a particular case, which is moreover quite general.

Consider a Riemannian space of $n$ dimensions such that, at least in a domain

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[1] The concern here, as in this whole Chapter, is *local* applications.

$\mathcal{D}$ of this space, the Ricci quadric with equation

$$R_{ij}X^iX^j = 1 \tag{13.1}$$

has its axes distinct in direction and that, furthermore, the magnitudes of the axes (or better the inverses of the squares of these axes) are independent functions of the coordinates $u^1, u^2, ..., u^n$. Any other space applicable onto this one will possess at least a domain $\overline{\mathcal{D}}$ in the interior of which the same geometric properties will necessarily be realised. We will look for the conditions of applicability of two such spaces.

Introduce in each of the two spaces a family of rectangular frames of reference that have at each point as basis vectors unit vectors in the principal directions of the space. At a particular point $A$ there will be $2^n$ ways of choosing the frame of reference depending on how to number the axes and of choosing the direction of these axes. The choice once made at a particular point will determine by continuity the choice to make at all the other points of the domain (if however this domain is simply connected, which we will assume).

With respect to these frames of reference, the components $R_{ij}$ of the Ricci tensor are all zero for $i \neq j$; the $n$ others $R_{11}, R_{22}, ..., R_{nn}$ are distinct by hypothesis and independent functions of the coordinates.

If the second space is applicable onto the first, there will exist at each point of $\overline{\mathcal{D}}$ a rectangular frame of reference of the type that has been indicated and such that the application that brings a point $\overline{M}$ of the second space onto the corresponding point $M$ of the first brings the frame of reference $(R_{\overline{M}})$ onto the frame of reference $(R_M)$ of the first. There will exist in the second space at each point functions $\overline{R}_{11}, \overline{R}_{22}, ..., \overline{R}_{nn}$ which will necessarily be distinct and independent functions of the coordinates and we will have by the application, assumed to exist, the relations

$$\overline{R}_{ii} = R_{ii} \qquad \text{(do not sum).} \tag{13.2}$$

This shows already that the application, if it exists, is defined by equations (13.2) which allows the coordinates $\overline{u}^{\,i}$ of the second space to be solved as functions of the coordinates $u^i$ of the first: there can only be *isolated* solutions to the problem.

But we can find in the following way *necessary and sufficient* conditions for the effective existence of an application.

**310.** Equalities (13.2) which realise the application lead to

$$d\overline{R}_{ii} = dR_{ii}. \tag{13.3}$$

Put

$$dR_{ii} = R_{ii|k}\omega^k, \qquad d\overline{R}_{ii} = \overline{R}_{ii|k}\overline{\omega}^{\,k};$$

$R_{ii|k}$ is the covariant derivative of the component $R_{ii}$ of the Ricci tensor in the $k^{\text{th}}$ principal direction.[2]

Since the application conserves the forms $\omega^i$, as well as each of the differentials $dR_{ii}$, we get the equations

$$\overline{R}_{ii|k} = dR_{ii|k} \qquad \text{(do not sum).} \tag{13.4}$$

*It is thus necessary for the application that the equations (13.2) lead to equations (13.4). These necessary conditions are also sufficient.* In fact equations (13.2) lead by hypothesis to equations (13.4), and, automatically, equations (13.3) lead to the relations

$$R_{ii|k}(\varpi^k - \omega^k) = 0 \qquad i = 1, 2, ..., n).$$

Now the determinant of these $n$ equations linear in $\varpi^1 - \omega^1, ..., \varpi^n - \omega^n$ is different from zero, without which the $n$ functions $R_{11}, R_{22}, ..., R_{nn}$ would not be independent. We thus see that relations (13.2) lead to

$$\varpi^i = \omega^i,$$

hence the equality of the two fundamental forms.

**311.** Application of the preceding criterion requires algebraic operations that we can *in part* avoid by substituting for the $n$ functions $R_{11}, R_{22}, ..., R_{nn}$ the following scalar tensors, rationally calculable with any system of Cartesian frames of reference,

$$A_1 = R_i{}^i, A_2 = R_i{}^j R_j{}^i, A_3 = R_i{}^j R_j{}^k R_k{}^i, ..., A_n = R_{i_1}{}^{i_2} R_{i_2}{}^{i_3} R_k{}^i ... R_{i_n}{}^{i_1}; \tag{13.5}$$

these tensors are none other than the sums of similar powers of the $n$ functions $R_{ii}$.[3] We can state the theorem in the following form:

**Theorem.** — *For two Riemannian spaces of $n$ dimensions for which the $n$ scalar tensors $A_1, A_2, ..., A_n$ are independent functions of the coordinates to be applicable, it is necessary and sufficient that the point correspondence which realises the equality one by one of these tensors in the two given spaces realise at the same time, by a suitable correspondence between the oriented principal directions of the two spaces, the equality one by one of the $n^2$ covariant derivatives of the $n$ tensors in the positive principal directions of the two spaces.*

We will not stop at the problem of algebra which still remains to be solved for the application of this theorem.

[2] In fact, we have for example

$$DR_{11} = dR_{11} + 2R_{1i}\omega_{1i};$$

now the only component $R_{1i}$which is not zero is $R_{11}$ and the corresponding factor $\omega_{1i} = \omega_{11}$ is zero. Actually $R_{11}$ is a scalar tensor (irrational).

[3] To appreciate this it is sufficient to see what these tensors become when we use rectangular frames of reference that have as axes the principal directions of the space.

Remember above all the remarkable conclusion that *knowing the Ricci tensor and its derived tensor*[4] *is sufficient to recognise the applicability of the two spaces, when the tensors (13.5) of these spaces are independent functions of the coordinates.* But it should not be concluded that, in the more general case, consideration of the Ricci tensor and of its derived tensors is enough to recognise the applicability of the two Riemannian spaces. If not two spaces of the same constant curvature of the second kind, and of the same number of dimensions, would be applicable one on the other, which is not so.

## II. — A problem in analysis.

**312.** To solve the problem of the application of Riemann spaces in all its generality, we will have to solve first the following problem of Analysis:

Problem. — *Given two systems of n independent linear differential forms, the first consisting of the forms*

$$\omega^1(u;du), \quad \omega^2(u;du), \quad ..., \omega^n(u;du)$$

*constructed from the n variables* $u^1, u^2, ..., u^n$ *and their differentials, the second consisting of the forms*

$$\varpi^1(v;dv), \quad \varpi^2(v;dv), \quad ..., \varpi^n(v;dv)$$

*constructed from the n variables* $v^1, v^2, ..., v^n$ *and their differentials, recognise whether it is possible to express the variables* $v^i$ *as functions of the variables* $u^i$ *so as to realise the equality one to the other of the forms of the two systems, and to determine these functions.*

**313.** Form the exterior derivatives $d\omega^i$ and $d\varpi^i$ of the given forms and express them, the first as exterior quadratic forms of the $n$ forms $\omega^1, \omega^2, ..., \omega^n$, assumed by hypothesis to be linearly independent, the latter as exterior quadratic forms of the forms $\varpi^1, \varpi^2, ..., \varpi^n$:

$$\left.\begin{aligned} d\omega^i &= \frac{1}{2}\, c_{kh}{}^i(u)[\,\omega^k\ \omega^h\,] \qquad && (c_{kh}{}^i = -c_{hk}{}^i), \\ d\varpi^i &= \frac{1}{2}\, \gamma_{kh}{}^i(u)[\,\varpi^k\ \varpi^h\,] \qquad && (\gamma_{kh}{}^i = -\gamma_{hk}{}^i). \end{aligned}\right\} \tag{13.6}$$

Any choice, for the $v^i$, of functions of $u^1, u^2, ..., u^n$, which realises the equalities $\varpi^i = \omega^i$, will realise the equalities $d\varpi^i = d\omega^i$ and consequently the equalities

$$\gamma_{kh}{}^i = c_{kh}{}^i. \tag{13.7}$$

A simple case, that we have already examined (n° 262), is that where the $c_{kh}{}^i$

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[4]

are constants: it is necessary, for the problem to be possible, that the ${\gamma_{kh}}^i$ are also constants and that there is equality to each other of the constants of the two series. We know then that there is an infinity of solutions that depend on $n$ arbitrary constants.

Apart from this simple case, take first the extreme case where $n$ of the functions ${c_{kh}}^i(u)$ are independent; it will be necessary that the functions ${\gamma_{kh}}^i(v)$ with the same indices also be independent and furthermore that the $\frac{n(n-1)}{2}$ equations (13.7) be compatible. The solution to the problem, if it exists, is then unique, or at least the problem admits only isolated solutions. But, by the reasoning already used in n° 310, we see, by differentiating equations (13.7) and putting

$$\begin{aligned} d{c_{kh}}^i &= {c_{kh}}^i{}_{|\ell}\, \omega^\ell, \\ d{\gamma_{kh}}^i &= {\gamma_{kh}}^i{}_{|\ell}\, \omega^\ell, \end{aligned}$$

that equations (13.7) must still have as a consequence the equations

$${\gamma_{kh}}^i{}_{|\ell} = {c_{kh}}^i{}_{|\ell}. \tag{13.8}$$

Conversely suppose that equations (13.7) are compatible with each other and lead to equations (13.8). They lead consequently to the relations

$${c_{kh}}^i{}_{|\ell}(u)(\varpi^\ell - \omega^\ell) = 0. \tag{13.9}$$

Now the rank of the matrix with $n$ columns whose entries in the $\ell^{\text{th}}$ column are the quantities ${c_{kh}}^i{}_{|\ell}(u)$ is equal to $n$ because the functions ${c_{kh}}^i(u)$ are independent of the variables $u^h$; relations (13.9) thus lead to

$$\varpi^i(v; dv) = \omega^i(u; du),$$

and any solution of equations (13.7) provide a solution of the given problem. These solutions are *isolated*.

**314.** We go now to the general case, and suppose that among the functions ${c_{kh}}^i(u)$, there are only $n_1 < n$ independent ones.

Suppose that, among the functions ${c_{kh}}^i{}_{|\ell}$ resulting from the differentiation of the ${c_{kh}}^i$, there are $n_2$ independent of each other and independent of the ${c_{kh}}^i$, which among the functions ${c_{kh}}^i{}_{|\ell m}$ resulting from the differentiation of the ${c_{kh}}^i{}_{|\ell}$, there rae $n_3$ independent of each other and of the preceding ones, and so on. There will come a moment where, after a certain number $p$ of differentiations, we will arrive at a set of functions such that a new differentiation does not yield functions independent of the ones preceding.

Two cases are possible:

**1°**. *The functions thus obtained are $n$ independent ones in number.* — It will be necessary and sufficient, for the problem to be possible, that the equations

$$\left.\begin{aligned} {\gamma_{kh}}^{i}(v) &= {c_{kh}}^{i}(u), \\ {\gamma_{kh}}^{i}{}_{|\ell}(v) &= {c_{kh}}^{i}{}_{|\ell}(u), \\ {\gamma_{kh}}^{i}{}_{|\ell m}(v) &= {c_{kh}}^{i}{}_{|\ell m}(u), \\ &\vdots \\ {\gamma_{kh}}^{i}{}_{|\ell_1\ell_2\ldots\ell_{p+1}}(v) &= {c_{kh}}^{i}{}_{|\ell_1\ell_2\ldots\ell_{p+1}}(u) \end{aligned}\right\} \tag{13.10}$$

are compatible, and that will be sufficient, because equations (13.10) differentiated yield $n$ linearly independent equations in $\varpi^1 - \omega^1, \varpi^2 - \omega^2, ...$, and consequently lead to $\varpi^i(v, dv) = \omega^i(u, du)$.

Since among the left hand sides of equations (13.10), there are $n$ which are independent functions of $v^h$, the problem admits only isolated solutions.

**2°**. *The functions thus obtained are $\nu < n$ independent ones in number.* — It will again be necessary that the equations (13.10) are compatible. Assuming this condition is fulfilled, differentiation of equations (13.10) other than those in the last line will give $\nu$ independent linear relations between $\varpi^1 - \omega^1, \varpi^2 - \omega^2, ..., \varpi^n - \omega^n$. There is no loss in generality if we assume that these equations can be solved with respect to the $\nu$ last forms $\varpi^{n-\nu+1} - \omega^{n-\nu+1}, ..., \varpi^n - \omega^n$.

To the compatibility of equations (10) it will therefore be sufficient to add the compatibility of the equations

$$\varpi^i(v; dv) = \omega^i(u; du) \qquad (i = 1, 2, ..., n - \nu) \tag{13.11}$$

among themselves and with equations (13.10).

Now, taking into account (13.10), the $n - \boldsymbol{u}$ equations (13.11) form a completely integrable system, because the exterior derivative of (13.11) give the relations

$${\gamma_{kh}}^{i}(v)[\,\varpi^k(v; dv)\ \varpi^h(v; dv)\,] = {c_{kh}}^{i}(v)[\,\omega^k(u; du)\ \omega^h(u; du)\,], \tag{13.12}$$

which are a consequence of relation (13.11) if we take into account equations (13.10) and of the fact that the differences $\varpi^{n-\nu+1} - \omega^{n-\nu+1}, ..., \varpi^n - \omega^n$ become zero by taking into account (13.11).

Ultimately *the compatibility of equations (13.10) thus gives the necessary and sufficient conditions for the problem to be possible, and the general solution of the problem depends on $n - \nu$ arbitrary constants.*

We see that we will only ever need to consider the derivatives of the functions ${c_{kh}}^{i}$ of order at most equal to $n$.

**315.** In applications, we will need to consider the case where the coefficients of the forms $\varpi^i$ are the same functions of the $v^k$ as the coefficients of the same indices of the forms $\omega^i$ are of the $u^k$. In this case, equations (13.10) are always compatible, since the admit the solutions $v^i = u^i$; the problem will only have a solution infinitely close to this one if the integer $\nu$ is smaller than $n$ and the general solution will then depend on $n-\nu$ arbitrary constants. This amounts to saying that the forms $\omega^i(u;du)$ are invariant under an infinity of transformations that depend on $n-\nu$ arbitrary constants; these transformations obviously generate a group. This remark will play the fundamental role in the search for the largest group of displacements of a given Riemannian space (Section IV).

## III. — The general problem of application of Riemannian spaces.

**316.** We will reduce this problem to the problem in Analysis considered in the previous Section. For that, given two Riemannian spaces of the same dimension, attach to each point of one all the rectangular systems of reference marks having this point as origin. For the first space, these systems of reference depend on $\dfrac{n(n+1)}{2}$ parameters, of which $n$ are the coordinates $u^i$ of the origin of the reference system and the $\dfrac{n(n-1)}{2}$ others are the parameters $\xi^1, \xi^2, ..., \xi^{n(n-1)/2}$ which fix the orientation of the system of reference. For the second space, these systems of reference will depend similarly on the coordinates $v^i$ of the origin and on the parameters $\eta^k$ analogous to the $\xi^k$ which fix the orientation of the system of reference.

The forms $\omega_i$ and $\omega_{ij}$ which define the elementary displacement takes us from a system of reference to an infinitely close system of reference of the first space, are constructed with the coordinates $u^i$ and the parameters $\xi^h$ as well as their differentials. But the $n$ forms $\omega_1, \omega_2, ..., \omega_n$ do not contain the $d\xi^h$, because they are zero when the origin of the system of reference remains fixed, that is, when the differentials $du^i$ are zero. On the contrary, the forms $\omega_{ij}$ can contain the differentials $d\xi^h$: they are even linearly independent with respect to these $\dfrac{n(n-1)}{2}$ differentials, because if we set to zero all the forms $\omega_i$ and $\omega_{ij}$, this means that the system of reference remains fixed and so that all the differentials $du^i$ and $d\xi^h$ are zero. Similar remarks naturally are valid for the corresponding forms $\varpi_i$ and $\varpi_{ij}$ of the second space.

**317.** That said, it is clear that if the two spaces are applicable, any application of one onto the other will map a system of reference of the first to a particular system of reference of the second, and the application will map forms $\omega_i$ to forms $\varpi_i$ and from $\omega_{ij}$ to forms to $\varpi_{ij}$. The problem therefore reduces to the following:

Problem. — Express the functions $v^i$ and $\eta^k$ as a function of the $u^i$ and $\xi^k$ in such a way as to satisfy the relations

$$\left.\begin{array}{lcl}\varpi_i\ (v,\eta;dv) & = & \omega_i\ (u,\xi;du),\\ \varpi_{ij}(v,\eta;dv,d\eta) & = & \omega_{ij}(u,\xi;du,d\xi).\end{array}\right\} \qquad (13.13)$$

Conversely any solution of these equations will give for the coordinates $v^1, v^2$, ..., $v^n$ of points of the second space functions of the coordinates $u^1, u^2, ..., u^n$ of points of the first space since, according to the first equations (13.13), the differentials $dv^i$ depend linearly only on the differentials $du^i$. And the point correspondence thus established between the two spaces will be an application since it will produce equality of the fundamental forms

$$(\varpi_1)^2 + (\varpi_2)^2 + \cdots + (\varpi_n)^2 \quad \text{and} \quad (\omega_1)^2 + (\omega_2)^2 + \cdots + (\omega_n)^2,$$

forms that do not depend on the rest of the variables $\eta^k$ and $\xi^k$.

**318.** We see that we are forced back to the problem in Analysis of the previous Section, the number $N$ of the differential forms here being $n + \dfrac{n(n-1)}{2} = \dfrac{n(n+1}{2}$. To apply the method that we have indicated, it is first necessary to calculate the exterior derivatives of the forms $\omega^i$ and $\omega_{ij}$. We know them. We have

$$\left.\begin{aligned} d\omega_i &= [\,\omega_k\ \omega_{ki}\,], \\ d\omega_{ij} &= [\,\omega_{ik}\ \omega_{ki}\,] + \tfrac{1}{2} R_{ijkh} [\,\omega^k\ \omega^h\,]. \end{aligned}\right\} \qquad (13.14)$$

We see here that *the first functions which arise and which play the role of the* ${c_{kh}}^i$ *of n° 313 are the components of the Riemannian curvature tensor.* Note, moreover, that these components do not depend only on the coordinates $u^i$, but also on the parameters $\xi^k$ of the systems of reference with respect to which they are calculated.

Consequently, we will have to calculate the differentials of these components $R_{ijkh}$ expressed linearly with respect to the forms $\omega_i$ and $\omega_{ij}$. We have obviously

$$dR_{ijkh} = R_{rjkh}\omega_{ir} + R_{irkh}\omega_{jr} + R_{ijrh}\omega_{kr} + R_{ijkr}\omega_{hr} + R_{rjkh|\ell}\omega_\ell, \qquad (13.15)$$

where the quantities $R_{ijkh|\ell}$ are the components of the derived tensor of Riemannian curvature; consequently we will have

$$\begin{aligned} dR_{ijkh|\ell} &= R_{rjkh|\ell}\omega_{ir} + R_{irkh|\ell}\omega_{jr} + R_{ijrh|\ell}\omega_{kr} \\ &\quad + R_{ijkr|\ell}\omega_{hr} + R_{rjkh|r}\omega_{\ell r} + R_{ijkh|\ell m}\omega_m, \qquad (15_1) \\ dR_{ijkh|\ell m} &= R_{rjkh|\ell m}\omega_{ir} + R_{irkh|\ell m}\omega_{jr} + R_{ijrh|\ell m}\omega_{kr} \\ &\quad + R_{ijkr|\ell m}\omega_{hr} + R_{rjkh|\ell r}\omega_{mr} + R_{ijkh|\ell mn}\omega_n, \qquad (15_2) \end{aligned}$$

and so on.

There will exist an integer $p$ for which the right hand sides of the equations $(15_p)$ will be linear combinations of the right hand sides of the equations (15), $(15_1)$, ..., $(15_p)$. This means that no component of the curvature tensor differentiated $p$ times is an independent function of the components of the curvature tensor and those of its $p-1$ first derivatives. When this integer $p$ is known, the condition for the applicability of the two spaces is the compatibility of the equations that express that the components of the curvature tensor and of its first $p$ derived tensors for the two spaces be pairwise equal to one another.

THEOREM. — *Given an $n$-dimensional Riemannian space, there is an integer $p$ that has the following property: For a second space of the same dimension to be applicable onto it, it is necessary and sufficient that there exist between these two spaces a rectangular system of reference to rectangular system of reference correspondence which realises the pairwise equality of the components of their Riemann-Christoffel tensors, and of their derived tensors of the first $p$ orders.*

**319.** The integer $p$ is equal to 1 if the components of the Riemann-Christoffel tensor are independent functions of the $\frac{n(n+1)}{2}$ variables $u^i$ and $\xi^k$. This is the case, for example, of the spaces examined in Section I at n°s 308-311; in fact, to say that the coefficients $R_{11}, R_{22}, ..., R_{nn}$ of the reduced equation of the Ricci quadric are independent functions, that is to say that the $\frac{n(n+1)}{2}$ components $R_{ij}(u, \xi)$ of the Ricci tensor are not related by any equation. The general theorem of n° 318 teaches us that the conditions for applicability of two spaces of the preceding kind involve involve only the components of the Riemann-Christoffel tensor and those of its derived tensor, whereas the theorem of n° 311 involves only the components of the Ricci tensor and its derived tensor. It must be concluded that the stated conditions in the general theorem of n° 318 may not all be independent; we will see other examples of this fact in n° 321.

**320.** In the case of two symmetric spaces of the same number of dimensions, the theorem in n° 318 gives a necessary and sufficient condition of a particularly simple application, namely that there exists a correspondence between the two spaces, rectangular system of reference to rectangular system of reference, which realises the pairwise equality of the components of the Riemann-Christoffel tensors of the two spaces. Since we know moreover that we can introduce a rectangular system of reference at each point in each of these spaces such that the components of the Riemann-Christoffel tensor are constants, the condition of applicability is simply that, once the choice of systems of reference has been made in one of the spaces, it can be made in the other in such a way as to realise the pairwise equality of the components of the two tensors.

**321.** In the case $n = 2$, we find the classic conditions for the application of two surfaces.[5] By calling the Riemannian curvature $K$, equations (15), $15_1$) and

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[5] See G. DARBOUX, *Leçons sur la théorie des surfaces*, t. III, Livre VII, Chap. II; E. CARTAN, *La théorie des groupes finis el continus et la Géométrie différentielle*, Chap. XII, n° 195, p. 227-230.

$(15_2)$ become

$$\left.\begin{aligned}
dK &= & K_1\omega_1 & \quad +K_2\omega_2,\\
dK_1 &= K_1\omega_{12} & +K_{11}\omega_1 & \quad +K_{12}\omega_2,\\
dK_2 &= -K_1\omega_{12} & +K_{12}\omega_1 & \quad +K_{22}\omega_2,\\
dK_{11} &= 2K_{12}\omega_{12} & +K_{111}\omega_1 & \quad +K_{112}\omega_2,\\
dK_{12} &= (K_{22}-K_{11})\omega_{12} & +K_{121}\omega_1 & \quad +K_{122}\omega_2,\\
dK_{22} &= -2K_{12}\omega_{12} & +K_{221}\omega_1 & \quad +K_{222}\omega_2.
\end{aligned}\right\} \tag{13.16}$$

The six quantities $K, K_1, K_2, K_{11}, K_{12}, K_{22}$ are functions of the coordinates $u, v$ of a point on the surface and of the parameter $\xi$, which defines the orientation of each system of reference with this point as origin. There are thus five functions of these six arguments that depend only on $u^1, u^2$. First, there is the Riemannian curvature $K$ itself. It's easy to find four others, namely

$$\begin{aligned}
\Delta_1 K &= (K_1)^2 + (K_2)^2,\\
\Delta_1(K, \Delta_1 K) &= 2(K_1^2 K_{11} + 2K_1K_2K_{12} + K_2^2K_{22}) = K_1(\Delta_1K)_1 + K_2(\Delta_1K)_2,\\
\Theta(K, \Delta_1 K) &= 2[K_1^2K_{12} + K_1K_2(K_{22}-K_{11}) - K_2^2K_{12}) = K_1(\Delta_1K)_2 - K_2(\Delta_1K)_1,\\
\Delta_2 K &= K_{11} + K_{22};
\end{aligned}$$

$\Delta_1$ and $\Delta_2$ are the symbols of the two differential parameters of Beltrami; $\Delta_1(U, V) = U_1V_1 + U_2V_2$, and the function $\Theta(U, V)$ is defined by the relation

$$[\,dU\ dV\,] = \Theta(U, V)[\,\omega_1\ \omega_2\,].$$

**322.** That said, leaving aside the surfaces of constant Riemannian curvature, which are applicable if they have the same curvature, several cases are possible:

**1°**. *The functions $K$ and $\Delta_1 K$ are independent.* — The reference functions $K, K_1, K_2$ are then also independent; the condition for the applicability of two surfaces in this first case is the existence of a *rectangular system of reference to rectangular system of reference* correspondence that realises pairwise the equality of each of the functions $K, K_1, K_2, K_{11}, K_{12}, K_{22}$. If only point functions are to be used, the required condition will be the existence of a *point-to-point* correspondence that realises pairwise the equality of each of the functions $K, \Delta_1K, \Delta_1(K, \Delta_1K), \Theta(K, \Delta_1K), \Delta_2K$. But in fact, these conditions are redundant. To abbreviate, put $\Delta_1K = P$. By putting

$$dP = P_1\omega_1 + P_2\omega_2 \qquad (\text{with } dK = K_1\omega_1 + K_2\omega_2),$$

we see that

$$\begin{aligned}
ds^2 = \omega_1^2 + \omega_2^2 &= \frac{(P_1dK - K_1dP)^2 + (P_2dK - K_2dP)^2}{(P_1K_2 - P_2K_1)^2}\\
&= \frac{\Delta_1P\,dK^2 - 2\Delta_1(P_1, K)dPdK + \Delta_1K\,dP^2}{(P_1K_2 - P_2K_1)^2}.
\end{aligned}$$

Application will be guaranteed if there exists a point correspondence that realises pairwise the equality of the functions

$$K,\ P,\ (K_1P_2 - K_2P_1),\ \Delta_1 P,\ \Delta_1(P,K);$$

but we need not take into account the function $(K_1P_2 - K_2P_1)^2$ because we have

$$(K_1P_2 - K_2P_1)^2 = \delta_1 K\, \Delta_1 P - [\Delta_1(P,K)]^2,$$

hence the

THEOREM. — *Given two surfaces for which the functions $K$ and $\Delta_1 K$ are independent, the necessary and sufficient condition for these two surfaces to be applicable is the existence of a point correspondence between the two surfaces that realises pairwise the equality of each of the functions*

$$K,\quad \Delta_1 K,\quad \Delta_1(K,\Delta_1 K),\quad \Delta_1(\Delta_1 K).$$

Less precisely, we can also say that the surfaces are applicable if $\Delta_1(K,\Delta_1 K)$ and $\Delta_1(\Delta_1 K)$ are the same functions of $K$ and of $\Delta_1 K$ for both surfaces. We see that the components of the curvature, twice differentiated, enters only through two functions in place of three, since the function $\Delta_2 K$ is not relevant.

**2$^o$.** *The first differential parameter of $K$ is a function of $K$.* — By putting $\Delta_1 K = f(K)$, we find easily That

$$\Delta_1(K,\Delta_1 K) = f(K)f'(K), \qquad \Theta(K,\Delta_1 K) = 0;$$

if for two surfaces $\Delta_1 K$ is the same function of $K$, then so is $\Delta_1(K,\Delta_1 K)$ and $\Theta(K,\Delta_1 K)$. That said, Case 2$^o$ can be subdivided into two others according as $\Delta_2 K$ is or is not an independent function of $K$.

**a.** *$\Delta_2 K$ is a function independent of $K$.* — The system of reference functions $K, K_1, K_2, K_{11}, K12, K_{22}$ are three independent functions, and it follows from the general theorem of n$^o$ 318 that the conditions for application of two surfaces of the class considered will involve the *third* covariant derivatives of $K$. By putting $\Delta_2 K = Q$, we see by a calculation similar to the one made in Case 1$^o$ that we have

$$ds^2 = \frac{\Delta_1 Q\, dK^2 - 2\Delta_1(K,Q)dKdQ + \Delta_1 K\, dQ^2}{(K_1Q_2 - K_2Q_1)^2},$$

and the same reasoning leads to

THEOREM. — *Given two surfaces of the class considered, the necessary and sufficient condition for them to be applicable is the existence of a point* *correspondence between the two surfaces resulting in the pairwise equality of each of the functions*

$$K, \quad \Delta_1 K, \quad \Delta_2 K, \quad \Delta_1(K, \Delta_2 K), \quad \Delta_1(\Delta_2 K).$$

**b.** $\Delta_2 K$, *as well as* $\Delta_1 K$, *is a function of* $K$. — According to the general theory, the conditions for application of two surfaces of this class involve only the covariant derivatives of $K$ of the first two orders. Any application will be given by a point correspondence that realises the pairwise equality of each of the functions $K, \Delta_1 K, \Delta_2 K$, which moreover amounts to saying that $\Delta_1 K$ and $\Delta_2 K$ must be the same functions of $K$ for the two surfaces. There is then an infinite number of applications subject to the additional condition that they realise the equality of the forms[6] $K_1\omega_2 - K_2\omega_1$, for the two surfaces. In fact this additional condition is necessary, and if it is satisfied, since we have already assumed the equality of the forms $K_1\omega_1 + K_2\omega_2 = dK$ for the two surfaces, we will have equality of the forms

$$(K_1^2 + K_2^2)ds^2 = \Delta_1 K \, ds^2,$$

and consequently the equality of the fundamental forms.

We can now note that there is a function $\rho(K)$ that makes the differential form $K_1\omega_2 - K_2\omega_1$ exact. By expressing the fact that the exterior differential of the form $\rho(K_1\omega_2 - K_2\omega_1)$ is zero, we find the condition

$$\rho'(K)\,\Delta_1 K + \rho(K)\,\Delta_2 K = 0, \qquad \frac{\rho'}{\rho} = -\frac{\Delta_2 K}{\Delta_1 K}.$$

Choose a specific solution to this equation; we can then state the

THEOREM. — *For two surfaces of the class considered to be applicable, it is necessary and sufficient that* $\Delta_1 K$ *and* $\Delta_2 K$ *be the same functions of* $K$ *for both surfaces. The applications are then given by point correspondences that realise the equality of the Riemannian curvature as well as of the exact differential form* $\rho(K_1\omega_2 - K_2\omega_1)$ *for the two surfaces. The applications thus depend on an arbitrary constant and they are obtained by quadratures.*

The surfaces of this class are, as we know, applicable onto a surface of

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[6] The form $K_1\omega_2 - K_2\omega_1$ because it is, up to sign, independent of the choice of rectangular system of reference; it is in fact the measure of the bivector determined by the gradient of the function $K$ and of the vector $\overrightarrow{dM}$. Denoting the components of $ds^2$ by $E, F, G$ (Gaussian notation), we have

$$K_1\omega_2 - K_2\omega_1 = \frac{1}{\sqrt{EG - f^2}} \begin{vmatrix} \dfrac{\partial K}{\partial u^1} & \dfrac{\partial K}{\partial u^2} \\ E\,du^1 + F\,du^2 & F\,du^1 + G\,du^2 \end{vmatrix}.$$

revolution; moreover, putting

$$\frac{K_1\omega_1 + K_2\omega_2}{\sqrt{\Delta_1 K}} = \frac{dK}{\sqrt{\Delta_1 K}} = dc, \qquad \rho(K_1\omega_2 - K_2\omega_1) = dy,$$

we have

$$ds^2 = dx^2 + F^2(x)dy^2, \qquad \text{with} \quad F(x) = \frac{1}{\rho\sqrt{\Delta_1 K}}.$$

## IV. — The largest group of displacements of a given Riemannian space.

**323.** The problem of finding the rigid displacements of a given space is a special case of the problem solved in the preceding Section. Only in equations (13) of n° 317, it must be assumed that the forms $\varpi_i$ and $\varpi_{ij}$ are constructed from their arguments in the same way as $\omega_i$ and $\omega_{ij}$ with theirs. The variables $v^1, v^2, ..., v^n$ and the variables $u^1, u^2, ..., u^n$ are the coordinates of two points in the same Riemannian space that correspond to each other through a rigid displacement of this space. The results we have obtained allow us to state the following general theorem:

THEOREM. — *Given a Riemannian space, by simple differentiations we can reduce the search for rigid displacements of this space to the search for transformations of rectangular systems of reference to rectangular systems of reference that leave invariant a certain number of functions of the parameters* $u^i, \xi^k$ *that individualise these frames and a number of Pfaffian expressions constructed from these parameters and their differentials. The structure constants of the largest continuous group of displacements in space are thus known without integration. The functions and Pfaffian forms to be considered involve only the components of the Riemann-Christoffel tensor, and its first* $p$ *derived tensors, where* $p$ *is an integer at most equal to* $\frac{n(n+1)}{2}$.

*We can also eliminate the parameters* $\xi^k$ *by algebraic operations (search for invariants with respect to the orthogonal group in* $n$ *variables) so that we need consider only point transformations that leave invariant functions of the coordinates* $u^i$ *alone and Pfaffian forms constructed from these coordinates and their differentials.*

**324.** Return now to equations (15), ($15_1$), ($15_2$), ... of n° 318. They allow us to recognise the order $r$ of the largest group $G$ of rigid displacements of the space, as well as the order $\rho$ of the largest group $g$ of rigid rotations around a point.

In fact, let $p$ be the integer corresponding to the given space. The number of independent linear forms with respect to the $\omega_i$ and the $\omega_{ij}$ which appear on the right hand sides of equations (15), ($15_1$),..., ($15_{p-1}$) is equal to $\frac{n(n+1)}{2} - r$;

it is in effect equal to the number of independent functions of the $u^i$ and the $\xi^k$ taken from among the components of the Riemann-Christoffel tensor and its first $p-r$ derived tensors, and this number is precisely the number of invariants of the $G$ group *considered as operating on the rectangular systems of reference of the space.* Consider now the same right hand sides of equations (15), ($15_1$), ..., ($15_{p-1}$), but leaving only the terms in $\omega_{ij}$; the number of independent linear forms thus obtained is equal to $\dfrac{n(n-1)}{2} - \rho$ because, at a generic point in space, it is equal to the number of functions of the independent $\xi^k$ that remain invariant to the rotation group at this point.

The number of group invariants, functions of point coordinates alone, is equal to the difference

$$\frac{n(n+1)}{2} - r - \left[\frac{n(n-1)}{2} - \rho\right] = n + \rho - r;$$

it is the difference between $n$ and the number of dimensions $r-\rho$ of the trajectories of the group.

We can arrive at the number $\rho$ and at the orthogonal group of rigid rotations at a point by a sequence of what could be called isotropy groups of orders 0, 1, 2, . . . of the space at this point.

The isotropy group of order zero is the group of rotations which leave invariant the components of the Riemann-Christoffel tensor. The components $\omega_{ij}$ of infinitesimal rotations of this group are precisely those that reduce the right hand sides of the equations (15) to zero, where we removes the terms in $\omega_{ij}$; the order $\rho_0$ of this isotropy group is equal to $\dfrac{n(n-1)}{2}$, reduced by the number of independent linear forms in $\omega_{ij}$ that enter into these equations.

The isotropy group of order 1 is the group of rotations which leave invariant the components of the Riemann-Christoffel tensor and of its derived tensor: its order $\rho_1$ is the difference between $\dfrac{n(n-1)}{2}$ and the number of independent linear forms in $\omega_{ij}$ which appear on the right hand sides of equations (15) and ($15_1$). We obtain also the infinitesimal rotations of the isotropy groups of successive orders $2, 3, ..., p-1$, as well as their orders $\rho_2, \rho_3, ..., \rho_{p-1}$, which are not increasing; we have obviously $\rho_{p-1} = \rho_p$, and this is the order $\rho$ of the group of rigid rotations at the point considered. Note that it is not sufficient that $\rho_q$ is equal to $\rho_{q-1}$ for us to arrive at the final rotation group, because by passing from equations ($15_{q-1}$) to equations ($15_q$), the number of *point* invariants may increase.

**325.** Let us add finally that if one can determine without integration the order of the largest group of displacements of a space as well as its structure, it is necessary, to obtain the group effectively, to integrate a completely integrable Pfaffian system, which reduces moreover to the integration of ordinary differential equations (Note V), an integration which can be subject to notable simplifications according to the structure of the group.

**326.** In the case of a symmetric space, we need consider only equations (15); since the right hand sides of these equations contain only terms in $\omega_{ij}$, there is no point invariant, the group is transitive and its order is equal to $\frac{n(n-1)}{2}$ minus the number of independent linear forms in w$\omega_{ij}$ which appear on the right hand sides of the equations. These linear forms, equated to zero, define the infinitesimal rigid rotations at a point.

## V. — The equations of Killing.

**327.** Almost all geometers who have dealt with the search for rigid displacements of a given Riemannian space have begun by searching for the *infinitesimal* displacements, and this by the integration of a certain differential system formed by what are called the *Killing equations.* In this section, we will deduce these equations.

Given a Riemannian space referred to a system of rectangular coordinate systems, denote by $\xi_i$ the components of the elementary displacement undergone by a point $M$ due to the action of an infinitesimal rigid displacement. Denote the symbol of this displacement by $\delta$; the component $\xi_i$ is the form $\omega_i$ where we have replaced the differentials $du^i$ of the coordinates by their infinitesimal increments $\delta u^i$ due to the displacement considered; we shall put

$$\xi_i = \omega_i(\delta) \quad \text{and} \quad \xi_{ij} = \omega_{ij}(\delta). \tag{13.17}$$

The structure equations

$$d\omega_i = [\,\omega_k\ \omega_{ki}\,]$$

can be written, using the two differentiation symbols $d$ and $\delta$, the first of which is an indeterminate symbol of differentiation

$$\delta\omega_i - d\xi_i = \xi_k\omega_{ki} - \xi_{ki}\omega_k,$$

or

$$\delta\omega_i = d\xi_i + \xi_k\omega_{ki} + \xi_{ik}\omega_k = D\xi_i + \xi_{ik}\omega_k = (\xi_{i|k} + \xi_{ik})\omega_k; \tag{13.18}$$

we have denoted the $k^{\text{th}}$ covariant, or absolute, derivative of the tensor $\xi_i$ by $\xi_{i|k}$.

We deduce from formulas (13.18) the relation

$$\frac{1}{2}\delta(ds^2) = \omega_i\,\delta\omega_i = (\xi_{i|k} + \xi_{ik})\omega_i\omega_k. \tag{13.19}$$

The infinitesimal displacement considered that leaves the fundamental form invariant, it follows from (19) that the quadratic form on the right hand side is identically zero and, consequently, that the coefficients $\xi_{i|k} + \xi_{ik}$ are antisymmetric with respect to the two indices $i, k$. Now $\xi_{ik} = -\xi_{ki}$: we therefore have the

THEOREM. — *For the vector field $\xi_i$ to define an infinitesimal rigid displacement, it is necessary and sufficient that its first derived tensor $\xi_{i|k}$ be a bivector.*

Our reasoning is based on the assumption that space is referred to a family of rectangular systems of reference, but the result is independent of the choice of local reference systems. We should thus have, in general,

$$\xi_{i|k} = -\xi_{k|i}. \tag{13.20}$$

**328.** Equations (13.20) lead to certain second order differential equations. Consider the case where we have chosen any Cartesian systems of reference, and find the exterior derivative of the equations

$$d\xi^i + \xi^k \omega_k{}^i = \xi^i{}_{|j}\omega^j;$$

we get (*cf.* n° 192)

$$\frac{1}{2}\,\xi^k R_k{}^i{}_{h\ell}[\omega^h\ \omega^\ell] = \xi^i{}_{|\ell h}[\omega^h\ \omega^\ell] = \frac{1}{2}(\xi^i{}_{|\ell h} - \xi^i{}_{|h\ell})[\omega^h\ \omega^\ell],$$

hence

$$\xi^i{}_{|\ell h} - \xi^i{}_{|h\ell} = \xi^k R_k{}^i{}_{h\ell} \quad \text{or} \quad \xi_{i|\ell h} - \xi_{i|h\ell} = \xi^k R_{kih\ell}.$$

By performing two successive cyclic permutations on the indices $i, h, \ell$, we obtain three equations

$$\begin{aligned}
\xi_{i|\ell h} - \xi_{i|h\ell} &= \xi^k R_{kih\ell},\\
\xi_{h|i\ell} - \xi_{h|\ell i} &= \xi^k R_{kh\ell i},\\
\xi_{\ell|hi} - \xi_{\ell|ih} &= \xi^k R_{k\ell ih},
\end{aligned}$$

from which we deduce easily, by taking into account (13.20),

$$\xi_{i|h\ell} = -R_{ih\ell m}\xi^m \quad \text{or} \quad \xi^i{}_{|h\ell} = R_h{}^i{}_{\ell m}\xi^m; \tag{13.21}$$

These equations are *Killing's equations*[7]

In the case of Euclidean space referred to a rectangular system of coordinates $x_i$, equations (13.20) and (13.21) become

$$\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0, \qquad \frac{\partial^2 \xi_i}{\partial x_h \partial x_j} = 0;$$

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[7] W. KILLING, *Ueber die Grundlagen der Geometrie* (*Journal de Crelle*, t. 109, 1892, p. 167); G. RICCI, *Sui gruppi continui di movimenti in una varietá qualunque a tre dimensioni* (*Mem. Soc. ital. Sc.*, t. 12, 1899, p. 77). See also M. LÉVY, *Sur la cinématique des figures contenues sur les surfaces courbes et en général dans les variétés planes ou courbes* (*C. R.*, t. 86, 18-8, p. 812-816).

integrated, they give

$$\xi_i = a_{ih} x_h + a_i, \qquad (a_{ij} = -a_{ji});$$

the components $a_i$ are related to an infinitesimal translation and the bivector $a_{ij}$ to an infinitesimal rotation about the origin.

# NOTE I

## ON THE AXIOM OF THE PLANE AND THE CAYLEY GEOMETRIES.

We assumed implicitly in the text (Chapter V), when dealing with the axiom of the plane, that the geodesic surfaces of a Riemannian space satisfy certain analytic conditions, which are worth examining more closely. In what follows, we will assume that the coefficients $g_{ij}$ of the fundamental form have continuous partial derivatives of the first order, which ensures the continuity of the quantities ${\Gamma_i}^k{}_j$. We will assume moreover that these quantities have properties that are strictly sufficient to ensure that:

**1°** the differential equations of the geodesics

$$\frac{d^2u^i}{ds^s} + {\Gamma_k}^i{}_h \frac{du^k}{ds}\,\frac{du^h}{ds} = 0 \tag{N1.1}$$

have one and only one solution corresponding to given initial conditions

$$u^i = (u^i)_0, \qquad \frac{du^i}{ds} = (v^i)_0, \qquad \text{for } s = 0;$$

**2°** in a sufficiently small domain of the space, there is one and only one geodesic that passes through two given points.

Naturally, we leave aside the search for the analytic conditions that the $g_{ij}$ must satisfy for this to be the case. Obviously, it is sufficient that they have continuous partial derivatives of the first two orders.

Accepting hypotheses 1° and 2°, we propose to show how the axiom of the plane leads to the possibility of geodesic representation on ordinary space.

We will assume that $n = 3$ and we will write $u, v, w$ in place of $u^1, u^2, u^3$.

### I – Preliminaries.

**1.** By putting

$$x = u'_0 s, \qquad y = v'_0 s, \qquad z = w'_0 s, \tag{N1.2}$$

where $u'_0, v'_0, w'_0$ denote the initial values of the derivatives of the unknown functions $u, v, w$, the equations for geodesics emanating from a given point $A\,(u_0, v_0, w_0)$

can be put into the form

$$\left.\begin{aligned} u-u_0 &= f(x,y,z),\\ v-v_0 &= g(x,y,z),\\ w-w_0 &= h(x,y,z). \end{aligned}\right\} \tag{N1.3}$$

In short, formulae (N1.3) define the representation of the Riemannian space on the *normal* Euclidean space (Chapter X).

Hypothesis $2^{\text{o}}$ states that equations (N1.3) can be solved with respect to $x, y, z$ (for sufficiently small $u-u_0, v-v_0, w-w_0$).

**2.** Suppose that, in the region considered of the Riemannian space, the coefficients ${\Gamma_k}^j{}_h$ are smaller in absolute value than a fixed number $M$. Consider an arc of a geodesic situated in this region. If $s$ is the curvilinear abscissa, we have

$$u-u_0 = u'_0 s + \frac{1}{2}\, s^2 u''(\theta s) \qquad (0 \leq \theta \leq 1).$$

Now we have

$$u''(\theta s) = -{\Gamma_1}^1{}_1(u,v,w)[u'(\theta s)]^2 - \cdots ,$$

where we denote by $u, v, w$ the coordinates of the point on the geodesic whose curvilinear abscissa is $\theta s$. When $u', v', w'$ take all possible values consistent with the condition that the $(u', v', w')$ is an unit vector, and this at the various points of the region considered, the right hand side remains less than a fixed quantity $hM$, where $h$ depends only on the coefficients $g_{ij}$. We thus have

$$\left.\begin{aligned} u-u_0 &= u'_0 s + \tfrac{1}{2}\,\theta_1 hMs^2,\\ v-u_0 &= v'_0 s + \tfrac{1}{2}\,\theta_2 hMs^2,\\ w-w_0 &= w'_0 s + \tfrac{1}{2}\,\theta_3 hMs^2, \end{aligned}\right\} \tag{N1.4}$$

with

$$|\theta_1| < 1, \qquad |\theta_2| < 1, \qquad |\theta_3| < 1.$$

We easily deduce from this that if we move on a geodesic surface at $A\,(u_0, v_0, w_0)$, and if $\ell_1, \ell_2, \ell_3$ are the covariant components of an unit vector normal to the surface at $A$, we have an inequality of the form

$$\begin{aligned} |\ell_1(u-u_0) + \ell_2(v-v_0) + \ell_3(w-w_0)& \\ < kM[(u-u_0)^2 + (v-v_0)^2 + (w-w_0)^2],& \end{aligned} \tag{N1.5}$$

where $k$ denotes a fixed coefficient.

**3.** Consider now a surface (S) that contains a given geodesic $(\gamma)$, *and which is geodesic at all its points on* $(\gamma)$. We will prove that *the unit vector normal to* $(S)$ *at a point on* $(\gamma)$ *remains normal to it when parallel-transported along* $(\gamma)$.

With no loss of generality, we can assume that the geodesic $(\gamma)$ is defined by the equations $u = v = 0$; according to (N1.1), this leads to the equations

$$\Gamma_3{}^1{}_3 = \Gamma_3{}^2{}_3 = 0$$

at all the points on $(\gamma)$. The covariant components of the unit vector normal to $(S)$ at a point on $(\gamma)$ are of the form $(\ell_1, \ell_2, 0)$. We want to show that we have, at all points on $(\gamma)$,

$$\left.\begin{aligned} \frac{d\ell_1}{dw} - \ell_1\Gamma_1{}^1{}_3 - \ell_2\Gamma_1{}^2{}_3 = 0, \\ \frac{d\ell_2}{dw} - \ell_1\Gamma_2{}^1{}_3 - \ell_2\Gamma_2{}^2{}_3 = 0, \\ -\ell_1\Gamma_3{}^1{}_3 - \ell_2\Gamma_3{}^2{}_3 = 0. \end{aligned}\right\} \tag{N1.6}$$

The third relation is self-evident. To prove the other two, take a particular point $A$ on $(\gamma)$ for which we can assume $w = 0$. We can change the variables in such a way that the quantities $\Gamma_i{}^k{}_j$ are all zero at $A$ (n$^{rmo}$ 84) and without the equations for $(\gamma)$ ceasing to be $u = v = 0$; under these conditions, it will thus be sufficient to show that we have at point $A$, by moving along $(\gamma)$,

$$\frac{d\ell_1}{dw} = 0,$$
$$\frac{d\ell_2}{dw} = 0.$$

Moreover, these two equations can be reduced to one only by virtue of the relation

$$g^{11}\ell_1{}^2 + 2g^{12}\ell_1\ell_2 + g^{22}\ell_2{}^2 = 1,$$

which gives, by differentiation and by remaining at point $A$,

$$(g^{11}\ell_1 + g^{12}\ell_2)\frac{d\ell_1}{dw} + (g^{12}\ell_1 + g^{22}\ell_2)\frac{d\ell_2}{dw} = 0.$$

If, for example, we suppose that at point $A$ the equation of the plane element tangent to $(S)$ is $du = 0$, $(\ell_2 = 0)$, we simply need to show that we have

$$\frac{d\ell_2}{dw} = 0,$$

or that the ratio $\dfrac{\ell_2}{w}$ tends towards zero as $w$ tends to zero.

Consider a particular geodesic $(\gamma')$ emanating from $A$ and on the surface $(S)$, with

$$u'_0 = 0, \qquad v'_0 = b \neq 0, \qquad w_0 = c \neq 0.$$

Given a positive number $n$, as small as we please; there will be a domain $(D)$ around the point $A$ inside which the continuous quantities $\Gamma_k{}^i{}_h$ will remain smaller than $\eta$ in absolute value. We will then have, according to (N1.4), for the geodesic $(\gamma')$,

$$\begin{aligned} u &= \phantom{bs +{}} \frac{1}{2}\,\theta_1 h\eta s^2, \\ v &= bs + \frac{1}{2}\,\theta_2 h\eta s^2, \\ w &= cs + \frac{1}{2}\,\theta_3 h\eta s^2. \end{aligned}$$

Since $w$ is a function of $s$ that has continuous derivatives of the first two orders, it is the same for $s$ considered as a function of $w$, and we can write, by introducing a fixed coefficient $h'$,

$$\left.\begin{array}{rcl} u &=& \theta'_1 h'\eta w^2, \\ v &=& \dfrac{b}{c}\,w + \theta'_2 h'\eta w^2 \end{array}\right\} \tag{N1.7}$$

with

$$|\theta'_1| < 1, \qquad |\theta'_2| < 1.$$

Give $w$ a fixed value $w_0$ and consider the corresponding point $M'$ of the geodesic $(\gamma')$ as belonging to the surface $(S)$, considered as geodesic at the point $M(0, 0, w_0)$ on $(\gamma)$. According to (N1.5), and denoting by $\ell_1, \ell_2, 0$ the covariant components of the unit vector normal to $(S)$ at point M, we will have

$$|\ell_1 u + \ell_2 v| < k\eta(u^2 + v^2),$$

or

$$|\ell_2\,\frac{b}{c}\,w_0 + h'\eta w_0^2(\theta'_1\ell_1 + \theta'_2\ell_2| < k\eta w_0^2\left[\left(\frac{b}{c} + \theta'_2 h'\eta w_0\right)^2 + {\theta'_1}^2 {h'}^2\eta^2 w_0^2\right],$$

or also

$$\left|\frac{b}{c}\;\frac{\ell_2}{w_0}\right| < h'\eta|\theta'_1\ell_1 + \theta'_2\ell_2| + k\eta\left[\left(\frac{b}{c} + \theta'_2 h'\eta w_0\right)^2 + {\theta'_1}^2 {h'}^2\eta^2 w_0^2\right].$$

Finally, by introducing a fixed number $H$ independent of $\eta$ and of $w_0$, of $\ell_1$ and of $\ell_2$, we can write

$$\left|\frac{b}{c}\;\frac{\ell_2}{w_0}\right| < H\eta.$$

This inequality shows that if $\eta$ is a given positive number which can be as small as we like, we can take $w_0$ to be small enough for the ratio $\dfrac{\ell_2}{w_0}$ to remain smaller in absolute value than $H\left|\dfrac{c}{b}\right|\eta$. This means that $\dfrac{\ell_2}{w}$ tends to zero with $w$, when the point $M$ on $(\gamma)$ tends towards $A$. This is what we wanted to prove.

**4.** The following consequence follows immediately from the preceding theorem:

*If two surfaces* $(S_1)$ *and* $(S_2)$ *intersect along a geodesic* $(\gamma)$ *and if they are both geodesic at all points on* $(\gamma)$*, they intersect at a constant angle.*

In fact, the two unit vectors normal to $(S_1)$ and $(S_2)$ at a point on $(\gamma)$ remain normal if we parallel-transport them along $(\gamma)$; consequently the angle of the normals to the two surfaces is constant all along $(\gamma)$.

## II. – The theorem of F. Schur.

**5.** It follows from the theorems just proved that the plane element tangent to a totally geodesic surface, that we defined as a surface that is geodesic at each of its points, varies continuously.

If all geodesic surfaces at a point $A$ are totally geodesic, it is easy to see that there always exists a surface that is geodesic at $A$ and which contains a given geodesic (in a region that is not too extended around $A$). In fact, let $(\gamma)$ be a geodesic, and $M$ one of its points. A geodesic $(\gamma')$ passes through points $A$ and $M$; consider the unit vector at $M$ normal to $(\gamma)$ and $(\gamma')$, and parallel-transport it from $M$ to $A$ along $(\gamma')$. There is a surface $(S)$ geodesic at $A$ and normal at this point to the vector obtained; the normal at $M$ to this surface, *which is geodesic all along* $(\gamma')$, will, according to n° 3, be normal at $M$, not only to $(\gamma')$, but to $(\gamma)$; the geodesic $(\gamma)$, tangent at $M$ to $(S)$, will therefore be entirely contained in the surface $(S)$, which is geodesic at $M$.

We are now in a position to prove Schur's theorem, according to which the space satisfies the axiom of the plane if there exist two points $A$ and $B$ such that any surface geodesic at one of these two points is totally geodesic (n° 112).

First, we can attach to each point $M$ in the space six quantities $x, y, z; x', y', z'$, as was done in n° 113; the first three are defined up to an arbitrary common factor, as are also the last three. In the text, we proved an important relationship between these quantities, based on the property that four totally geodesic surfaces passing through AB have the same cross ratio at $A$ and at $B$. The proof of this property given in the text is not valid here. But, thanks to the theorem proved in n° 4, this property is obvious, because the angles at which the four surfaces intersect at $A$ are the same as those at which they intersect at $B$.

We can thus accept the general result recalled at the beginning of n° 114. *We can attach to each point of the space four homogeneous coordinates* $X, Y, Z, T$ *such that any totally geodesic surface passing through* $A$ *is defined by an equation that is linear in* $X, Y, Z$*, and any totally geodesic surface passing through* $B$ *by an equation that is linear in* $X, Y, T$. Consequently, any geodesic that is at the intersection of two of these surfaces, is defined by a system of equations that is linear in $X, Y, Z, T$. In other words, *the Riemannian space admits a geodesic* *representation in ordinary space, in which geodesics are represented by lines.*

**6.** What is no longer obvious is that the planes of ordinary space are the images of geodesic surfaces in Riemannian space, because the non-homogeneous coordinates $\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}$ of ordinary space are functions of the coordinates $u, v, w$ of of the Riemannian space, *whose nature we do not know*. Consequently, it is not certain that geodesics tangent to the same plane element of the Riemannian space have as their images straight lines in the same plane in ordinary space.

This property is nevertheless exact at point $A$ and point $B$, according to equations (8) of n$^{\text{o}}$ 113. To prove this in the general case, take a point $P$ in Riemannian space, and its image $P'$ in ordinary space. Any geodesic issuing from $P$ is defined by the mutual ratios of three quantities $(du, dv, dw)$; the image line will also be defined by the mutual ratios of three quantities $d\xi, d\eta, d\zeta$, where for example we denote the ratios $\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}$ by $\xi, \eta, \zeta$. On the one hand, consider $du, dv, dw$, and on the other, $d\xi, d\eta, d\zeta$ as the homogeneous coordinates of two points $m, m'$ on a plane $(\Pi)$. Let $a$ and $b$ be the points corresponding to the geodesics $PA, PB$; let $a'$ and $b'$ be the points corresponding to the image straight lines $P'A', P'B'$.

Any plane element at $P$ in the Riemannian space will be represented in the plane $(\Pi)$ by a straight line $d$, and any plane element at $P'$ in ordinary space by a straight line $d'$. If the straight line $d$ passes through $a$, the plane element at $P$, tangent to a totally geodesic surface passing through $A$, has as its image a plane element at $P'$ which is part of a plane passing through $A'$; consequently, *to any straight line $d$ passing through $a$ there corresponds a straight line $d'$ passing through $a'$*. Similarly, *to any straight line $d$ through $b$ there corresponds a straight line $d'$ through $b'$*. Finally, according to the above remark, the cross ratio of four straight lines $d$ through $a$ (or through $b$) is equal to the cross ratio of the four corresponding straight lines $d'$ through $a'$ (or through $b'$).

**7.** That said, let $\delta$ be any straight line in the plane $(\Pi)$, which corresponds to a surface $\Sigma$ that is geodesic at $P$; we look for the locus of the points $m'$ that correspond to the various points $m$ of $\delta$. The cross ratio of the four lines $(a \cdot m_0, m_1, m_2, m_3)$ is equal to the cross ratio at $A$ of four geodesic surfaces that contain the geodesic $AP$; it is thus equal to the cross ratio of these four surfaces at $P$, and consequently to the cross ratio of the four curves of intersection of these surfaces with $\Sigma$. Obviously we find the same result by taking the cross ratio of the four straight lines $(b \cdot m_0, m_1, m_2, m_3)$. It follows that we will also have

$$(a' \cdot m'_0, m'_1, m'_2, m'_3) = (b' \cdot m'_0, m'_1, m'_2, m'_3);$$

thus the points $m'_1, m'_2, m_3$ are in a straight line. *To $\delta$ therefore there corresponds a straight line $\delta'$.*

Since the correspondence $(m, m')$ in the plane $(\Pi)$ has the property that every straight line corresponds to a straight line, it follows that every plane element of the Riemannian space corresponds to a plan element of ordinary space. Consequently, *surfaces in the Riemannian space whose images are planes in ordinary space are totally geodesic surfaces.* The axiom of the plane is thus satisfied in the Riemannian space.

**8.** There is more. The point correspondence $(m, m')$ of the plane $(\Pi)$ is projective. In other words, if we move on any geodesic through a point $P$, we have

$$\frac{du}{a\,d\xi + b\,d\eta + c\,d\zeta} = \frac{dv}{a'\,d\xi + b'\,d\eta + c'\,d\zeta} = \frac{dw}{a''\,d\xi + b''\,d\eta + c''\,d\zeta}\,.$$

The cosine of the angle of two directions, in the ordinary space where the representation is made, will therefore be of precisely the same form as in a Cartesian system of coordinate for which the isotropic cone would be a specific cone of the second order.

We might add moreover that if, when we move along the straight line $d\eta = d\zeta = 0$, the quantities $\frac{\Delta u}{\Delta \xi}, \frac{\Delta v}{\Delta \xi}, \frac{\Delta w}{\Delta \xi}$ tend towards definite limits; in other words, *the coordinates $u, v, w$ have first-order partial derivatives with respect to $\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}$ and conversely.*

To move on from this to Cayley geometries, there is no longer any difficulty in reasoning as in the text (n° 156-157). *Any Riemannian space that satisfies the axiom of the plane is thus locally Euclidean, spherical or hyperbolic.*

# NOTE II

## ON THE LINEAR RIEMANNIAN CURVATURE.

In Chapter VII we showed that the Riemannian curvature shows up when we develop the space along a closed contour. For a contour bounding a very small area about a given point, this curvature depends on the orientation of the area; for given orientation, it is proportional to the area. We can say briefly that the Riemannian curvature is a *superficial* quantity (that is, attached to an element of surface in the space).

We assumed for this that the coefficients $g_{ij}$ of the fundamental form have continuous partial derivatives of the first two orders.

We shall see that things look different if we do not make this assumption. Consider the simplest case where, in a certain domain of the Riemannian space, the $g_{ij}$ have continuous partial derivatives of the first two orders, except at all points of a surface $(\Sigma)$ crossing the domain. We will assume that the $g_{ij}$ has continuous partial derivatives of the first order at all points on the surface, *but with discontinuity of the normal derivative.* More precisely, at a point $M$ of $(\Sigma)$, the function $g_{ij}$ has a well defined derivative in all directions, but the derivatives taken in the two opposite directions normal to $(\Sigma)$ do not have the same value, whereas the derivatives in two opposite directions tangent to $(\Sigma)$ are equal.

There is nothing to prevent us from assuming that the surface $(\Sigma)$ is defined by the equation $u^3 = 0$. We will call the two faces of $(\Sigma)$ the *positive face* and the *negative face*. If we fix the indices $i$ and $j$, the covariant vector

$$\left(\frac{\partial g_{ij}}{\partial u^k}\right)_+ - \left(\frac{\partial g_{ij}}{\partial u^k}\right)_-$$

is, by hypothesis, normal to the surface; its first two covariant components ($k =$ 1 and 2) are therefore zero. If we denote by $h_{ij}$ the sudden change in the normal derivative when we go from the negative side to the positive side, we have

$$\left(\frac{\partial g_{ij}}{\partial u^k}\right)_+ - \left(\frac{\partial g_{ij}}{\partial u^k}\right)_- = \frac{1}{\sqrt{g^{33}}}\, h_{ij}. \tag{N2.1}$$

If we denote by $H_{ikj}$ the difference between the two values of $\Gamma_{ikj}$ on the positive face and on the negative face of $(\Sigma)$, we obtain without difficulty the values

$$\left.\begin{aligned} H_{ikj} &= 0 && (i,k,j=1,2),\\ H_{i3j} &= -H_{3ij} = -\frac{1}{2}\,\frac{h_{ij}}{\sqrt{g^{33}}} && (i,j=1,2),\\ H_{33i} &= 0, \qquad H_{3i3} = \frac{h_{i3}}{\sqrt{g^{33}}} && (i,j=1,2),\\ H_{333} &= \frac{1}{2}\,\frac{h_{ij}}{\sqrt{g^{33}}} \end{aligned}\right\} \tag{N2.2}$$

That said, consider a small arc of the curve $MM'$ on the surface $(\Sigma)$; begin with a vector $(X^i)$ of origin $M$ and parallel-transport it along the arc of the curve $MM'$ *on the negative side of* $(\Sigma)$, then along the arc of the curve $M'M$ *on the positive side of* $(\Sigma)$. By the first transport, the vector has components $(Y^i)$ given by the formulae

$$Y^i = X^i - X^k (\Gamma_k{}^i{}_r)_- \; du^r;$$

by the second transport, the vector has components $(Z^i)$ given by the formulae

$$Z^i = Y^i - Y^k (\Gamma_k{}^i{}_r)_+ \; du^r = X^i + X^k H_k{}^i{}_r \; du^r. \tag{N2.3}$$

Note that, according to (2), the $H_{kir}$, where the third index $r$ is different from 3, satisfy the relations $H_{kir} = -H_{ikr}$. It follows that the *parallel transport considered has caused the vector to undergo an infinitesimal rotation of covariant components*

$$a_{ij} = H_{ijr} du^r.$$

We find immediately that

$$\left.\begin{aligned} a_{12} &= 0,\\ a_{13} &= -\frac{1}{2\sqrt{g^{33}}}(h_{11}du^1 + h_{12}du^2)\\ a_{23} &= -\frac{1}{2\sqrt{g^{33}}}(h_{12}du^1 + h_{22}du^2) \end{aligned}\right\} \tag{N2.4}$$

We thus see that *a rotation about an axis tangent to* $(\Sigma)$ *is associated with each elementary arc of a curve on* $(\Sigma)$ *which is what we could call the linear Riemannian curvature of the space in the direction of this arc of the curve.*

If we denote by $ds$ the length of the arc of the curve, and the direction cosines of its tangent by $\alpha^1, \alpha^2$, an easy calculation shows that the inner product of the vector $\overrightarrow{MM'}$ with the bivector that represents the rotation is equal to

$$\begin{aligned} &\frac{1}{2}\left[h_{11}(du^1)^2 + 2h_{12}du^1du^2 + h_{22}(du^2)^2\right]\\ &\qquad = \frac{1}{2}\left[h_{11}(\alpha^1)^2 + 2h_{12}\alpha^1\alpha^2 + h_{22}(\alpha^2)^2\right] ds^2. \end{aligned}$$

The scalar quantity

$$K = \frac{1}{2}\left[h_{11}(\alpha^1)^2 + 2h_{12}\alpha^1\alpha^2 + h_{22}(\alpha^2)^2\right] \tag{N2.5}$$

is the Riemannian curvature of the space in the direction $(\alpha^1, \alpha^2)$; this expression involves the discontinuities of the normal derivatives of the three coefficients $g_{11}, g_{12}, g_{22}$ which define the metric on the surface. If we place a very small length $\varepsilon$ on the normals of the surface in each direction at the various points of the arc of the curve, we obtain two new arcs of a curve of lengths $d\sigma_+$ and $d\sigma_-$. We have

$$K = \lim_{\varepsilon\to 0}\frac{d\sigma_+^2 + d\sigma_-^2 - 2ds^2}{2\varepsilon\, ds^2} = \lim_{\varepsilon\to 0}\frac{d\sigma_+ + d\sigma_- - 2ds}{\varepsilon\, ds}.$$

*The curvature $K$ is again the sum of the coefficients of dilatation of an arc of a curve on the surface when we move it perpendicularly to the surface in each of the two directions.*

# NOTE III

## ON NORMAL SPACES WITH NEGATIVE OR ZERO RIEMANNIAN CURVATURE.

Using Riemann normal coordinates, we can establish remarkable properties of normal Riemann spaces with variable Riemannian curvature, in the case where this curvature is negative or zero at any point and in any plane direction.

### I – Preliminaries. Properties of $ds^2$ in normal coordinates.

**1.** We have given (n° 56) the definition of a normal Riemannian space. We will assume here an additional hypothesis, which may in fact be unnecessary; we will assume that in each part of the space, represented analytically by means of a system of coordinates $u^i$, the coefficients $g_{ij}$ of the fundamental form have continuous partial derivatives of the first *three* orders [8] where the form is, of course, positive definite. The $\Gamma_i{}^k{}_j$ thus have continuous partial derivatives of the first two orders and the $R_{ijrs}$ continuous partial derivatives of the first order. According to the classical theorems on differential equations, the quantities $u^i - (u^i)_0$, considered as functions of the normal coordinates (n° 213)

$$x^1 = \alpha^1 s, \qquad ..., \qquad x^n = \alpha^n s$$

relative to the point $(u^i)_0$, have continuous partial derivatives of the first two orders. The same is true of the components of a vector obtained by transporting a fixed vector in parallel along a geodesic emanating from the point $(u^i)_0$.

**2.**

### II. – The simply connected covering space.

### III. – Geodesics of simply connected spaces.

### IV. – Normal spaces that are not simply connected.

### V. – Closed geodesics of normal spaces that are not simply connected.

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[8] Instead of *two*, as we assumed in n° 52.

# NOTE IV

## GEODESICS OF NORMAL RIEMANNIAN SPACES.

### I. – An existence theorem.

**1.** We have shown, in Note III, the theorem according to which any two points of a normal space with everywhere negative or zero curvature can be joined by a geodesic. The proof used explicitly the hypothesis concerning the sign of the curvature. This theorem is nevertheless true for all normal spaces.

We will make the same analytical hypotheses as were made in Note III. We will first prove the following preliminary theorem, of purely local scope.

THEOREM 13.1 *Given a point $O$ of a Riemannian space, let $\Sigma_R$ be the hypersphere locus of points $M$ of the space whose distance $[OM]$ to the point $O$ is less than or equal to $R$. There exists a number $R$ with the following property: there exists a geodesic arc and only one joining $O$ to any point $M$ of $\Sigma_R$ without exiting $\Sigma_R$, and this arc has for length the distance $[OM]$.*

### II. – The theorem of F. Schur.